Julia Sets of Positive Measure V

The Bounce

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Following X. Buff and A. Chéritat.

THE CENTRAL THEOREM

S Inou-Shishiku bound

$\Theta = \{\theta = [a_1, \dots, a_n, \dots] \mid (\forall i)\ a_i \geq S\}$

Fix $\theta \in \Theta$ bounded type, $\theta = [a_1, \dots, a_n, \dots]$

$\theta' = \theta'(k, N) = [a_1, \dots, a_k, N, S, \dots, S, \dots]$

$\Delta = \Delta_\theta$, $\Delta' = \Delta_{\theta'}$ Siegel disc

Central Theorem: $m(\Delta \setminus K_{\theta'})$ can be made arbitrarily small by choosing k and then N big enough

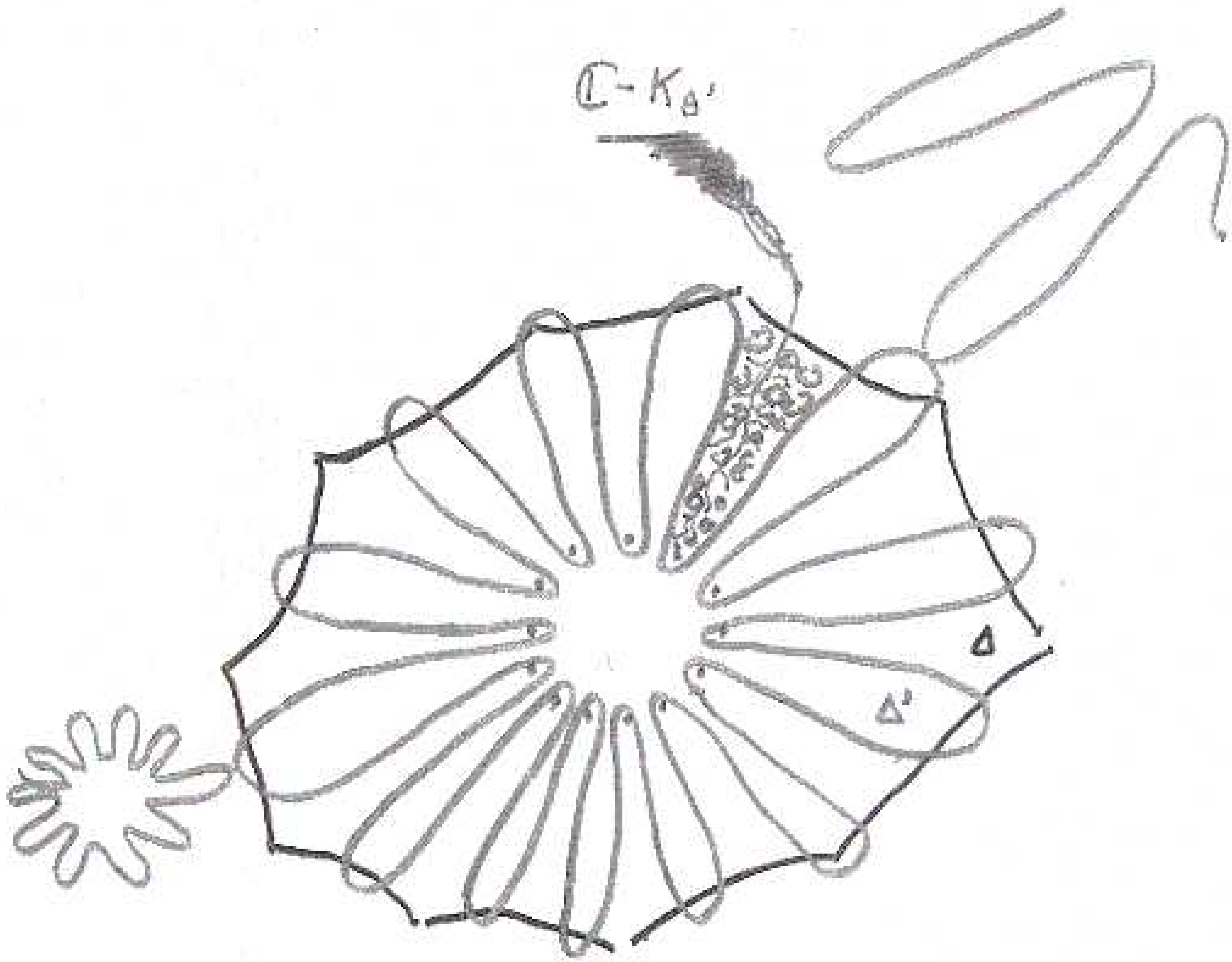
$(\forall \varepsilon > 0)(\exists k_0)(\forall k \geq k_0)(\exists N_0)(\forall N \geq N_0)\ m(\Delta \setminus K_{\theta'}) < \varepsilon$

Corollary 1. $(\forall \varepsilon_1, \varepsilon_2, \varepsilon_3)\ \exists \theta'$

- $|\theta' - \theta| < \varepsilon_1$
- $f_{\theta'}$ has a cycle in $D(0, \varepsilon_2)$
- $m(K_\theta \setminus K_{\theta'}) < \varepsilon_3$

Corollary 2: $\forall \varepsilon > 0\ \exists \hat{\theta}$ Cremer

$$m(K_{\hat{\theta}}) \geq m(K_\theta) - \varepsilon$$



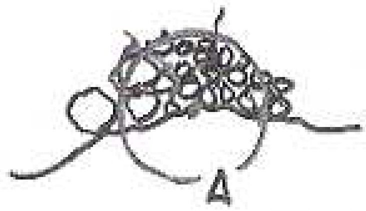
C - K Δ'
Δ
Δ'

THE TOLL BELT

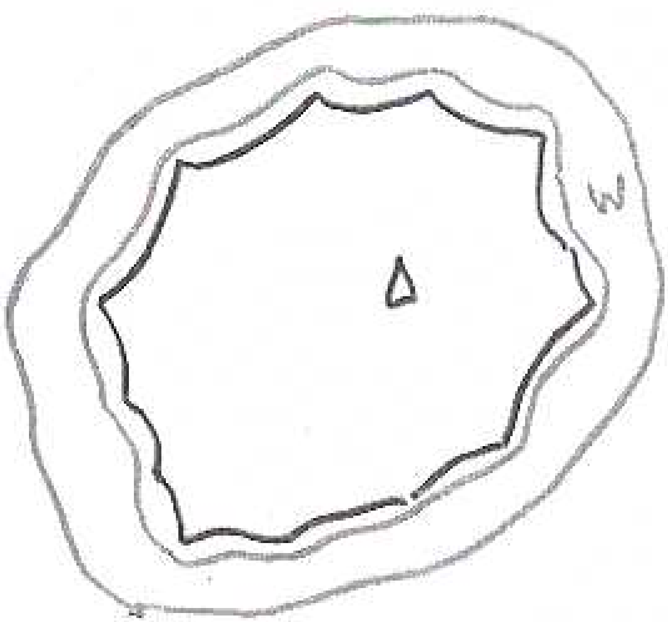
<u>Notation</u>: $\mathrm{dens}_X Y = \dfrac{m(X\cap Y)}{m(X)}$ m = Lebesgue measure

$$\eta_{McM}(\rho) = \sup_{z\in\partial\Delta} \mathrm{dens}_{D(z,\rho)}(\mathbb{C}\smallsetminus K_\theta)$$

Theorem (McMullen): $\eta_{McM}(\rho)\to 0$ when $\rho\to 0$



$\delta>0$ chosen small



Toll Belt:

$$W = W(\theta,\delta) = \{z\in\mathbb{C} \mid 2\delta < d(z,\overline{\Delta}) < 9\delta\}$$

PROPERTIES of the TOLL BELT

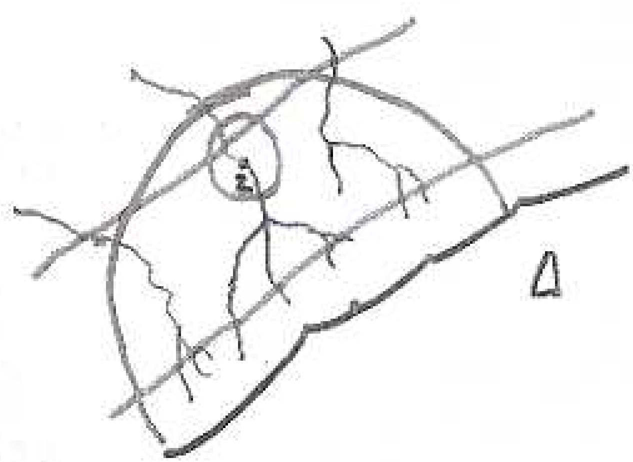
Prop. 1: For θ' close to θ, a point $z \in \Delta$ cannot escape to ∞ under $f_{\theta'}$ without stepping in W.

$(\exists \varepsilon > 0)(\forall \theta' \,|\, |\theta' - \theta| < \varepsilon)(\forall z \in \Delta \setminus K_{\theta'})(\exists n)\ f_{\theta'}^{n}(z) \in W$.

Pf: $\Delta, W \subset D(0,2)$. $|f'_\theta| < 4$. $f_\theta(\bar\Delta) = \bar\Delta$.
$f_\theta(\mathrm{Int}\ \partial W) \subset\subset \mathrm{Int}(\partial^+ W)$. Same for θ'. □

Prop 2: $(\forall z \in W)\ \mathrm{dens}_{D(z,\delta)}(\mathbb{C} \setminus K_\theta) \geq \eta_W$.

$$\eta_W = 100\, \eta_{MLM}(10\delta)$$



A REQUIREMENT on θ'

Requirement 1 :

$$\partial(P_{\theta'}, \bar\Delta) = \partial(\bar\Delta', \bar\Delta) < \frac{\delta}{2}$$

∂ = Hausdorff semi distance: $\partial(X, Y) = \sup_{x \in X} d(x, Y)$

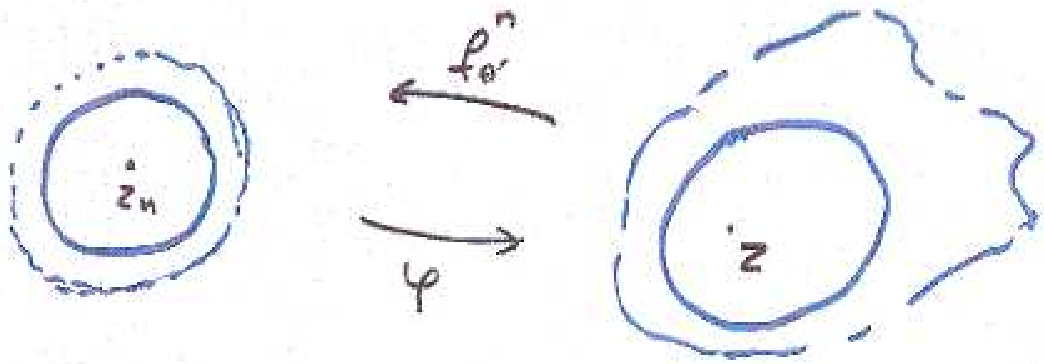
Possible by Buff's talk.

Prop. 3 : Under requirement 1,

$$z \in \mathbb{C}\ ,\quad z_n = f_{\theta'}^{n}(z) \in W \Rightarrow$$

$$\exists \varphi : D = D(z_n, \delta) \xrightarrow{\cong} V_z \text{ nbd } z$$

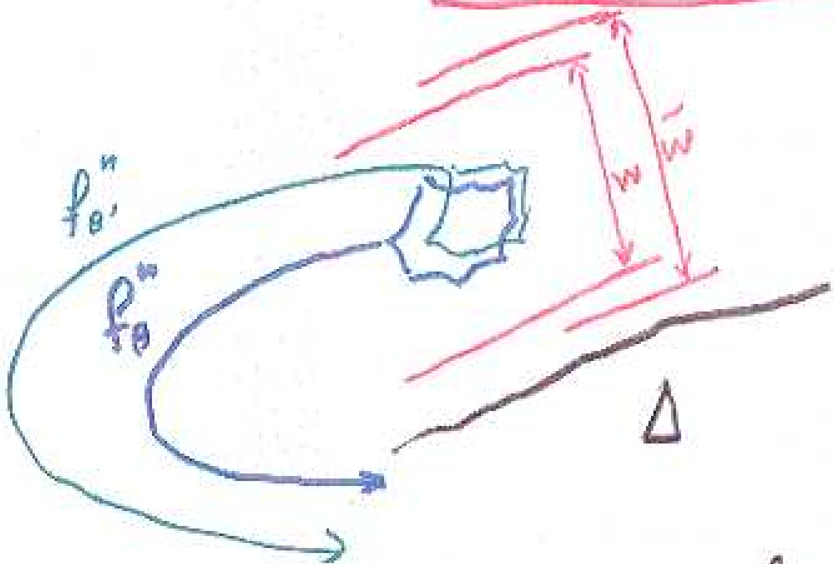
- φ is a branch of $f_{\theta'}^{-n}$
- φ extends univalently to $D(z_n, \frac{3}{2}\delta)$



Koebe :

$$\frac{\sup_D |\varphi'|}{\inf_D |\varphi'|} \le K \qquad \text{absolute constant.}$$

SECOND REQUIREMENT



$\tilde W = \{ z \mid \delta < d(z,\Delta) < 10\delta \}$

$$K_\theta^\ell = \{ z \in K_\theta \mid \quad f_\theta^\ell(z) \in \Delta \}$$

$$Y(\theta',\ell) = \{ z \in \tilde W \mid (\exists n \le \ell) \ \ f_{\theta'}^n(z) \in \Delta \}$$

$Y(\theta',\ell) \longrightarrow K_\theta^\ell$ when $\theta' \to \theta$

$(\forall L \subset\subset K_\theta^\ell)$ $Y(\theta',\ell) \supset L$ if $|\theta' - \theta|$ small enough.

$m(\tilde W \cap K_\theta) \setminus Y(\theta',\ell) \longrightarrow 0$ when $\begin{matrix} \theta' \to \theta \\ \ell \to \infty \end{matrix}$

<u>Requirement 2 on ℓ and θ'</u>:

- $(\forall z \in W)$ $\mathrm{dens}_{D(z,\delta)}(\mathbb{C} - K_\theta^\ell) < 2\eta_W$
- $(\forall z \in W)$ $\mathrm{dens}_{D(z,\delta)}(\mathbb{C} - Y(\theta',\ell)) < 2\eta_W$

DISTORTION ESTIMATE

K : Koebe constant for $\frac{3}{2}$:

$\forall \varphi : D = D(a,r) \to \mathbb{C}$ which extends univalently to $D(a, \frac{3}{2} r)$

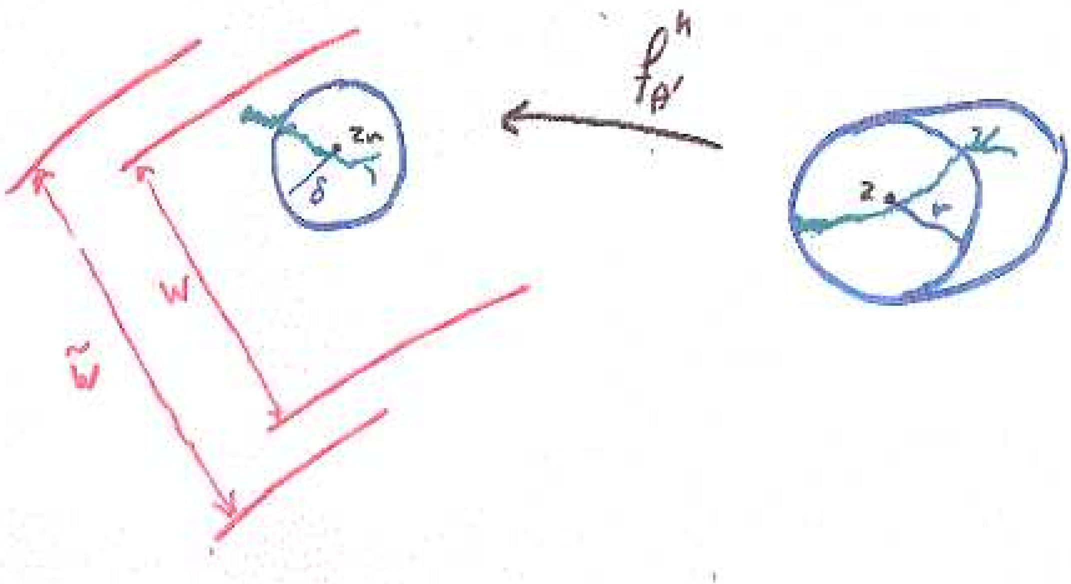
$$\frac{\sup_D |\varphi'|}{\inf_D |\varphi'|} \leq K .$$

Prop 4: Under requirements (1) and (2),

$$z \in \mathbb{C} ,\quad z_n = f^n_{\theta'}(z) \in W$$

$$\Downarrow$$

$$(\exists r) \quad \mathrm{dens}_{D(z,r)} \left(\mathbb{C} - f_{\theta'}^{-n}(Y(\theta', \ell))\right) \leq 2K^4 \eta_W$$



MEASURE DENSITIES

<u>Notations</u>: • $m|_X$ = Lebesgue measure on X, extended by 0.

• $Y(\theta', \ell) = \{ z \in \tilde{W} \mid (\exists n \leq \ell) \quad f_{\theta'}^n(z) \in \Delta \}$

• $Y^*(\theta', \ell) = \{ z \in \tilde{W} \mid (\exists n \leq \ell) \quad f_{\theta'}^n(z) \in \Delta \cap \Delta' \}$

By Chéritat's talk

Theorem: when k and $N \to \infty$,

$$\liminf \; m|_{\Delta'} \geq \frac{1}{2} \, m|_{\Delta} \quad \text{(weak convergence)}$$

Corollary: For ℓ fixed, k and $N \to \infty$,

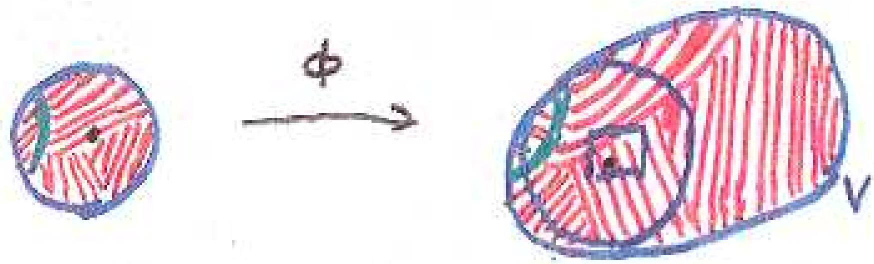
$$\liminf \; m|_{Y^*(\theta', \ell)} \geq \frac{1}{2} \, m|_{K_\theta^\ell} \, .$$

THIRD REQUIREMENT

Requirement 3: $\forall z \in W$

- $\mathrm{dens}_{D(z,\delta)} \, Y^*(\theta', \ell) > \frac{1}{3} \, \mathrm{dens}_{D(z,\delta)} \, K_\theta^\ell$ and moreover

- $\forall \phi : D(z,\delta) \xrightarrow{\cong} V$ which extends univalently to $D(z, \frac{3}{2}\delta)$
  $\mathrm{dens}_V \, \phi(Y^*) > \frac{1}{3} \, \mathrm{dens}_V \, \phi(K_\theta^\ell)$
  and moreover



- For the biggest disc $D = D(\phi(z), r) \subset V$
  $\mathrm{dens}_D \, \phi(Y^*) > \frac{1}{3} \, \mathrm{dens}_V \, \phi(K_\theta^\ell)$
  and moreover

- For any square $Q \subset D$ with side $\geq \frac{r}{2\sqrt{2}}$
  $\mathrm{dens}_Q \, \phi(Y^*) > \frac{1}{3} \, \mathrm{dens}_Q \, \phi(K_\theta^\ell)$.

THE BOUNCE

Fix $A = [a_1, \ldots, a_n, \ldots] \in \Theta$ bounded type

$A' = [a_1, \ldots, a_n, N, s, \ldots, s, \ldots]$

$\Delta = \Delta_A$, $\Delta' = \Delta_{A'}$, $W = W(A, \delta)$ toll belt.

- $X_p = \{ z \in \Delta \mid z$ bounces $\geq p$ times between Δ and W under $f_{A'}$ $\}$

$(\exists\, n_0 = 0 < n_1 < \cdots < n_p)$ $f_{A'}^{n_j} \in \left| \begin{array}{l} \Delta \text{ for } j \text{ even}, \\ W \text{ for } j \text{ odd}. \end{array} \right.$

- $X_p^* = X_p - X_{p+1}$

Remarks: $\Delta - K_{A'} \subset \bigsqcup X_{2p+1}^*$,

$X_0^* = \Delta \cap \Delta'$.

Proposition 5 : Under requirements (1), (2), (3),

a) $\mathrm{dens}_{X_{2p+1}} (X_{2p+1}^*) \leq \Lambda \eta_W$

b) $\mathrm{dens}_{X_{2p}} (X_{2p}^*) \geq \frac{1}{3}$

$\Lambda = 16 \pi K^4$ absolute constant,

$\Lambda \eta_W \simeq 10\,000\, \eta_{M,M} (10\,\delta)$

LEMMA à la VITALI

Lemma: $U \subset \mathbb{C}$ openset, $Z \subset U$, $\alpha \in [0,1]$

$$(\forall z \in U)(\exists r)\quad \text{dens}_{D(z,r)} Z \le \alpha$$

$$\Downarrow$$

$$\text{dens}_U Z \le 8\pi\alpha$$

- Dyadic squares: $\left[\frac{p}{2^k}, \frac{p+1}{2^k}\right[\times \left[\frac{p'}{2^k}, \frac{p'+1}{2^k}\right[$

Pf of Lemma: For $z \in U$, Q_z = biggest dyadic square contained in $D_z = D(z, r_z)$, $\ni z$.

$$m(Q_z) \ge \frac{1}{8\pi}\, m(D_z),$$

$$\text{dens}_Q Z \le 8\pi\alpha.$$

The maximal Q_z form a partition of a set containing U. $\square$

Pf of Prop 5: a) apply lemma.

b) get directly a partition of X_{2p} in dyadic squares. $\square$

FINAL ESTIMATES

- $m(X_{2p+2}) \leq m(X_{2p+1}) \leq \frac{2}{3}\, m(X_{2p})$
- $m(X_{2p+1}) \leq \frac{2}{3}\, m(X_{2p}) \leq \left(\frac{2}{3}\right)^{p+1} m(\Delta)$
- $m(X^*_{2p+1}) \leq \Lambda \eta_W \, m(X_{2p+1}) \leq \left(\frac{2}{3}\right)^{p+1} \Lambda \eta_W \, m(\Delta)$
- $m(\Delta - K_{\theta'}) = \sum m(X^*_{2p+1}) \leq 2 \Lambda \eta_W \, m(\Delta)$

This can be made arbitrarily small
by choosing δ small enough.

Then one has to choose k, N, ℓ (in this order)
big enough so that requirements (1), (2), (3)
are satisfied.

$\boxed{\text{q.e.d. for Central Theorem}}$