The Fields Institute Seminar Series on Quantitative Finance

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Economic Capital Allocation, Risk Contributions & Diversification in Credit Portfolios

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Summary

- *Concentration Risk ←→ Diversification*
  - Concentration risk: arguably the most important cause of major problems in banks, according to the BCBS
  - Diversification: key tool for managing the risk of credit portfolios
- *Economic Capital Allocation*
  - Modelling concentration/diversification → optimal EC allocation
  - Applications: pricing, profitability & limits, optimal risk-return portfolios and strategies, performance measurement and risk based compensation
- This seminar: overview of measurement of diversification & risk contributions in credit portfolios – application to capital allocation

Presenting joint work with: H. Mausser, A. Kreinin (Algorithmics), J.C. Garcia Cespedes, J.A. de Juan Herrero (BBVA), D. Saunders (University of Waterloo)

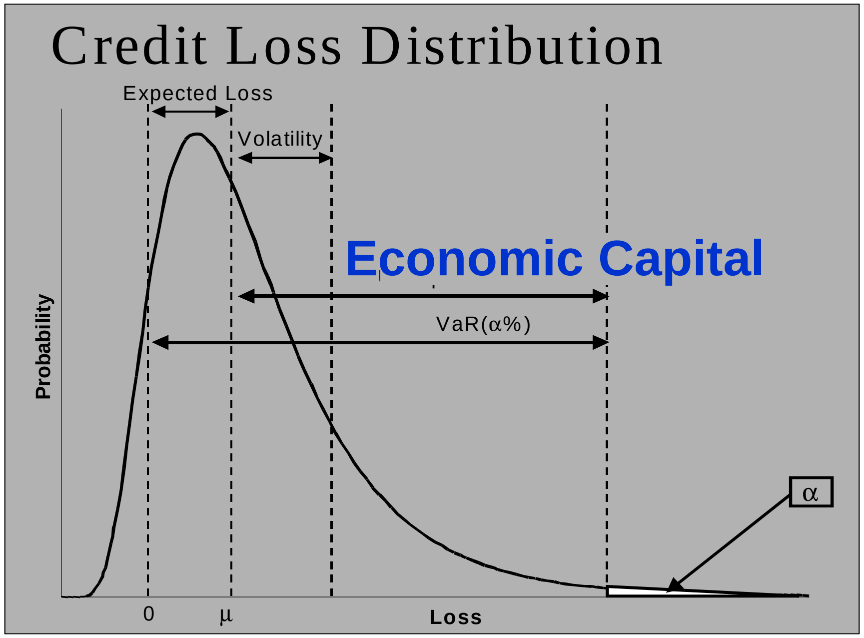
Key points

1. Marginal risk contributions: additive & reflect benefits of diversification
   - Capital allocation related to diversification/concentration risk
2. Risk measure – substantial impact on capital allocation
   - VaR & expected shortfall (ES) contributions avoid inefficient allocations associated with volatility-based methods
   - Quantile level → significant impact on allocations
3. VaR & ES contributions can be calculated analytically (simple models)
   - Fast calculation, understand capital allocation strategies better
   - Diversification Factor → analytical method
4. Monte Carlo methods – risk contributions in realistic credit models
   - Challenging at extreme quantiles
   - Improvements exploiting conditional independence framework, quantile estimators (VaR), variance reduction techniques (Importance Sampling)

Introduction: Economic Capital

- Capital acts as a buffer to absorb large unexpected losses
  - Protect depositors and other claim holders
  - Provide confidence to external investors and rating agencies on the financial health and viability of the firm

- Types of capital
  - Book capital
  - Regulatory capital
  - Economic capital



Capital Allocation

- In addition to measuring EC, management requires general methodologies to allocate it equitably among various components
  - Activities, business units, asset classes, obligors, transactions, etc.
  - Capital *attribution* and capital *allocation*

- Capital is (should be) sub-additive → portfolio diversification
  - Example: portfolio loss distribution is Normal

$$\sigma^2(y) = \sigma^2(x_1) + \sigma^2(x_2) + 2\sigma(x_1)\sigma(x_2)\rho$$

How do we allocate capital then?

Capital Allocation

- **Stand-alone capital**: diversification benefits not passed down to sub-portfolios/business units – each expected to operate on a stand-alone basis
  - Non-additive: sum of stand-alone EC for individual sub-portfolios may exceed the total EC for the portfolio
- **Incremental capital** (discrete marginal contributions) – difference between EC for the portfolio and EC for the portfolio without the sub-portfolio.
  - Capital released if sub-portfolio were sold or added (natural measure for evaluating the risk of acquisitions or divestitures)
  - Non-additive
- **Marginal capital** (or diversified contributions) – "optimal" level of portfolio risk-taking achieved only when diversification benefits are allocated down
  - Assign to each sub-portfolio/BU the economic capital allocation closer to its marginal contribution to the total economic capital
  - Additive: sum of diversified capital for all sub-portfolios = total EC for portfolio

Additive (Marginal) Risk Contributions

- Portfolio Losses

$$L = \sum_{i=1}^{N} L_i = \sum_{i=1}^{N} l_i x_i$$

- Additive decomposition of a portfolio risk measure, ρ(*L*)

$$\rho(L) = \sum_{i=1}^{N} C_i^{\rho}$$

C_i^{ρ} is the risk contribution of obligor/position *i*

- The relative risk contribution of obligor $R_i^{\rho} = \dfrac{C_i^{\rho}}{\rho(L)}$

- If $\rho(L)$ is homogeneous of degree one $C_i^{\rho} = x_i \dfrac{\partial \rho}{\partial x_i}$.

Capital Allocation Measures

- **Volatility** – most common approach
  - Generally ineffective for credit → losses far from Normal

$$C_i^{\sigma} = \frac{\operatorname{cov}(L_i, L)}{\sigma(L)}$$

- **VaR**-based measure - natural choice

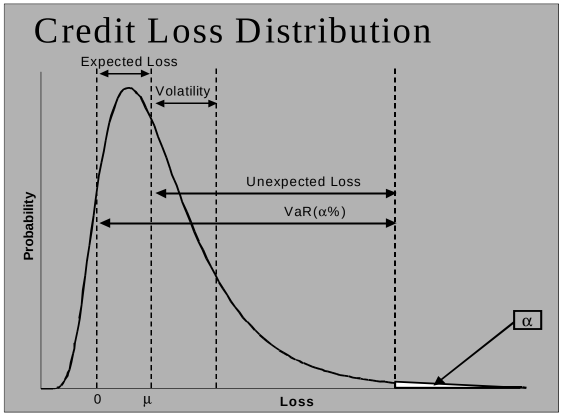
$$C_i^{VaR_\alpha} = E\left(L_i \,\middle|\, L = VaR_\alpha(L)\right)$$

  - Conceptual shortcoming: it is not a coherent risk measure
  - Computational shortcoming (accurate and stable contributions in simulation

- **Expected Shortfall** (ES)

$$C_i^{ES_\alpha} = E\left(L_i \,\middle|\, L \geq VaR_\alpha(L)\right)$$

  - Coherent risk measure & lends itself to stable optimization
  - ES contributions more easily computed from simulation
  - But… does not correspond to the standard definition of Capital



Coherent Capital Allocation

Axiomatic approach (Kalkbrener et al 2004)

1. *Linear (additive) allocation: capital allocated to union of sub-portfolios = sum of capital amounts allocated to individual sub-portfolios*
2. *Diversifying allocation: capital allocated to sub-portfolio X of a larger portfolio Y* $\leq$ *capital of X considered as a stand-alone portfolio*
3. *Continuous allocation: small increase in a position only has a small effect on the risk capital allocated to that position*

- These three axioms uniquely determine a capital allocation scheme – essentially a *marginal capital allocation*
- Any allocation satisfying these axioms → *coherent risk measure*
  - ES yields a linear, diversifying and continuous capital allocation
  - VaR yields an additive but not a diversifying allocation

Analytical Risk Contributions

1. Risk Contributions: Single-Factor Credit Model

- In the presence of diversification – marginal capital contributions are in general not *portfolio-invariant*

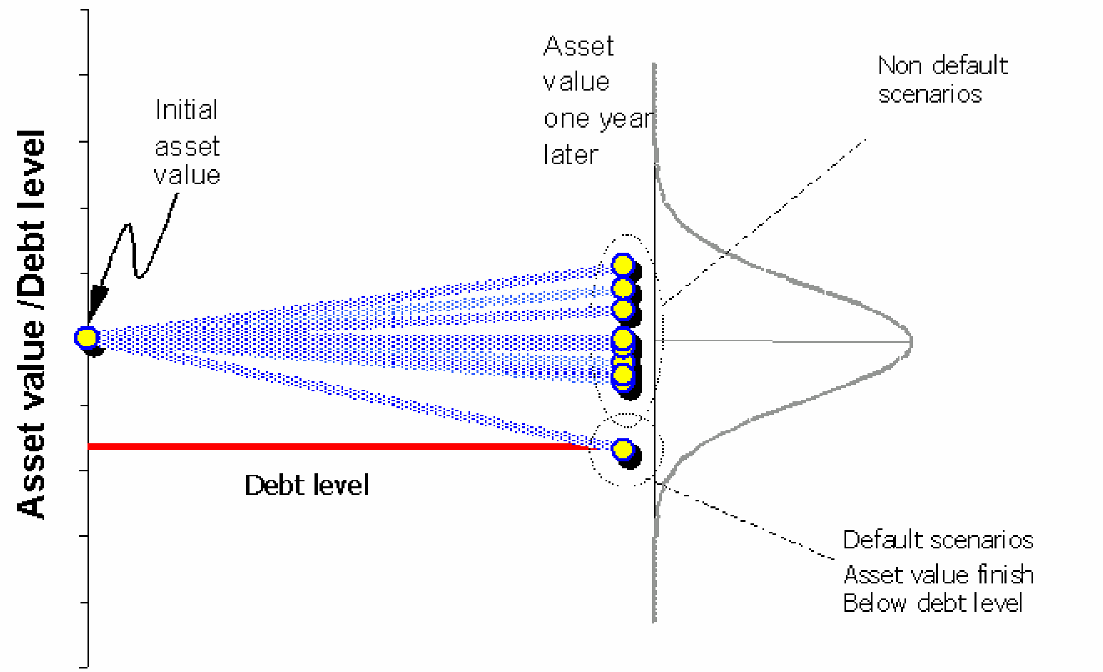
- If EC is defined in terms of a VaR measure, two conditions are necessary and sufficient to guarantee portfolio-invariant contributions (Gordy 2003):
  - *Asymptotically fine-grained* (*AFG)* portfolio: no single exposure accounts for more than a small share of total exposure
  - *Single Factor (SF)* model

- *SF-AFG* portfolio model is the basis of Basel II model

The Merton Model for Credit Risk

- A latent variable drives the default process (creditworthiness idex)
  - e.g. the “asset value” of the firm
- When the “asset value” falls below a certain level (debt level), the company defaults.

• CWI modeled using a normal distribution (assumption can be easily relaxed)



Source: Juan Carlos Garcia Cespedes, BBVA

Basel II Model – Single Factor, AFG Portfolio

- Obligor creditworthiness: $Y_j = \sqrt{\rho_j}\, Z + \sqrt{1-\rho_j}\, \varepsilon_j$

$$EC_\alpha = \sum_{j=1}^{N} C_j^{EC_\alpha}$$

$$C_j^{EC_\alpha} = LGD_j \cdot EAD_j \cdot \left[N\left(\frac{N^{-1}(PD_j) - \sqrt{\rho_j}\, z^\alpha}{\sqrt{1-\rho_j}} \right) - PD_j \right]$$

PD = obligor's probability of default

LGD = loss given default

ρ = one-factor asset correlation

EAD = Exposure at default

α = confidence level (e.g. 0.001)

z^α = α-percent of a standard Normal variable

The capital contribution does not depend on the composition of the rest of the portfolio.

Basel II IRB Model & Diversification

- Basel II minimum credit capital requirements are a big & very important step forward for financial services regulation
  - Based on 99.9% systemic credit risk and single-factor Merton model
  - Closed form formulae and additive risk contributions

- The two key shortcomings
  - Only systemic risk → granularity adjustment
  - <u>Does not fully recognize diversification</u>
    - Basel II diversification through calibration of single-factor model

2. Multi-factor Diversification and Contributions

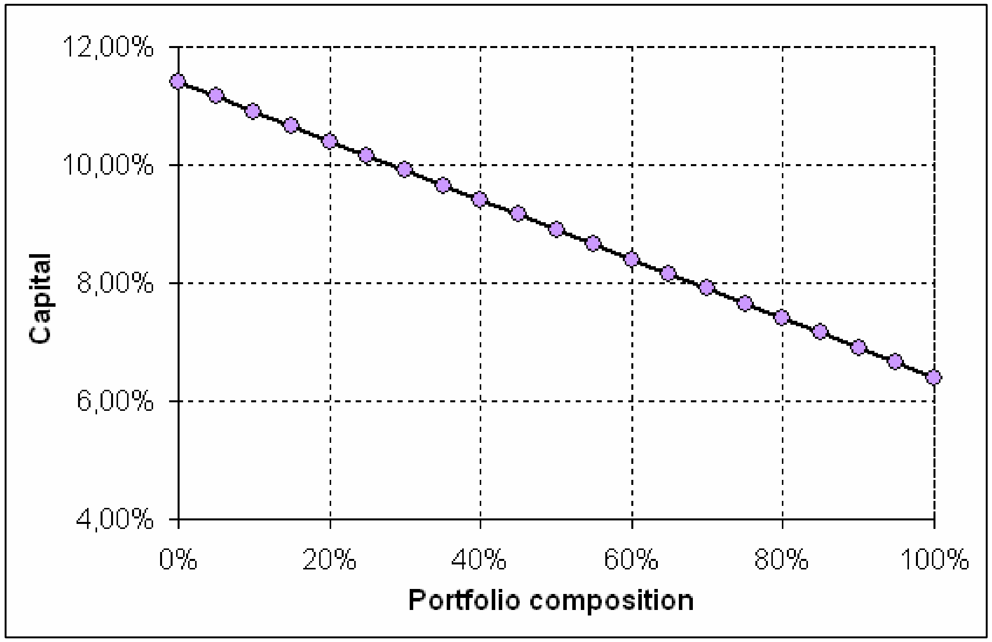
Example: Diversification for Two "Sectors"

- Portfolio comprised of two large homogeneous subportfolios in two sectors: PD= 1% and PD= 3%
  - Assume asset correlation inside each sector is 20%

$$C = x_1 \cdot \left[N\left(\frac{N^{-1}(PD_1) - \sqrt{\rho_1}\, z^{\alpha}}{\sqrt{1-\rho_1}} \right) - PD_1 \right] + x_2 \cdot \left[N\left(\frac{N^{-1}(PD_2) - \sqrt{\rho_2}\, z^{\alpha}}{\sqrt{1-\rho_2}} \right) - PD_2 \right]$$

- Basel II Capital (one-factor) as a function of portfolio mix (% of low-PD sector) is a straight line

- Adding 1% PD CPs always reduces capital

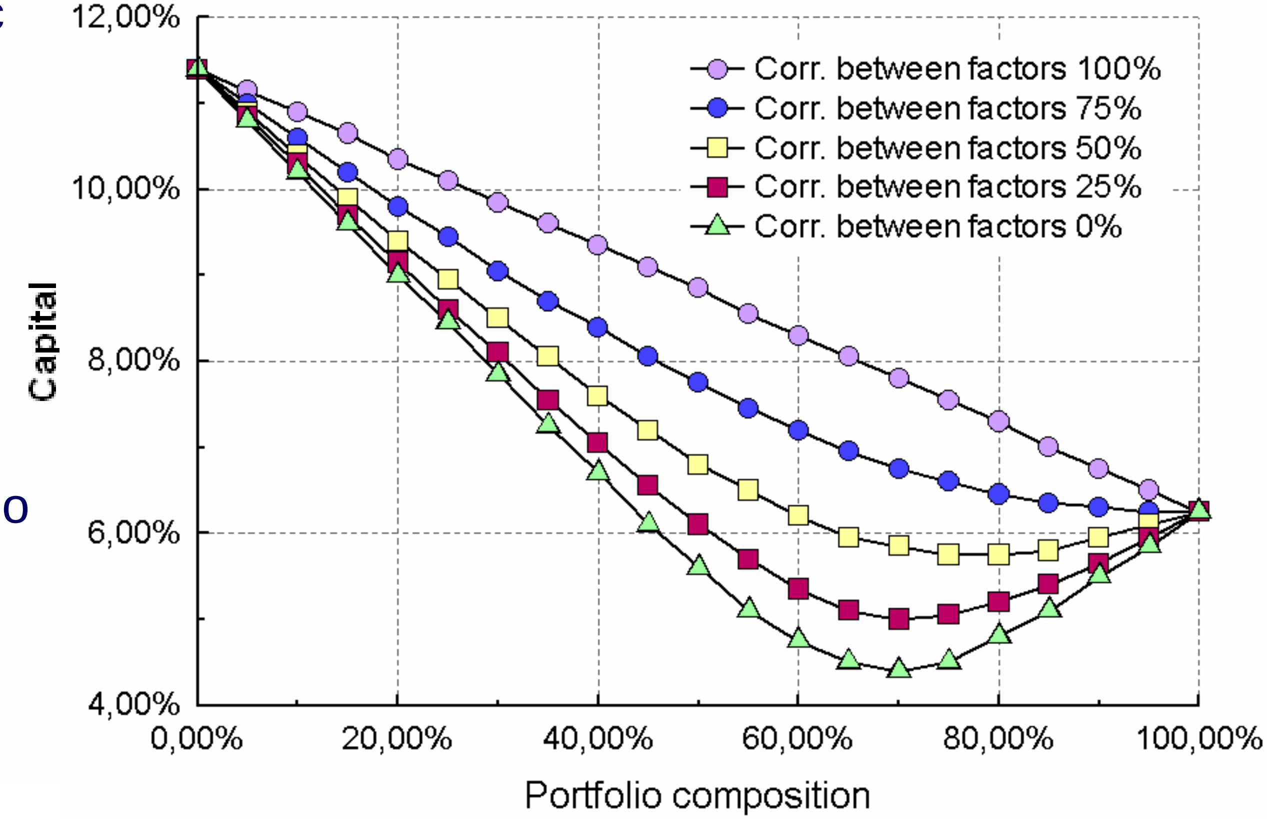


Source: Juan Carlos Garcia Cespedes, BBVA

Capital Allocation for Two "Sectors"

Assume a different systemic factor drives defaults in each sector

- Capital is monotonic in correlations between the two factors
- Capital is not necessarily monotonic in portfolio composition
  - Adding 3% PD counterparties is not always bad



Source: Juan Carlos Garcia Cespedes, BBVA

Diversification Factor

- Portfolio stand-alone capital = sum of SA capital of all positions

$$C_{SA} = C_1 + C_2 + \ldots + C_n$$

  - In the Basel II context, it can be though of as the capital from a one-factor model

- Economic capital: $EC = DF \times C_{SA}$

DF is called the ***diversification factor***

- What should *DF* depend on?
  - Relative size of positions (large concentrations)
  - Cross correlations of assets and sectors (sector diversification)

Diversification Factor Model (Garcia et al 2005)

- Portfolio stand-alone capital = sum of SA Capital of all positions

$$C_{SA} = C_1 + C_2 + \ldots + C_n$$

  - C_i's the capital from a one-factor model

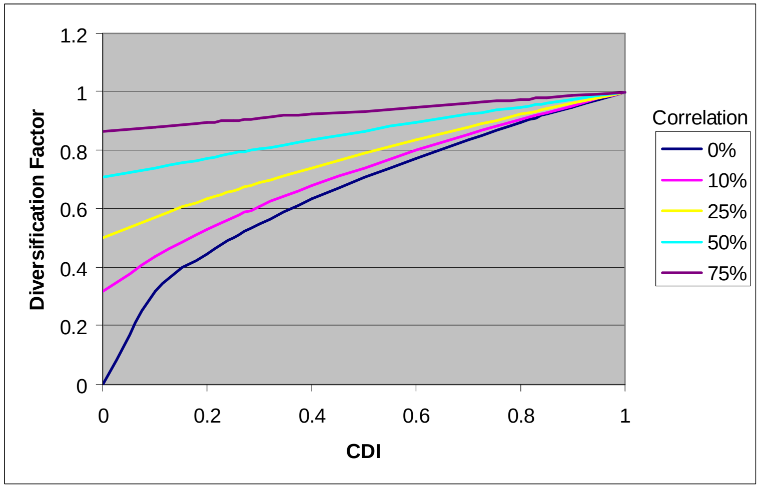
$$\boldsymbol{EC = DF \times C_{SA}} \qquad \boldsymbol{DF = DF\,(CDI, \beta)}$$

Diversification factor a function of

- *CDI*: *credit diversification index*
  - Size concentration risk

$$CDI = \frac{\sum_k C_k^{\,2}}{(C_{SA})^2} = \sum_k w_k^{\,2}$$

- β : *average cross sector correlation*



Capital Diversification Index CDI

$$CDI = \frac{\sum_k C_k^{\ 2}}{\left(C^{1f}\right)^2} = \sum_k w_k^{\ 2}$$

CDI = Herfindahl concentration index applied to *sector capital charges*

Indication of the portfolio diversification across sectors (no correlation)

- Inverse of the CDI → "effective number of sectors" in the portfolio
- e.g., for the two factors, the CDI ranges between 0.5 (maximum diversification) and 1 (maximum concentration)

- Intuition: for uncorrelated sectors, the std. deviation of portfolio losses is

$$\sigma_P = \sqrt{CDI} \sum_k \sigma_k$$

- For correlated sectors

$$\sigma_P = \sqrt{\left(1-\tilde{\beta}\right)CDI + \tilde{\beta}} \sum_k \sigma_k$$

$\tilde{\beta}$ correlation of sector losses (**not** the asset correlation)

Diversification Factor Model

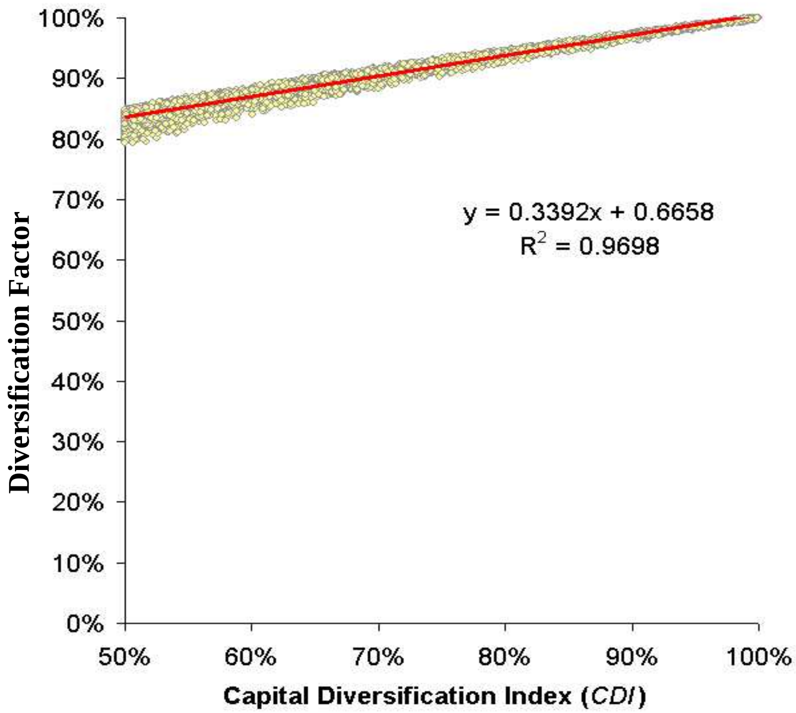
Numerical calibration of *DF* model using MC simulation

Example: two-factors & β=60%

- A linear model fits the data very well
- R^2 of 0.97; SE = 8 bps

$$DF(CDI, \beta = 0.6) = 0.6658 + 0.3392 \cdot CDI$$

Capital Diversification index	Diversification Factor Adjustment
50%	84%
55%	85%
60%	87%
65%	89%
70%	90%
75%	92%
80%	94%
85%	95%
90%	97%
95%	99%
100%	100%

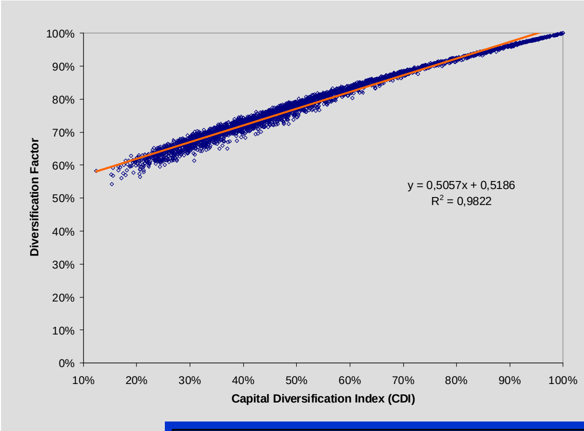


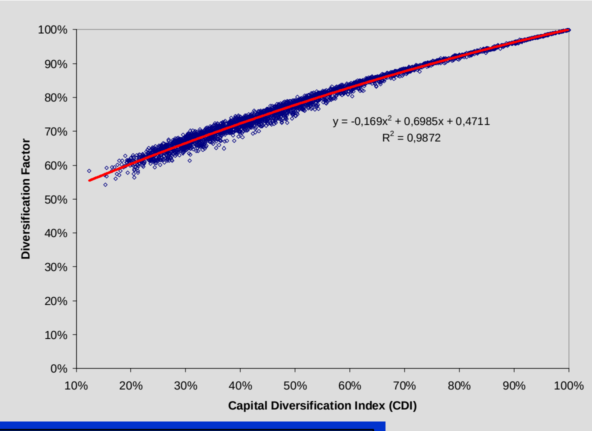
DF* as a function of the *CDI
(Two-factor, β = 60%)

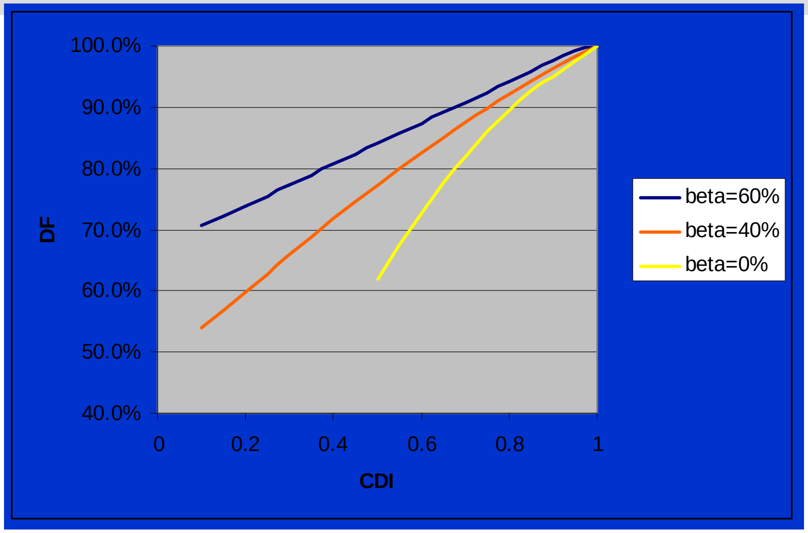
Diversification Factor Model

Numerical calibration of *DF* model using MC simulation

CDI	60%	40%
0.1	70.7%	53.9%
0.15	72.2%	57.0%
0.2	73.9%	59.9%
0.25	75.5%	62.8%
0.3	77.2%	65.9%
0.35	78.9%	68.8%
0.4	80.6%	71.8%
0.45	82.3%	74.6%
0.5	84.0%	77.1%
0.55	85.7%	79.9%
0.6	87.4%	82.5%
0.65	89.1%	85.0%
0.7	90.8%	87.5%
0.75	92.5%	89.8%
0.8	94.1%	92.1%
0.85	95.8%	94.2%
0.9	97.5%	96.2%
0.95	99.2%	98.2%
1	100.0%	100.0%







Example: Diversification Factor

- Portfolio:70% of stand-alone capital in sub-portfolio P1 and 30% in P2
- Economic capital – 86% of stand-alone capital → Diversification

	Capital One-Factor	SA Capital Contributions %	Unadjusted Capital Contributions
P1	70.0	70.0%	60.4
P2	30.0	30.0%	25.9
Total	100.0	100%	86.3
		CDI	**0.58**
		DF	**86.3%**

$EC = DF \times C_{SA}$

Example: Diversification Factor & Capital Allocation

- Portfolio:70% of stand-alone capital in sub-portfolio P1 and 30% in P2
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Total	100.0	100%	86.3
		CDI	**0.58**
		DF	**86.3%**

$$EC = DF \ X \ C_{SA}$$

Capital Allocation:

$$EC = DF_1 x C_1 + DF_2 x C_2 + \ldots + DF_n x C_n$$

DF_k **= marginal diversification factors**

Each sub-portfolio contributes differently to diversification

- On a marginal basis, the portfolio is more concentrated in P1, therefore it should be expected to get less diversification benefits

$$DF_1 < DF_2$$

Example: Diversification Factor & Capital Allocation

- Portfolio with 70% of stand alone capital in P1 and 30% in P2
- Overall capital – 86% of stand-alone capital (due to diversification)

	Capital One-Factor	SA Capital Contributions %	Unadjusted Capital Contributions	Marginal Sector Diverisfication Factor	Marginal Sector Capital Contributions	Marginal Sector Capital Contributions %
P1	70.0	70.0%	60.4	**0.94**	66.1	**76.6%**
P2	30.0	30.0%	25.9	**0.67**	20.2	**23.4%**
Total	100.0	100%	86.3		86.3	**100%**
		CDI	**0.58**			
		DF	**86.3%**			

- Consistent with a marginal risk allocation
  - The smaller portfolio contributes more to the overall diversification – gets a diversification factor of 67%
  - Larger portfolio gets a 94% factor
  - Capital contributions of the portfolios are 66.1 and 20.2 (summing to 86.3)

2. Capital Allocation in *DF* Model (Multi-Factor)

- EC is a homogeneous on the on the individual SA capital of each sector → marginal decomposition of the form

$$EC = \sum_{k=1}^{K} \frac{\partial EC}{\partial C_k} \cdot C_k$$

DF model analytically tractable – simple formulae & decomposition

$$EC = \sum_{k=1}^{K} DF_k \cdot C_k \quad \rightarrow \quad DF_k = \frac{\partial EC}{\partial C_k}$$

$$DF_k = DF + 2\frac{\partial DF}{\partial CDI} \cdot \left[\frac{EC_k}{EC^{sf}} - CDI \right] + 2\frac{\partial DF}{\partial \bar{\beta}} \cdot \frac{1-\left(EC_k / EC^{sf}\right)}{1-CDI} \cdot \left[\bar{Q}_k - \bar{\beta}\right]$$

Marginal DF = *portfolio* + *sector size* + *sector correlation*

$$DF_k = DF + \Delta DF_{Size} + \Delta DF_{Corr}$$

Examples

1. Behaviour of VaR and ES contributions using a one-factor model and a granular portfolio.
   - Choice of quantile can have significant impact on capital allocation
2. Impact of multi-factor diversification on capital & contributions
   - Single-factor vs. two-factor capital allocations
   - Sensitivity of the marginal allocations to the size of their components and the level of diversification

Contributions: uniform exposures portfolio

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	10	100%	11.00%	0.15	31.3%	25.2%
2	10	100%	10.00%	0.15	28.4%	24.0%
3	10	100%	9.00%	0.15	25.6%	22.7%
4	10	100%	2.00%	0.15	5.7%	9.1%
5	10	100%	1.50%	0.15	4.3%	7.5%
6	10	100%	1.00%	0.15	2.8%	5.7%
7	10	100%	0.30%	0.15	0.9%	2.4%
8	10	100%	0.20%	0.15	0.6%	1.8%
9	10	100%	0.10%	0.15	0.3%	1.0%
10	10	100%	0.05%	0.15	0.1%	0.6%
Total	100				3.5	19.3

(15% intra-sector correlation)

Capital 15.8

86% EL
72% VaR

Contributions: uniform exposures portfolio

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	10	100%	11.00%	0.15	31.3%	25.2%
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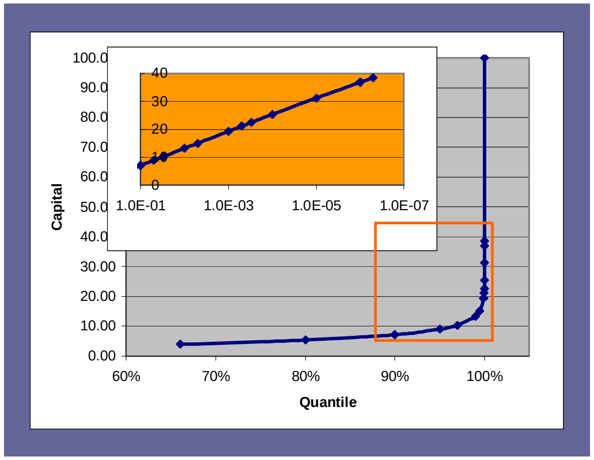
3	10	100%	9.00%	0.15	25.6%	22.7%
4	10	100%	2.00%	0.15	5.7%	9.1%
5	10	100%	1.50%	0.15	4.3%	7.5%
6	10	100%	1.00%	0.15	2.8%	5.7%
7	10	100%	0.30%	0.15	0.9%	2.4%
8	10	100%	0.20%	0.15	0.6%	1.8%
9	10	100%	0.10%	0.15	0.3%	1.0%
10	10	100%	0.05%	0.15	0.1%	0.6%
Total	100				3.5	19.3
(15% intra-sector correlation)					Capital	15.8



- Roughly an exponential law in the tail
- Losses at 100% = 100

Contributions: uniform exposures portfolio

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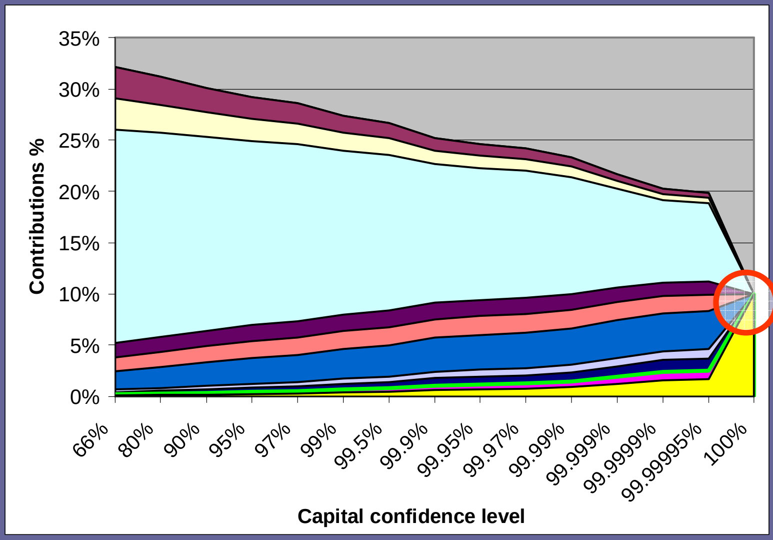
10	10	100%	0.05%	0.15	0.1%	0.6%
Total	100				3.5	19.3
(15% intra-sector correlation)					**Capital**	**15.8**

100% level

- Every sector contributes 10% of losses
- Credit quality does not play a role – only size
- At all other levels credit quality is important

	Quantile					
	90%	99%	99.9%	99.99%	99.999%	100%
1	30.1%	27.4%	25.2%	23.3%	21.7%	10%
2	27.7%	25.7%	24.0%	22.4%	21.0%	10%
3	25.3%	24.0%	22.7%	21.4%	20.2%	10%
4	6.4%	8.0%	9.1%	10.0%	10.6%	10%
5	4.9%	6.4%	7.5%	8.5%	9.2%	10%
6	3.3%	4.6%	5.7%	6.6%	7.4%	10%
7	1.0%	1.7%	2.4%	3.1%	3.8%	10%
8	0.7%	1.2%	1.8%	2.3%	2.9%	10%
9	0.3%	0.7%	1.0%	1.5%	1.9%	10%
10	0.2%	0.4%	0.6%	0.9%	1.2%	10%
1/CDI	4.2	4.7	5.2	5.8	6.2	10.0

Risk contributions in tail of the distribution



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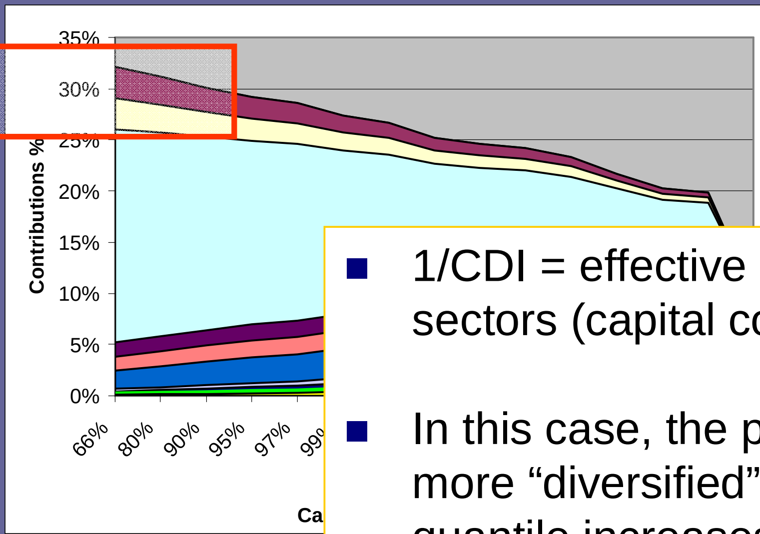
6	10	100%	1.00%	0.15	2.8%	5.7%
7	10	100%	0.30%	0.15	0.9%	2.4%
8	10	100%	0.20%	0.15	0.6%	1.8%
9	10	100%	0.10%	0.15	0.3%	1.0%
10	10	100%	0.05%	0.15	0.1%	0.6%
Total	100				3.5	19.3
(15% intra-sector correlation)					Capital	15.8

- At 66% level – 3 sector (lowest credit quality) contribute 87% VaR
- Goes down with quantile level
  - At 99.9% - 72% contribution
  - At 99.999% - 63% contribution
- Quantile increase → low quality sectors' capital shifts to high quality

	Quantile					
	90%	99%	99.9%	99.99%	99.999%	100%
1	30.1%	27.4%	25.2%	23.3%	21.7%	10%
2	27.7%	25.7%	24.0%	22.4%	21.0%	10%
3	25.3%	24.0%	22.7%	21.4%	20.2%	10%
4	6.4%	8.0%	9.1%	10.0%	10.6%	10%
5	4.9%	6.4%	7.5%	8.5%	9.2%	10%
6	3.3%	4.6%	5.7%	6.6%	7.4%	10%
7	1.0%	1.7%	2.4%	3.1%	3.8%	10%
8	0.7%	1.2%	1.8%	2.3%	2.9%	10%
9	0.3%	0.7%	1.0%	1.5%	1.9%	10%
10	0.2%	0.4%	0.6%	0.9%	1.2%	10%
1/CDI	4.2	4.7	5.2	5.8	6.2	10.0

Risk contributions in tail of the distribution



- 1/CDI = effective number of sectors (capital contributions)
- In this case, the portfolio looks more "diversified" as the quantile increases

Contributions: non- uniform exposures portfolio

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	2	100%	11.00%	0.15	25.9%	16.2%
2	2	100%	10.00%	0.15	23.6%	15.4%
3	2	100%	9.00%	0.15	21.2%	14.6%
4	2	100%	2.00%	0.15	4.7%	5.9%
5	5	100%	1.50%	0.15	8.8%	12.1%
6	5	100%	1.00%	0.15	5.9%	9.2%
7	5	100%	0.30%	0.15	1.8%	3.8%
8	10	100%	0.20%	0.15	2.4%	5.7%
9	30	100%	0.10%	0.15	3.5%	10.0%
10	37	100%	0.05%	0.15	2.2%	7.1%
Total	100				0.85	6.0

Capital	**5.2**

- Much smaller than in previous case

- Exposures proportionate to credit quality
  - Previous case: allocation varied with quantiles but rankings remained the same
  - This case is more complex - opposing effects of distributions of credit quality & exposure sizes

Contributions: non- uniform exposures portfolio

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	2	100%	11.00%	0.15	25.9%	16.2%
2	2	100%	10.00%	0.15	23.6%	15.4%
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4	2	100%	2.00%	0.15	4.7%	5.9%
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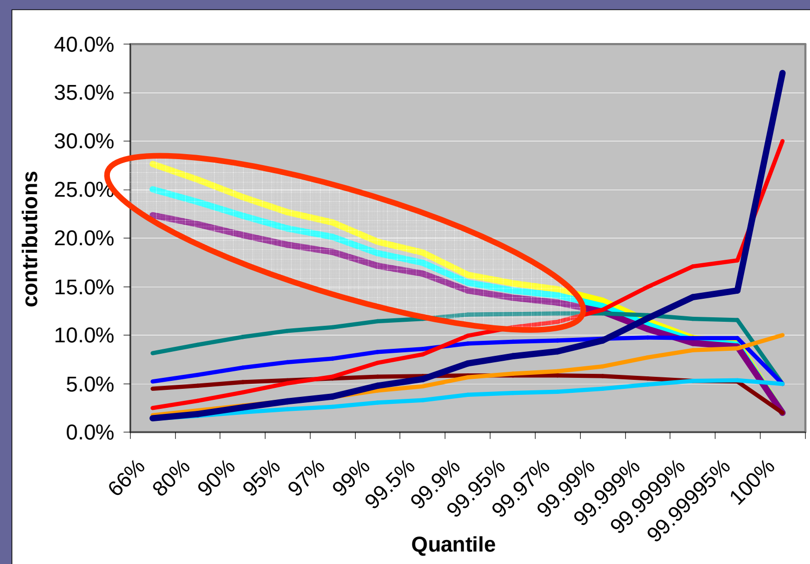
8	10	100%	0.20%	0.15	2.4%	5.7%
9	30	100%	0.10%	0.15	3.5%	10.0%
10	37	100%	0.05%	0.15	2.2%	7.1%
Total	100				0.85	6.0

Capital	**5.2**

- Lower quantiles: lowest quality sectors are the biggest contributors
  - 66% level → 75% contributions
  - 99% → 55%
  - 99.99% → 39%

	Quantile						
	90%	99%	99.9%	99.99%	99.999%	99.9999%	100%
1	24.2%	19.6%	16.2%	13.5%	11.4%	9.7%	2%
2	22.3%	18.4%	15.4%	13.0%	11.1%	9.5%	2%
3	20.3%	17.2%	14.6%	12.4%	10.6%	9.2%	2%
4	5.2%	5.7%	5.9%	5.8%	5.6%	5.3%	2%
5	9.8%	11.4%	12.1%	12.3%	12.1%	11.7%	5%
6	6.7%	8.3%	9.2%	9.6%	9.8%	9.7%	5%
7	2.1%	3.1%	3.8%	4.5%	4.9%	5.3%	5%
8	2.8%	4.3%	5.7%	6.8%	7.7%	8.5%	10%
9	4.2%	7.1%	10.0%	12.6%	15.0%	17.1%	30%
10	2.6%	4.8%	7.1%	9.5%	11.8%	13.9%	37%

1/CDI	5.9	7.4	8.5	9.1	9.2	9.0	4.1



Risk contributions in tail of the distribution

Contributions: non- uniform exposures portfolio

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	2	100%	11.00%	0.15	25.9%	16.2%
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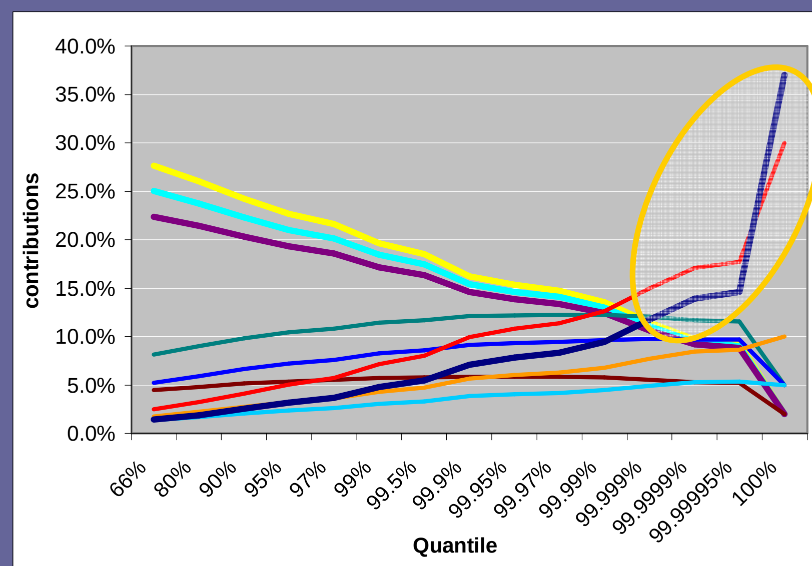
9	30	100%	0.10%	0.15	3.5%	10.0%
10	37	100%	0.05%	0.15	2.2%	7.1%
Total	100				0.85	6.0

Capital	**5.2**

- High quantiles: bigger sectors tend to become the biggest contributors
  - 100% level → 77% contributions

	Quantile						
	90%	99%	99.9%	99.99%	99.999%	99.9999%	100%
1	24.2%	19.6%	16.2%	13.5%	11.4%	9.7%	2%
2	22.3%	18.4%	15.4%	13.0%	11.1%	9.5%	2%
3	20.3%	17.2%	14.6%	12.4%	10.6%	9.2%	2%
4	5.2%	5.7%	5.9%	5.8%	5.6%	5.3%	2%
5	9.8%	11.4%	12.1%	12.3%	12.1%	11.7%	5%
6	6.7%	8.3%	9.2%	9.6%	9.8%	9.7%	5%
7	2.1%	3.1%	3.8%	4.5%	4.9%	5.3%	5%
8	2.8%	4.3%	5.7%	6.8%	7.7%	8.5%	10%
9	4.2%	7.1%	10.0%	12.6%	15.0%	17.1%	30%
10	2.6%	4.8%	7.1%	9.5%	11.8%	13.9%	37%
1/CDI	5.9	7.4	8.5	9.1	9.2	9.0	4.1

Risk contributions in tail of the distribution



Contributions: non- uniform exposures portfolio

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
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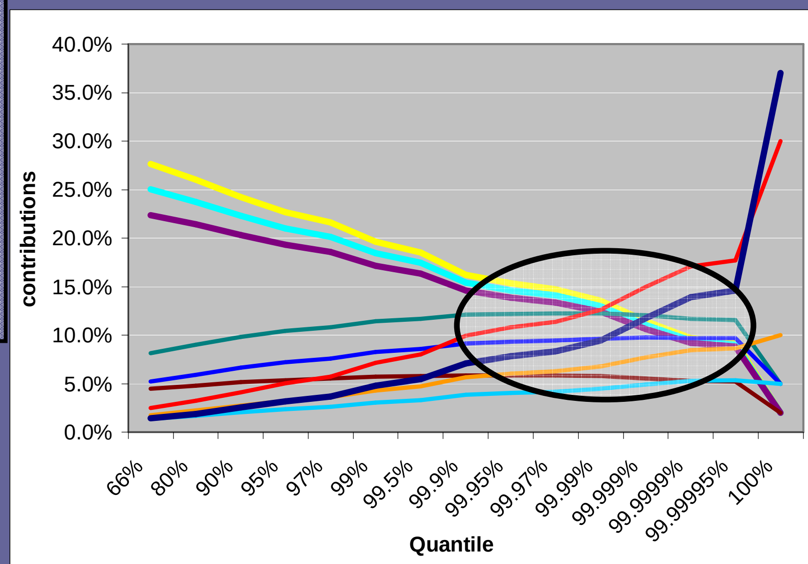
8	10	100%	0.20%	0.15	2.4%	5.7%
9	30	100%	0.10%	0.15	3.5%	10.0%
10	37	100%	0.05%	0.15	2.2%	7.1%
Total	100				0.85	6.0

Capital	**5.2**

- Ranking of sectors changes with quantile

	Quantile						
	90%	99%	99.9%	99.99%	99.999%	99.9999%	100%
1	24.2%	19.6%	16.2%	13.5%	11.4%	9.7%	2%
2	22.3%	18.4%	15.4%	13.0%	11.1%	9.5%	2%
3	20.3%	17.2%	14.6%	12.4%	10.6%	9.2%	2%
4	5.2%	5.7%	5.9%	5.8%	5.6%	5.3%	2%
5	9.8%	11.4%	12.1%	12.3%	12.1%	11.7%	5%
6	6.7%	8.3%	9.2%	9.6%	9.8%	9.7%	5%
7	2.1%	3.1%	3.8%	4.5%	4.9%	5.3%	5%
8	2.8%	4.3%	5.7%	6.8%	7.7%	8.5%	10%
9	4.2%	7.1%	10.0%	12.6%	15.0%	17.1%	30%
10	2.6%	4.8%	7.1%	9.5%	11.8%	13.9%	37%
1/CDI	5.9	7.4	8.5	9.1	9.2	9.0	4.1



Risk contributions in tail of the distribution

Contributions: non- uniform exposures portfolio

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	2	100%	11.00%	0.15	25.9%	16.2%
2	2	100%	10.00%	0.15	23.6%	15.4%
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4	2	100%	2.00%	0.15	4.7%	5.9%
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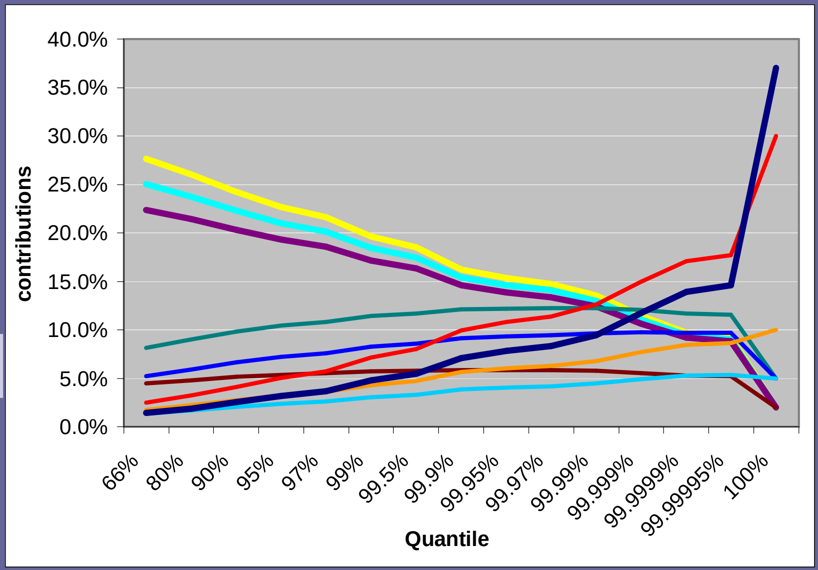
6	5	100%	1.00%	0.15	5.9%	9.2%
7	5	100%	0.30%	0.15	1.8%	3.8%
8	10	100%	0.20%	0.15	2.4%	5.7%
9	30	100%	0.10%	0.15	3.5%	10.0%
10	37	100%	0.05%	0.15	2.2%	7.1%
Total	100				0.85	6.0

Capital	**5.2**

- (1/CDI) effective number of sectors is not monotonic in the quantile
  - Peaks at about 99.999% level (where the "dispersion" is the smallest and the portfolio looks "most diversified")

	Quantile						
	90%	99%	99.9%	99.99%	99.999%	99.9999%	100%
1	24.2%	19.6%	16.2%	13.5%	11.4%	9.7%	2%
2	22.3%	18.4%	15.4%	13.0%	11.1%	9.5%	2%
3	20.3%	17.2%	14.6%	12.4%	10.6%	9.2%	2%
4	5.2%	5.7%	5.9%	5.8%	5.6%	5.3%	2%
5	9.8%	11.4%	12.1%	12.3%	12.1%	11.7%	5%
6	6.7%	8.3%	9.2%	9.6%	9.8%	9.7%	5%
7	2.1%	3.1%	3.8%	4.5%	4.9%	5.3%	5%
8	2.8%	4.3%	5.7%	6.8%	7.7%	8.5%	10%
9	4.2%	7.1%	10.0%	12.6%	15.0%	17.1%	30%
10	2.6%	4.8%	7.1%	9.5%	11.8%	13.9%	37%
1/CDI	5.9	7.4	8.5	9.1	9.2	9.0	4.1

Risk contributions in tail of the distribution



$\mathcal{R}^2$

The Diversification Factor & Capital Contributions

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	10	100%	11.00%	0.15	31.3%	25.2%
2	10	100%	10.00%	0.15	28.4%	24.0%
3	10	100%	9.00%	0.15	25.6%	22.7%
4	10	100%	2.00%	0.15	5.7%	9.1%
5	10	100%	1.50%	0.15	4.3%	7.5%
6	10	100%	1.00%	0.15	2.8%	5.7%
7	10	100%	0.30%	0.15	0.9%	2.4%
8	10	100%	0.20%	0.15	0.6%	1.8%
9	10	100%	0.10%	0.15	0.3%	1.0%
10	10	100%	0.05%	0.15	0.1%	0.6%
Total	100				3.5	19.3

Capital	**15.8**

- Each sector is driven by a single (different) factor
  - 15% inter-sector correlation
- *CDI* = 0.18
  - 5.6 effective sectors
  - HI (exposures) = 0.1
- Single-Factor Model corresponds to 100% correlation of sector factors

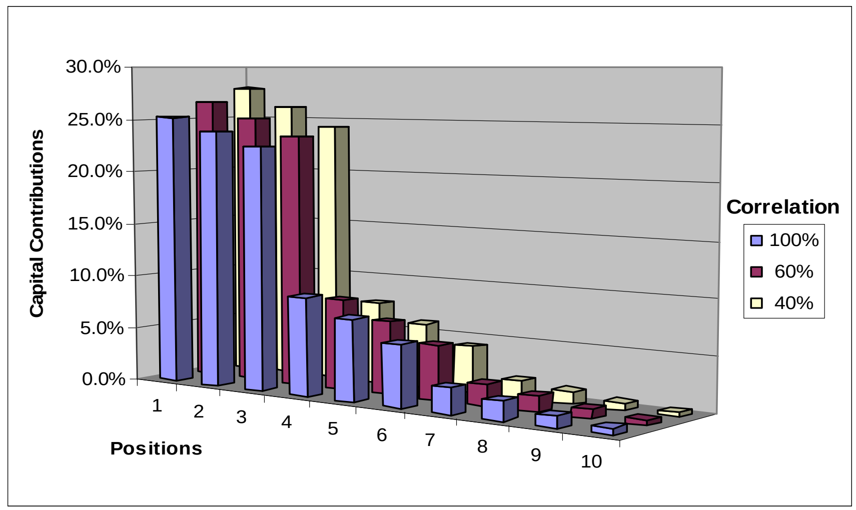
The Diversification Factor & Capital Contributions

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	10	100%	11.00%	0.15	31.3%	25.2%
2	10	100%	10.00%	0.15	28.4%	24.0%
3	10	100%	9.00%	0.15	25.6%	22.7%
4	10	100%	2.00%	0.15	5.7%	9.1%
5	10	100%	1.50%	0.15	4.3%	7.5%
6	10	100%	1.00%	0.15	2.8%	5.7%
7	10	100%	0.30%	0.15	0.9%	2.4%
8	10	100%	0.20%	0.15	0.6%	1.8%
9	10	100%	0.10%	0.15	0.3%	1.0%
10	10	100%	0.05%	0.15	0.1%	0.6%
Total	100				3.5	19.3
					Capital	15.8

				Capital %		
Positions	EAD	EL%	VaR %	Corr=100%	Corr=60%	Corr=40%
1	10%	31.3%	25.2%	23.9%	25.1%	26.7%
2	10%	28.4%	24.0%	23.0%	24.0%	25.3%
3	10%	25.6%	22.7%	22.0%	22.8%	23.8%
4	10%	5.7%	9.1%	9.9%	9.2%	8.3%
5	10%	4.3%	7.5%	8.3%	7.5%	6.6%
6	10%	2.8%	5.7%	6.3%	5.7%	4.8%
7	10%	0.9%	2.4%	2.7%	2.4%	1.9%
8	10%	0.6%	1.8%	2.0%	1.7%	1.4%
9	10%	0.3%	1.0%	1.2%	1.0%	0.8%
10	10%	0.1%	0.6%	0.7%	0.6%	0.5%
Total	100	3.52	19.33	15.81	11.58	9.28

CDI	0.180
Sectors	5.6

DF	100%	73.2%	40.0%
Slope		0.34	0.59



- Single factor model
  - "intra-sector" corr=100%
  - DF = 100%
- Diversification → capital reductions
  - Corr= 60%→ DF= 73%
    (27% lower capital)
  - Corr= 40%→ DF= 40%
    (60% lower capital)

The Diversification Factor & Capital Contributions

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	10	100%	11.00%	0.15	31.3%	25.2%
2	10	100%	10.00%	0.15	28.4%	24.0%
3	10	100%	9.00%	0.15	25.6%	22.7%
4	10	100%	2.00%	0.15	5.7%	9.1%
5	10	100%	1.50%	0.15	4.3%	7.5%
6	10	100%	1.00%	0.15	2.8%	5.7%
7	10	100%	0.30%	0.15	0.9%	2.4%
8	10	100%	0.20%	0.15	0.6%	1.8%
9	10	100%	0.10%	0.15	0.3%	1.0%
10	10	100%	0.05%	0.15	0.1%	0.6%
Total	100				3.5	19.3
					Capital	15.8

Positions	EAD	EL%	VaR %	Capital % Corr=100%	Corr=60%	Corr=40%
1	10%	31.3%	25.2%	23.9%	25.1%	26.7%
2	10%	28.4%	24.0%	23.0%	24.0%	25.3%
3	10%	25.6%	22.7%	22.0%	22.8%	23.8%
4	10%	5.7%	9.1%	9.9%	9.2%	8.3%
5	10%	4.3%	7.5%	8.3%	7.5%	6.6%
6	10%	2.8%	5.7%	6.3%	5.7%	4.8%
7	10%	0.9%	2.4%	2.7%	2.4%	1.9%
8	10%	0.6%	1.8%	2.0%	1.7%	1.4%
9	10%	0.3%	1.0%	1.2%	1.0%	0.8%
10	10%	0.1%	0.6%	0.7%	0.6%	0.5%
Total	100	3.52	19.33	15.81	11.58	9.28

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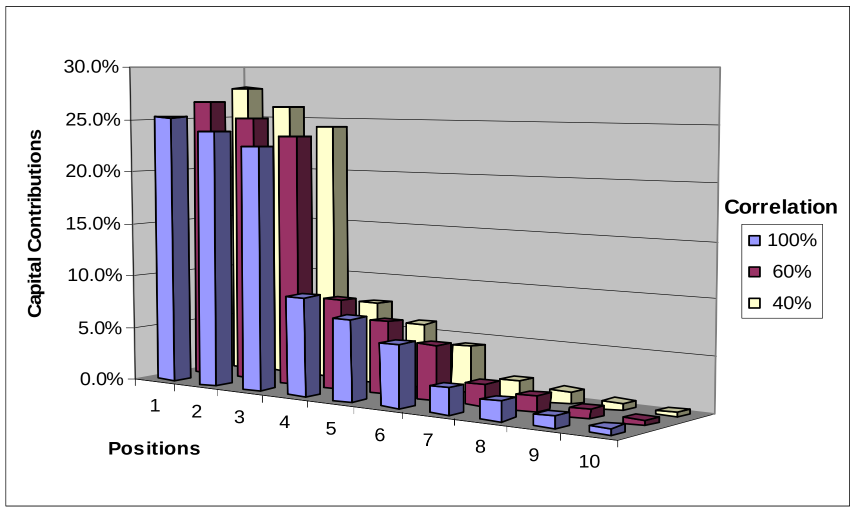

- SF model (100% correlation) - sector contributions = SA capital
- Diversification → DF_k (*marginal diversification factors*)

  Depend on:
  - Relative sector size (SA capital)
  - intra-sector correlation

    (in this example the same for all sectors)

The Diversification Factor & Capital Contributions

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	10	100%	11.00%	0.15	31.3%	25.2%
2	10	100%	10.00%	0.15	28.4%	24.0%
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8	10	100%	0.20%	0.15	0.6%	1.8%
9	10	100%	0.10%	0.15	0.3%	1.0%
10	10	100%	0.05%	0.15	0.1%	0.6%
Total	100				3.5	19.3
					Capital	**15.8**

- Smaller sectors contribute more to overall diversification
  - % allocations **<** SA contrib.
- Bigger sectors → bigger % contrib.
- Effect grows with diversification
  - smaller corr. → higher effect
- e.g. three largest portfolios
  - SF model → 69% contribution
  - Corr=40% → 76% contribution

Positions	EAD	EL%	VaR %	Capital % Corr=100%	Capital % Corr=60%	Capital % Corr=40%
1	10%	31.3%	25.2%	23.9%	25.1%	26.7%
2	10%	28.4%	24.0%	23.0%	24.0%	25.3%
3	10%	25.6%	22.7%	22.0%	22.8%	23.8%
4	10%	5.7%	9.1%	9.9%	9.2%	8.3%
5	10%	4.3%	7.5%	8.3%	7.5%	6.6%
6	10%	2.8%	5.7%	6.3%	5.7%	4.8%
7	10%	0.9%	2.4%	2.7%	2.4%	1.9%
8	10%	0.6%	1.8%	2.0%	1.7%	1.4%
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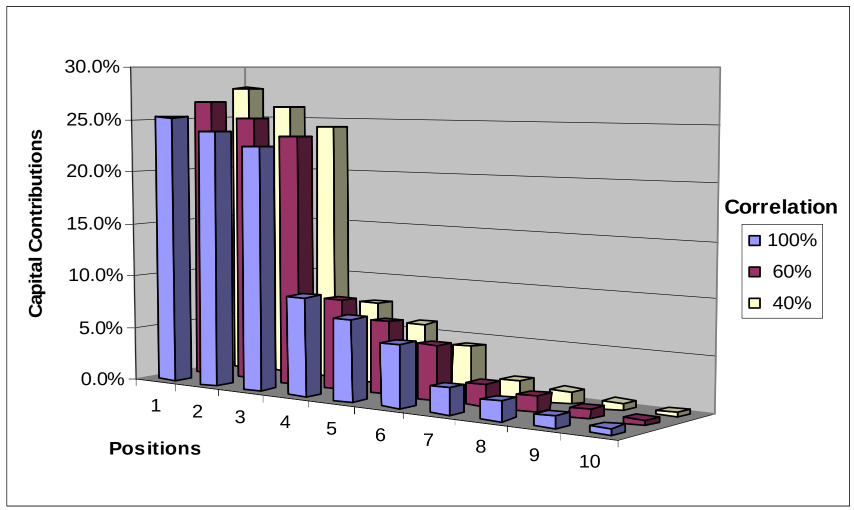

DF as Management Tool

Calibrated model to EC model - Implied parameters

- The model can be fitted to full multi-factor EC model
  - Implied parameters → risk & concentration indicators
- Implied fitted model can be used
  - Analytical sensitivities
  - Communication tool – understand underlying problem better
  - Fast model for real-time calculation or extrapolation
    - e.g. akin to implied volatility surface with BS model or implied correlation skew in CDOs

Summary -Diversification Factor Model

- Simple multi-factor adjustment to one-factor Basel II model
- Intuitive capital allocation (risk contributions) & sensitivities
  - Diversification factor at the portfolio level and also for contributions
  - Contributions further attributed to size and correlation components
- Tabulated *DF* – function of two variables
  - Size concentration
  - Average cross-sector correlation
- Applications
  - Potential regulatory application (Basel II)
  - Effective credit portfolio decision management support tools

Some Remarks on *DF* model

- Literature
  - Michael Pykhtin (2004) – general analytical model (similar techniques as in the GA)
  - Dirk Tasche (2005) – analytical capital contributions and “diversification index”
  - Several papers discuss the calibration of the underlying, simplified, multi-factor model (e.g. Gordy and Heitfield (2004) Credit Suisse (2004), etc.), and the analytics behind the portfolio model
- *DF* model vs. calibration of one-factor model in Basel II
  - Application of actual DF for regulatory purposes needs adjustment (up) to account for calibration of one-factor model (which, for already accounts for some diversification within the sample data)
- Recalibration of model is generally expensive
  - Although we used MC simulation for the multi-factor model, semi-analytical tools are also available
- *DF* model captures largely systemic risk (sector and geographical concentrations)
  - Extensions using perhaps granularity adjustment technique (Gordy) – “name concentrations”
- Perceived limited codependence structure of underlying multi-factor model (one economy-wide systemic factor links all sectors)
  - Mostly a calibration issue - possible extensions to more than one factor to link sectors

Concluding Remarks

*Capital & credit portfolio management → **active management***

- **Capital allocation** – intuitive and must detect concentrations
  - Positions: sectors, CPs, assets
  - Systemic risk factors: macro-economic or financial (Rosen & Saunders 2006)
- **Risk decomposition**: default vs. economic; systemic vs. idiosyncratic; horizon;
  - Understand concentration: Size (individual position) and correlation
- **Sensitivities & stress testing**: credit correlations, obligors/assets, PD/LGD/EAD
- **Real-time marginal capital** calculation and allocation
- **Risk & return optimization**
- **Communication tool** – intuition to understand underlying problem better
  - Provide useful summary measures – risk & concentration indicators
  - Reconciliation of economic and regulatory capital

Thank You!

www.fields.utoronto.ca/~drosen

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Presenter’s Bio

Dr. Dan Rosen is currently a visiting research fellow at the *Fields Institute for Research in Mathematical Sciences* an adjunct professor at the *University of Toronto's graduate program in Mathematical Finance*. In addition, he serves as risk and capital management consultant to a number of financial institutions in Europe, North America and Latin America.

Up to July 2005, Dr. Rosen had a successful ten-year career at *Algorithmics* Inc., where he held senior management roles in strategy and business development, research and financial engineering, and product marketing. In his latest role as Vice President of Strategy and Business Development, he was responsible for setting the strategic direction of Algorithmics' solutions, business models for new initiatives and strategic alliances. Since joining Algorithmics in 1995, he headed up the design, positioning and marketing of credit risk and capital management solutions, market risk management tools, operational risk, and advanced simulation and optimization techniques, as well as their application to several industrial settings.

Dr. Rosen lectures extensively around the world on enterprise risk and capital management, credit risk, market risk, and financial engineering. He has authored numerous papers on quantitative methods in risk management, applied mathematics, operations research, and has coauthored two books and various chapters in risk management books. Dr. Rosen is the regional director in Toronto of PRMIA (Professional Risk Management International Association) and authored two chapters of the Professional Risk Manger Handbook, a member of the credit risk steering committee of the IAFE, and one of the founders of RiskLab, an international network of research centers in Financial Engineering and Risk Management, initiated by Algorithmics and the University of Toronto. He holds several degrees, including an M.A.Sc. and a Ph.D. in Chemical Engineering from the University of Toronto.