Theorem (conjectured in 1929, Oppenheim; proved in 1986, M.) Let Q be an indefinite irrational quadratic form in $n \geq 3$ variables. Then for every $\varepsilon > 0$ there exists $x \in \mathbb{Z}^n, x \neq 0$ such that

$$|Q(x)| < \varepsilon .$$

Slightly stronger statement:
Under the same conditions
$Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

Conjecture. If $\hat{Q}$ is a product of three linearly independent linear forms on $\mathbb{R}^3$ and

$$\inf_{v \in \mathbb{Z}^3, v \neq 0} |\hat{Q}(v)| > 0$$

then $\hat{Q}$ is a multiple of a rational form.

Theorem (M., 1986). Any bounded orbit of $SO(2,1)$ in $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ is compact.

Conjecture. Any bounded orbit of

$$D = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} \middle| \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, \lambda_1 \lambda_2 \lambda_3 = 1 \right\}$$

in $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ is compact.

$SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ is isomorphic to

$$\Omega_3 \overset{def}{=} \left\{ \begin{array}{c} \text{the space of unimodular} \\ \text{lattices in } \mathbb{R}^3 \end{array} \right\}$$

Conjecture. Let L be the product of n linear forms on $\mathbb{R}^n$. Suppose that $n \geq 3$ and L is not a multiple of a form with integer coefficients. Then

$$\inf_{v \in \mathbb{Z}^n, v \neq 0} L(v) = 0$$

$$x_1^2 - \lambda x_2^2 = (x_1 - \sqrt{\lambda}\, x_2)(x_1 + \sqrt{\lambda}\, x_2)$$

Conjecture. Let $n \geq 3$, $G = SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z})$, and let D denote the group of all diagonal matrices in G. Then any bounded orbit of D in G/Γ is closed.

$$\lfloor x \rfloor = \min_{k \in \mathbb{Z}} |x-k|$$

Littlewood conjecture (~1930). For any two irrational numbers $\alpha, \beta \in \mathbb{R}$

$$\liminf_{n\to\infty} n\lfloor n\alpha \rfloor \lfloor n\beta \rfloor = 0 .$$

$$n(n\alpha - m_1)(n\beta - m_2) = \hat{Q}(n, m_1, m_2)$$

$\{x\}$ the fractional part of x.

$$\liminf_{n\to\infty} n\{n\alpha\}\{n\beta\} = 0 \quad ?$$

not true $\alpha = \sqrt{2}$, $\beta = 2 - \sqrt{2}$

ν a positive continuous function on the sphere

$$\{v \in \mathbf{R}^n \mid \|v\| = 1\}$$

$$\Omega = \{v \in \mathbf{R}^n \mid \|v\| < \nu(v/\|v\|)\}$$

$T\Omega$ the dilate of Ω by T

$$N_{Q,\Omega}(a,b,T) \stackrel{\text{def}}{=} \#\{x \in \mathbf{Z}^n \mid x \in T\Omega \text{ and } a < Q(x) < b\}$$

$$V_{Q,\Omega}(a,b,T) \stackrel{\text{def}}{=} \mathrm{Vol}\{x \in \mathbf{R} \mid x \in T\Omega \text{ and } a < Q(x) < b\}$$

$$V_{Q,\Omega}(a,b,T) \sim \lambda_{Q,\Omega}(b-a)T^{n-2}$$

where

$$\lambda_{Q,\Omega} = \int_{L\cap\Omega} \frac{dA}{\|\nabla Q\|},$$

L is the light cone $Q = 0$ and dA is the area element on L.

Let $\mathcal{O}(p,q)$ denote the space of quadratic forms of signature (p,q) and discriminant ± 1.

Theorem. *(Dani, Margulis 1993) (i) Let $p \geq 2$ and $q \geq 1$. Then for any irrational $Q \in \mathcal{O}(p,q)$ and any interval (a,b)*

$$\liminf_{T\to\infty} \frac{N_{Q,\Omega}(a,b,T)}{V_{Q,\Omega}(a,b,T)} \geq 1.$$

Moreover, this bound is uniform over compact sets of forms: if $\mathcal{K}$ is a compact subset of $\mathcal{O}(p,q)$ which consists of irrational forms, then

$$\liminf_{T\to\infty} \inf_{Q\in\mathcal{K}} \frac{N_{Q,\Omega}(a,b,T)}{V_{Q,\Omega}(a,b,T)} \geq 1.$$

(ii) If $p > 0, q > 0$ and $n = p + q \geq 5$, then for any $\varepsilon > 0$ and any compact subset $\mathcal{K}$ of $\mathcal{O}(p,q)$ there exists $c = c(\varepsilon, \mathcal{K}) > 0$ such that for all $Q \in \mathcal{K}$ and $T > 0$

$$N_{Q,\Omega}(-\varepsilon,\varepsilon,T) \geq c V_{Q,\Omega}(-\varepsilon,\varepsilon,T).$$

Theorem. *(Eskin, Margulis, Mozes 1998) If* $p \geq 3$*,* $q \geq 1$ *and* $n = p + q$ *then, as* $T \to \infty$

$$N_{Q,\Omega}(a, b, T) \sim \lambda_{Q,\Omega}(b - a)T^{n-2}$$

for any irrational form $Q \in \mathcal{O}(p, q)$.

Theorem. *(Eskin, Margulis, Mozes 1998) Let* Ω *be the unit ball, and let* $q = 1$ *or* 2*. Then for every* $\varepsilon > 0$ *and every interval* (a, b) *there exists an irrational form* $Q \in \mathcal{O}(2, q)$ *and a constant* $c > 0$ *such that for an infinite sequence* $T_j \to \infty$

$$N_{Q,\Omega}(a, b, T_j) > cT_j^q(\log\ T_j)^{1-\varepsilon}.$$

Theorem. *(Eskin, Margulis, Mozes 1998) Let* $\mathcal{K}$ *be a compact subset of* $\mathcal{O}(p,q)$ *and* $n = p + q$*. Then, if* $p \geq 3$ *and* $q \geq 1$ *there exists a positive constant* $c = c(\mathcal{K}, a, b, \Omega)$ *such that for any* $Q \in \mathcal{K}$ *and all* $T > 1$

$$N_{Q,\Omega}(a, b, T) < cT^{n-2}.$$

If $p = 2$ *and* $q = 1$ *or* $q = 2$*, then there exists a constant* $c = c(\mathcal{K}, a, b, \Omega)$ *such that for any* $Q \in \mathcal{K}$ *and all* $T > 2$

$$N_{Q,\Omega}(a, b, T) < cT^{n-2} \log\ T.$$

We say that a quadratic form $Q \in \mathcal{O}(2,2)$ is *extremely well approximable by split rational forms*, to be abbreviated as EWAS, if for any $N > 0$ there exist a split integral form Q' and a real number $\lambda > 2$ such that $\|\lambda Q - Q'\| \leq \lambda^{-N}$. If the ratio of two nonzero coefficients of Q is Diophantine then Q is not EWAS.

$x \in \mathbf{R}$ is callled *Diophantine* if there exists $N > 0$ such that $|qx - p| > q^{-N}$ for any integers p and q.

Theorem. *(Eskin, Margulis, Mozes) The asymptotic formula*

$$N_{Q,\Omega}(a, b, T) \sim \lambda_{Q,\Omega}(b - a)T^{n-2}$$

holds if $Q \in \mathcal{O}(2,2)$ is not EWAS and $0 \notin (a, b)$.

Let Δ be a lattice in $\mathbf{R}^2$ and let $M = \mathbf{R}^2/\Delta$ denote the associated flat torus. The eigenvalues of the Laplacian on M are the values of the binary quadratic form $q(m,n) = 4\pi^2\|mv_1 + nv_2\|^2$, where $\{v_1, v_2\}$ is a $\mathbf{Z}$-basis for the dual lattice Δ^*. We label these eigenvalues (with multiplicity) by

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \ldots$$

$$|\{j \mid \lambda_j(M) \leq T\}| \sim c_M T, c_M = (\text{area } M)/4\pi$$

$$R_M(a,b,T) \stackrel{\text{def}}{=}$$

$$\frac{|\{(j,k)|\lambda_j(M) \leq T, \lambda_k(M) \leq T, j \neq k, a \leq \lambda_j(M) - \lambda_k(M) \leq b\}}{T}$$

Theorem. *(Eskin, Margulis, Mozes 2001) Let M be a flat 2-torus rescaled so that one of the coefficients in the associated binary quadratic form is 1. Let A_1, A_2 denote the two other coefficients of q. Suppose that there exists $N > 0$ such that for all triples of integers (p_1, p_2, q)*

$$\max_{i=1,2} |A_i - \tfrac{p_i}{q}| > \tfrac{1}{q^n}.$$

Then for any interval (a, b) which does not contain 0,

$$(*) \qquad \lim_{t\to\infty} R_M(a, b, T) = c_M^2(b-a)$$

In particular, if one of the A_i is Diophantine, then (*) holds, and therefore the set of $(A_1, A_2) \subset \mathbf{R}^2$ for which (*) does not hold has zero Hausdorff dimension.

$$E_s = \{x \in \mathbf{R}^d \mid Q(x) \leq s\}$$

vol B denotes the Lebesgue measure of B

$\mathrm{vol}_{\mathbf{Z}}$ B denotes the number of points in $B \cap \mathbf{Z}^d$

$$\Delta(r, a) \stackrel{\text{def}}{=} \left| \frac{\mathrm{vol}_{\mathbf{Z}}(rE_1 + a) - \mathrm{vol}\ rE_1}{\mathrm{vol}\ rE_1} \right|$$

Theorem. *(Götze) Assume that Q is positive definite and $d \geq 5$. Then*

$$\sup_{a \in \mathbf{R}^d} \Delta(r, a) = O(r^{-2})$$

as $r \to \infty$.

Theorem. *(Götze) Assume that Q is positive definite and $d \geq 5$. Then*

$$\sup_{a \in \mathbf{R}^d} \Delta(r, a) = o(r^{-2})$$

iff Q is irrational.

For $d \geq 9$ both theorems had been proved earlier by Bentkus and Götze (1999).

Corollary. *For η let E_s^η denote the elliptic shell $E_{s+\eta} \setminus E_s$. Then for any fixed $\eta > 0$*

$$\sup_{a \in \mathbf{R}^d} \frac{\mathrm{vol}_{\mathbf{Z}}(E^\eta_{r^2} + a)}{\mathrm{vol} E^\eta_{r^2}} = 1 + o(1),$$

as $r \to \infty$ holds uniformly in the center a provided that $d \geq 5$ and Q is irrational and positive definite.

Proves Davenport-Lewis conjecture about gaps between values of a positive definite form at integral points.

Let $0 \leq v_1 \leq v_2 < \ldots$ denote an enumeration of the values of $Q(m - a)$, $m \in \mathbf{Z}^d$, in increasing order. Then

$$\lim_{n \to \infty} (v_{n+1} - v_n) = 0.$$