Liquidity Risk with Coherent Risk Measures

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Outline

- Describe financial risk and coherent risk measures (representation by "scenarios" or probabilities)

- Incorporate liquidity risk into risk measurement (definition of "acceptable" portfolio)

- Consider an example of modeling liquidity (liquidity based restrictions)

- Consider a liquidation strategy by (finite) random time

Financial Risk

- Possible Outcomes $\Omega = \{\omega_1, \omega_2, ..., \omega_K\}$

- Random future value $X : \Omega \to \mathbb{R}$ (normalized with respect to a risk-free asset)

- A set of "acceptable" future values X

- A risk measure is a mapping ρ from random variables to real numbers.

Risk measures and acceptance sets

$\rho(X)$ specifies how much capital is required to make a position acceptable,

i.e. $\rho(X) \leq 0 \;\Rightarrow\; X$ is acceptable

The acceptance set associated with a risk measure ρ is

$$\mathcal{A}_\rho = \{X \mid \rho(X) \leq 0\}.$$

Value at Risk

VaR, Value at Risk, is a commonly used risk measure.

$$VaR_\alpha(X) = -\inf\{m | P[X \leq m] > \alpha\}$$

Shortcomings: it controls the frequency of failures but not their economic consequences

VaR is not subadditive: it could happen that

$$VaR_\alpha(X + Y) > VaR_\alpha(X) + VaR_\alpha(Y)$$

Coherent measures of risk

A risk measure ρ is called **coherent** if it satisfies the following axioms

1.**Subadditivity**

$$\rho(X+Y) \leq \rho(X) + \rho(Y)$$

2.**Positive homogeneity** If $\lambda \geq 0$

$$\rho(\lambda X) = \lambda \rho(X)$$

3.**Translation invariance** For all $m \in \mathbb{R}$

$$\rho(X+m) = \rho(X) - m$$

4.**Monotonicity**

$$X \leq Y \Rightarrow \rho(Y) \leq \rho(X)$$

A risk measure ρ is called **convex** if it satisfies the following axioms

1.**Convexity** For $0 \leq \lambda \leq 1$

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$$

2.**Translation invariance** For all $m \in \mathbb{R}$

$$\rho(X + m) = \rho(X) - m$$

3.**Monotonicity**

$$X \leq Y \Rightarrow \rho(Y) \leq \rho(X)$$

The associated acceptance set $\mathcal{A}$ is convex, monotone and closed.

Representation of risk measures

If measure of risk ρ is convex, then there exists a set $\mathcal{S}$ of probability measures P^i on Ω and constants f^i such that

$$\rho(X) = -\inf_{P^i \in \mathcal{S}} \left\{ E_{P^i}[X] - f^i \right\}.$$

For coherent measures of risk, the constants are zero.

The acceptance set is

$$\mathcal{A}_\rho = \{X \mid E_{P^i}[X] \geq f^i \text{ for all } i\}.$$

Choose a set of scenarios and corresponding risk limits. Let a financial position X be "acceptable" if

$$E_{P^i}[X] \geq f^i$$

for every i.

The resulting risk measure is coherent/convex.

Q. Suppose that a trader borrows a million and uses up for the stock of a single company.

– Is it correct to value the holdings of this trader at the present per-share price?

– Would the position of this trader be acceptable?

Model

On a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$

- a risk-free asset

- traded risky assets $S^1, S^2, \cdots, S^N$

- trading strategy $\pi_t = (\pi_t^0, \pi_t^1, \cdots, \pi_t^N)$ where π_t^0 is the number of units of the risk-free asset (the amount of cash holding) and π_t^n is the number of shares of asset $S^n \, (n = 1, ..., N)$ held at each time t

Consider a set of test measures $\{P^i, i \in I\}$ and risk limits $f^i \in \mathbb{R}$ where each P^i is absolutely continuous with respect to P.

We assume for "admissible" trading strategies (subject to liquidity based restrictions)

- no additional cashflows is generated

- "mark-to-market" value is bounded below

Acceptable portfolio

Definition A portfolio X is *acceptable* if there exist an " admissible" trading strategy π_t and a date T for which X can be decomposed (by trading) into a cash-only position and a positive portfolio by date T:

(i) $\pi_T^n = 0$ for all $1 \leq n \leq N$

where π_T^n denotes the number of shares of asset S^n held at date T, and

(ii)

$$E_{P^i}[e^{-rT}\pi_T^0] \geq f^i$$

for every $i \in I$.

A positive portfolio means that it entails only nonnegative cashflows in the future.

The acceptance set

The set $\mathcal{A}$ of all acceptable portfolios is

- **convex**
  i.e., if X is acceptable and Y is acceptable, then so is $\lambda X + (1 - \lambda)Y$ for $0 \leq \lambda \leq 1$.

- **monotone**
  i.e., if X is acceptable and $X \leq Y$ ($Y - X$ produces nonnegative cash flows), then Y is acceptable.

- **not necessarily positive homogeneous**
  i.e., X is acceptable but $2X$ might not be acceptable.

Proof of convexity

Since X is acceptable, there exists an admissible trading strategy ϕ_t for X which satisfies $\phi^n_{T_1} = 0\,(1 \leq n \leq N)$ and $e^{-rT_1}\phi^0_{T_1}$ is acceptable for some T_1.

Since Y is also acceptable, there exist an admissible trading strategy ψ_t and date T_2 for Y.

Set $T = \max\{T_1, T_2\}$ (assume $T = T_1$). Let

$$\pi_t = \lambda\phi_t + (1-\lambda)\psi_{t\wedge T_2}$$

Then, for this strategy π_t the portfolio $\lambda X + (1-\lambda)Y$ is decomposed into a cash-only position and a positive portfolio by date T, and the discounted value of cash-only part is

$$\lambda e^{-rT_1}\phi^0_{T_1} + (1-\lambda)e^{-rT_1}e^{r(T_1-T_2)}\psi^0_{T_2}$$

which is an acceptable random variable.

Modeling liquidity

Consider a market consisting of

- a risk-free asset (the interest rate r)

- a risky asset following a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

- the actual price traded in the market

$$P_t^{\pm} = S_t \pm \frac{\lambda}{2} S_t = (1 \pm \frac{\lambda}{2}) S_t$$

  where P^+, P^- represent the prices for buyers and for sellers respectively, and λ indicates the bid-ask spread

- a set of scenarios and risk limits $\{P^i, i \in I\}$ and $\{f^i, i \in I\}$

Admissible trading strategies

Define trading strategies (π_t^0, π_t^1) to be all $\{\mathcal{F}_t\}$-adapted processes with left continuous paths that have right limits.

π_t^0 denotes the amount held in cash and π_t^1 the number of shares of asset S held at time t.

Assume the firm cannot "liquidate" too fast:

Definition A trading strategy π_t is *admissible* if it satisfies

$$|\pi_{t_1}^1 - \pi_{t_2}^1| \leq \epsilon \, |t_1 - t_2|$$

and keeps the wealth ("mark-to-market" value) bounded below.

Then

$$\pi_t^1 = \Pi_t^+ - \Pi_t^-$$

where Π_t^+ is interpreted as the cumulative number of shares of asset S bought and Π_t^- as the cumulative number sold until time t.

$$\begin{aligned} d\pi_t^0 &= r\pi_t^0 dt - d(\Pi_t^+)P_t^+ + d(\Pi_t^-)P_t^- \\ &= r\pi_t^0 dt - (d\pi_t^1)S_t - \frac{\lambda}{2}(d(\Pi_t^+) + d(\Pi_t^-))S_t \end{aligned}$$

Definition A trading strategy $\pi_t = (\pi_t^0, \pi_t^1)$ is said to be *self-financing* if it satisfies

$$d(\pi_t^0 + \pi_t^1 S_t) = r\pi_t^0 dt + \pi_t^1 dS_t - \frac{\lambda}{2}(d(\Pi_t^+) + d(\Pi_t^-))S_t$$

The wealth process

The wealth ("mark-to-market" value) process W is written as

$$\begin{aligned} dW(t) &= r\pi_t^0 dt + \pi_t^1 dS_t - \frac{\lambda}{2}(d(\Pi_t^+) + d(\Pi_t^-))S_t \\ &= r\{W(t) - \pi_t^1 S_t\}\, dt + \pi_t^1 dS_t - \frac{\lambda}{2}(d(\Pi_t^+) \\ &\qquad +d(\Pi_t^-))S_t \end{aligned}$$

Lemma 1 The discounted wealth process is a supermartingale under any $Q \in \mathcal{Q}$

where $\mathcal{Q}$ is the set of probability measures absolutely continuous with respect to P, under which the (discounted) asset price process is a local martingale. Assume $\mathcal{Q} \cap \{P^i, i \in I\} \neq \emptyset$.

Proof

$$d(e^{-rt}W(t)) = e^{-rt}\{-r\pi_t^1 S_t dt + \pi_t^1 dS_t - \frac{\lambda}{2}(d(\Pi_t^+) + d(\Pi_t^-))S_t\}$$

$$e^{-rt}W(t) = W(0) + \int_0^t \pi_u^1 d(e^{-ru}S_u) - \frac{\lambda}{2}\int_0^t e^{-ru}S_u(d(\Pi_t^+) + d(\Pi_t^-))$$

The stochastic integral (with respect to $e^{-rt}S_t$) is a local martingale.

A continuous local martingale which is bounded below is a supermartingale.

The last term (without the minus sign) is non-negative and nondecreasing.

Theorem 2 For any fixed date T, there is a constant K such that:

if the initial "mark-to-market" value of a portfolio is less than K, then the portfolio cannot be decomposed into an acceptable cash-only position and a positive portfolio by time T.

Proof

Suppose a portfolio X is acceptable. Then there must exist an admissible trading strategy π_t for which $\pi_T^1 = 0$ and $e^{-rT}\pi_T^0$ satisfies

$$E_{P^i}[e^{-rT}\pi_T^0] \geq f^i$$

for every P^i.

On the other hand, if P^i belongs to $\mathcal{Q}$,

$$\begin{aligned} E_{P^i}[e^{-rT}\pi_T^0] &\leq E_{P^i}[e^{-rT}W(T)] \\ &\leq W(0) \end{aligned}$$

Let $K = \max\{f^i : P^i \in \mathcal{Q}\}$.

If the initial "mark-to-market" value $W(0)$ is less than K, then the portfolio cannot be acceptable.

Liquidation by (finite) random time

Assume that $\mu > \frac{\sigma^2}{2}$.($r = 0$) Assume $S_0 = 1$ and initial shares of stock held $\pi_0^1 = 1$.

- **Step 1.** Hold the stock until the stock holdings are worth $L(\gg 1)$

  Set

$$\begin{aligned} \sigma_1 &= \inf\{t : S_t = L\} \\ &= \inf\{t : (\mu - \frac{\sigma^2}{2})t + \sigma B_t = \ln L\} \\ &< \infty \end{aligned}$$

$$E_P[e^{-\theta\sigma_1}] = \exp\{-\frac{\ln L}{\sigma^2}(\sqrt{(\mu - \frac{\sigma^2}{2})^2 + 2\sigma^2\theta} - (\mu - \frac{\sigma^2}{2})\}$$

for any fixed $\theta > 0$.

$$E_P[\sigma_1] = \frac{1}{\mu - \frac{\sigma^2}{2}} \ln L.$$

- **Step 2.** Sell the stock at rate ϵ until the stock holdings are worth 1

  Set

$$\begin{aligned}\tau_1 &= \inf\{t \geq \sigma_1 : \pi_t^1 S_t = 1\} \\ &= \inf\{t \geq \sigma_1 : (1 - \epsilon(t - \sigma_1))S_t = 1\}\end{aligned}$$

  The change in the value of cash holding

$$\begin{aligned}d\pi_t^0 &= -d(\Pi_t^+)P_t^+ + d(\Pi_t^-)P_t^- \\ &= \epsilon(1 - \frac{\lambda}{2})S_t\end{aligned}$$

  for $\sigma_1 < t < \tau_1$.

  Thus the total amount of changes in the value of cash holding

$$Y_1 = \int_{\sigma_1}^{\tau_1} d\pi_t^0 = \epsilon(1 - \frac{\lambda}{2}) \int_{\sigma_1}^{\tau_1} S_t \, dt$$

- **Step 3.** Wait until the stock holdings are worth L

  For $t \geq \tau_1$

  $$S_t = S_{\tau_1} \exp\{(\mu - \frac{\sigma^2}{2})(t - \tau_1) + \sigma(B_t - B_{\tau_1})\}$$

  By the strong Markov property of Brownian motion, the distribution of $\frac{S_t}{S_{\tau_1}}$ (conditioned on $\mathcal{F}_{\tau_1}$) is the same as the distribution of S_u, where $u = t - \tau_1$.

  Let

  $$\begin{aligned} \sigma_2 &= \inf\{t \geq \tau_1 : \pi^1_{\tau_1} S_t = L\} \\ &= \inf\{t \geq \tau_1 : \frac{S_t}{S_{\tau_1}} = L\} \\ &< \infty \end{aligned}$$

  since $\pi^1_{\tau_1} S_{\tau_1} = 1$.

- **Step 4.** Sell the stock at rate $\frac{\epsilon}{S_{\tau_1}}$ until the stock holdings are worth 1

  Set

$$\begin{aligned} \tau_2 &= \inf\{t \geq \sigma_2 : \pi_t^1 S_t = 1\} \\ &= \inf\{t \geq \sigma_2 : (\pi_{\tau_1}^1 - \frac{\epsilon}{S_{\tau_1}}(t - \sigma_2))S_t = 1\} \end{aligned}$$

  Then, for $\sigma_2 < t < \tau_2$

$$\begin{aligned} d\pi_t^0 &= -d(\Pi_t^+)P_t^+ + d(\Pi_t^-)P_t^- \\ &= \frac{\epsilon}{S_{\tau_1}}(1 - \frac{\lambda}{2})S_t \end{aligned}$$

  Thus, the amount transferred into cash holdings

$$Y_2 = \int_{\sigma_2}^{\tau_2} d\pi_t^0 = \epsilon(1 - \frac{\lambda}{2}) \int_{\sigma_2}^{\tau_2} \frac{S_t}{S_{\tau_1}}\, dt$$

- **Step 5.** Repeat to produce $Y_3, Y_4, \cdots$

Assume that $\mu > \frac{\sigma^2}{2}$.($r = 0$) Let X be a portfolio whose initial shares of stock is 1.

Theorem 3 There exist an admissible trading strategy and a (finite) stopping time τ^* such that X is decomposed into a cash-only position and a positive portfolio by date τ^*, and

$$E_{P^i}[\pi^0_{\tau^*}] \geq f^i$$

for all $i \in I$.

Proof

Consider

$Y_1 = \epsilon(1-\frac{\lambda}{2}) \int_{\sigma_1}^{\tau_1} S_t \, dt, \; Y_2 = \epsilon(1-\frac{\lambda}{2}) \int_{\sigma_2}^{\tau_2} \frac{S_t}{S_{\tau_1}} \, dt, ...$

By the law of large numbers

$$Y_1 + Y_2 + \cdots \to \infty$$

almost surely under P.

Let

$$Z_t = \pi_0^0 + \sum_{\tau_m \le t} Y_m$$

Then $Z_t \le \pi_t^0$ and $Z_t \to \infty$ a.s. under P.

Let

$$\tau^* = \inf\{t : Z_t \ge 2\bar{f}\}$$

where $\bar{f} = \max\{f^i, i \in I\}$. Then $\tau^* < \infty$ a.s. and

$$P\{\omega : Z_{\tau^*} < \bar{f}\} = 0$$

Proof continued

Since each P^i is locally absolutely continuous with respect to P, there exists an increasing sequence $\{T_n\}$ of stopping times such that:
(i) $P^i\{\lim T_n = \infty\} = 1$
(ii) $P^i|_{\mathcal{F}_{T_n}}$ is absolutely continuous with respect to $P|_{\mathcal{F}_{T_n}}$ for all n.

For the localizing sequence $\{T_n\}$ (depending on P^i)

$$\begin{aligned} P^i\{Z_{\tau^*} < \bar{f}\} &= \lim_{n\to\infty} P^i\left\{\{Z_{\tau^*} < \bar{f}\} \cap \{T_n \geq \tau^*\}\right\} \\ &= 0 \end{aligned}$$

Then

$$E_{P^i}[\pi^0_{\tau^*}] \geq E_{P^i}[Z_{\tau^*}] \geq \bar{f} \geq f^i$$

for every $i \in I$.

Summary

- Liquidity risk is incorporated into risk measurement.

- A notion of acceptable portfolio is established.

- An example of modeling liquidity is presented.

- The requirement of finite fixed time for liquidation is necessary in the regulation of liquidity risk.

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