Mom Technology

and
Volumes
of
Hyperbolic 3-manifolds

Joint with
Robert Meyerhoff
Peter Milley

Mostow [1968] **Rigidity** If ρ_1, ρ_2 hyperbolic metrics on a closed hyperbolic n-manifold N, $n \geq 3$ then $\exists$ isometry

$$f : N_{\rho_1} \longrightarrow N_{\rho_2} \quad f \simeq id.$$

(G-Meyerhoff N. Thurston 2003
$n=3$ f isotopic to id.)

Corollary Volume is a topological invariant for closed hyp manifolds $n \geq 3$

True $n=2$ Gauss-Bonnet

Theorem (Thurston - 1977
Extending Gromov, Jorgensen)

Volumes of complete finite volume hyperbolic 3-manifolds are a well-ordered closed subset of $\mathbb{R}$. Order type ω^ω

Only finitely many manifolds can have the same volume.

(Wang) Volumes of hyp. n-manif. discrete if $n \geq 4$

Hyperbolic Complexity Conjecture

~1980

(Thurston, Weeks, Matveev-Fomenko

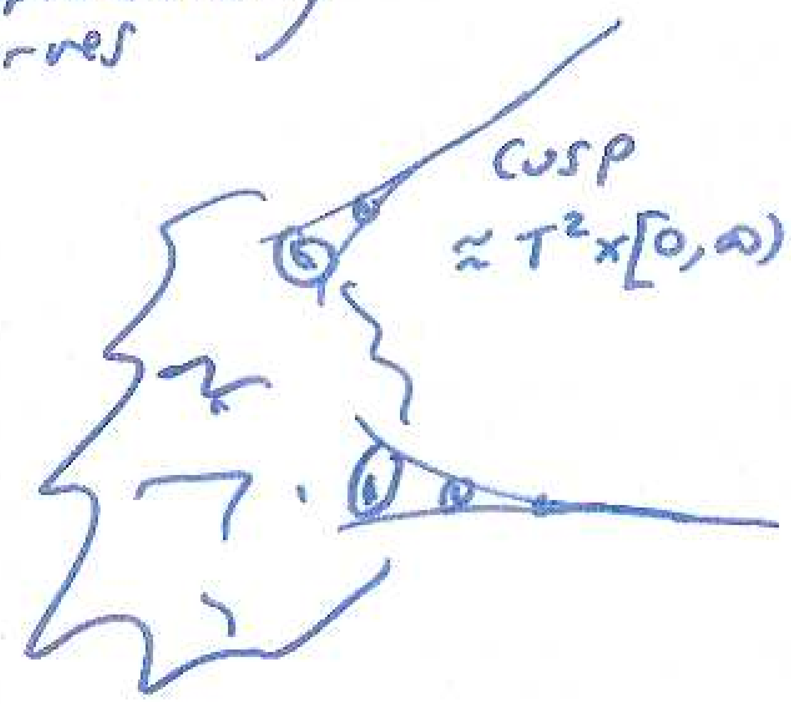
The complete low volume hyperbolic 3-manifolds are obtained by filling cusped hyperbolic 3-manifolds of small topological complexity.

Cusped Hyperbolic 3-manifold

Topology

$M^3 - N$(simple closed curves)

Geometry



Filling (opposite of Drilling)

"Fill" in cusp by adding a solid torus

The <u>weeks</u> <u>manifold</u> W

is the smallest known

closed hyperbolic 3-manifold.

Volume $(W) \approx .9427\ldots$

(1998) (Culler-Hersonsky-Shalen)

If Rank $H_2(M) \geq 3$, then $Vol(M) > Vol(W)$

(1986) (Meyerhoff) Identified

minimal Vol. cusped orbifold.

(1987) (Adams) The Gieseking

manifold is the cusped

(non orientable) manifold of

least volume $= 1.01\ldots$

(1998) (Cao-Meyerhoff) The Figure 8

+ Sister are smallest OR. cusped

manifolds. Vol $= 2.02\ldots$

Lower Bounds for vol(smallest)

1979	Meyerhoff	0.0006
1986	Meyerhoff	0.0008
1991	Gehring-Martin	0.0010
1996	G-Meyerhoff-N. Thurston	16668...
1999	Przeworski	0.2766
7/2000	Przeworki	0.2814
	Marshall Martin	0.2855
10/2000	[MM]+[Prez]	0.2907
2001	Agol	0.32
2002	Przeworski	0.3315
2005	Agol-Dunfield (using Perelman)	0.67
2006	G-Meyerhoff-Milley (using [ADT])	0.89

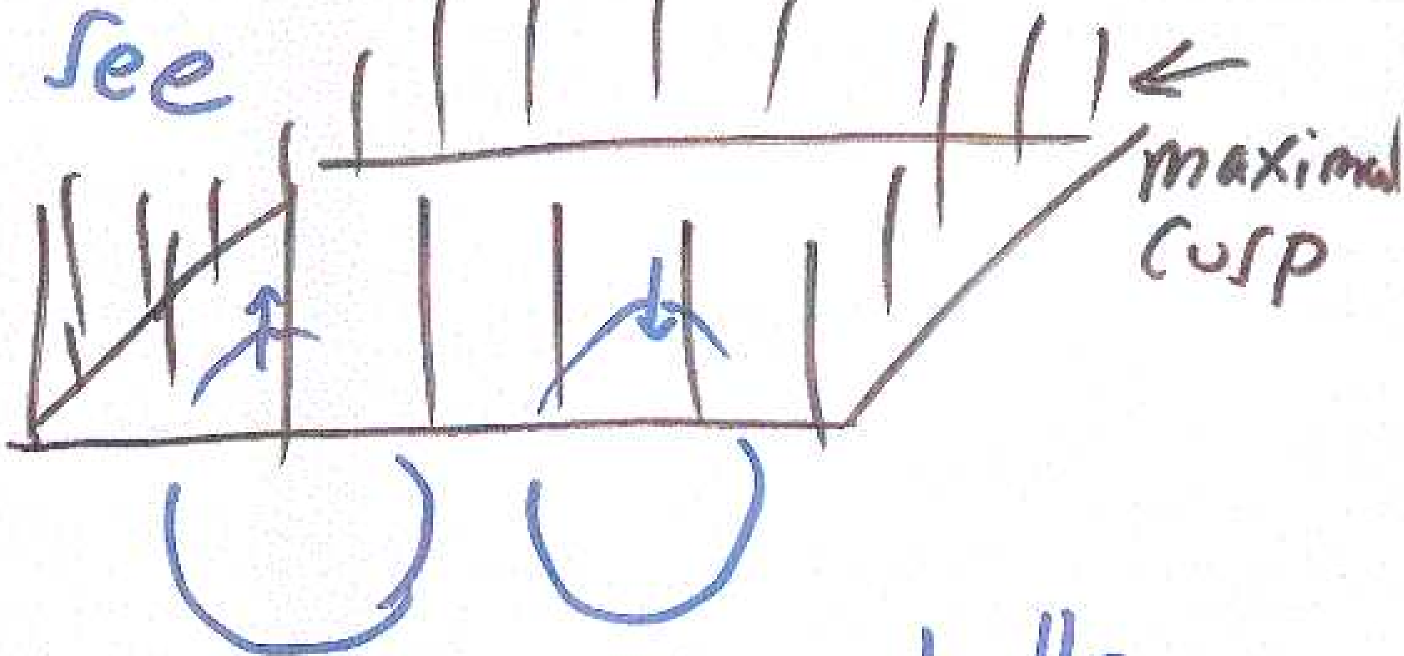
Lower bound for volume of a cusped manifold is the volume of a maximal cusp.

$$\mathrm{Vol}(\mathrm{cusp}) = \tfrac{1}{2}(\mathrm{Area}\ \partial\ \mathrm{cusp})$$

$$\geq \tfrac{1}{2}\left(\pi (1/2)^2\right) = \pi/8$$

Adams Ball

If N^3 1-cusped hyp Manifold, then in $\mathbb{H}^3$ see



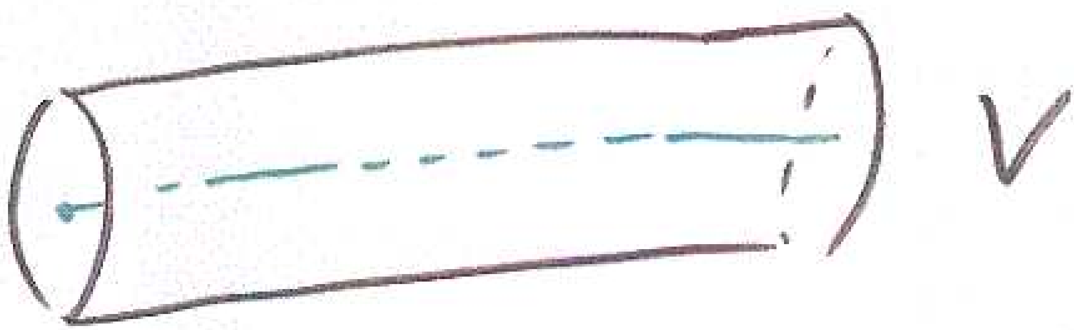
Translated horoballs ineq. under $\mathbb{Z}\oplus\mathbb{Z}$ action

Fact ~~[illegible]~~

~~[illegible]~~ $\mathrm{Vol}(\text{cusp}) = \frac{1}{2}(\text{Area } \partial \text{ cusp})$

(Gehring-Martin) If V is a $D^2 \times S^1$ R-Neighborhood of a geodesic then

$$\mathrm{Vol}(V) = \frac{1}{2}\tanh(R)\,\mathrm{Area}(\partial V)$$



when $R = \frac{\log(3)}{2}$ then

$$\mathrm{Vol}(V) = \frac{1}{4}\mathrm{Area}(\partial V)$$

~1994

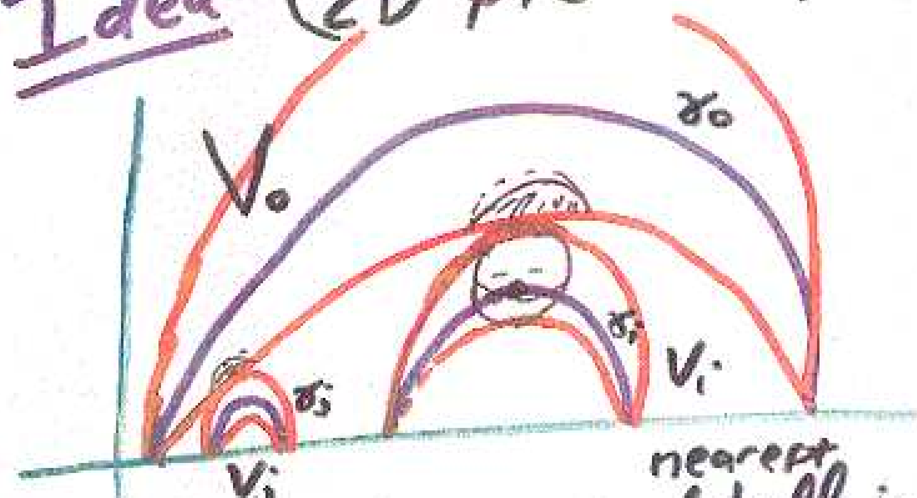
Gehring-Martin Technology

or

[How to estimate the volume of a maximal tube V about a geodesic knowing only its radius r]

[GM] $\mathrm{Vol}(V) = \frac{1}{2}\tanh(r)\,\mathrm{Area}(\partial V)$

Idea (2D picture)



Find max ellipse

Ellipse [illegible]

$\forall \gamma_i$ ($\in q_i$) Consider maximal nearest ball in V_i

Project Ball to ∂V_0. Estimate area of shadows. Apply Formul

Theorem (G-Meyerhoff-N. Thurston)

If M closed, orientable, hyperbol
γ shortest geodesic, then

either i) tube radius$(\gamma) \geq \frac{\log(3)}{2}$

or ii) $\mathrm{Vol}(M) \geq \mathrm{Vol}_3 \approx 1.01\ldots$

Proof ~~via~~ with Rigorous computer assistance.

It sufficed to analyze a compact region of $\mathbb{C}^3$. Chopped region into $\sim 500{,}000{,}000$ subboxe

Eliminated all but 6 boxes by one of $\sim 32{,}000$ reasons.

Six Boxes contained the thin tubed manifolds.

Przeworski: Technology

Let V be a maximal tube about a geodesic, $\{V_i\}$ lifts to $\mathbb{H}^3$.

Prez Shadow = proj of V_i into ∂V.

Theorem (Prez) If Rad$(V) \geq .42$ then these shadows have disjoint interiors.

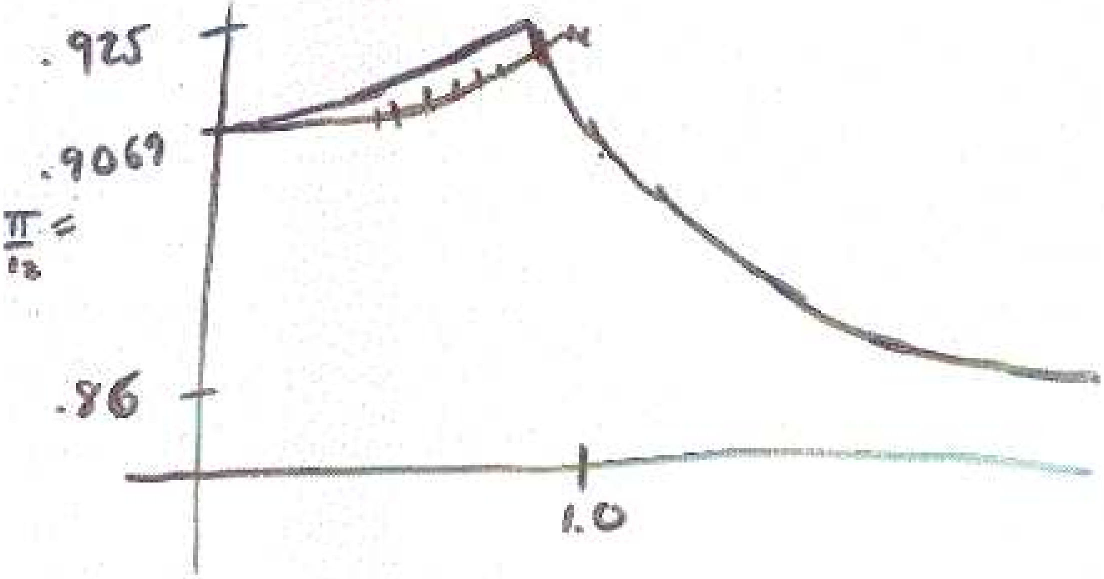
If V = max tube about a shortest geod. then (using [GMT]) Vol(shortest) $> .2766$

This technology uses the complex distance rather than just real distance.

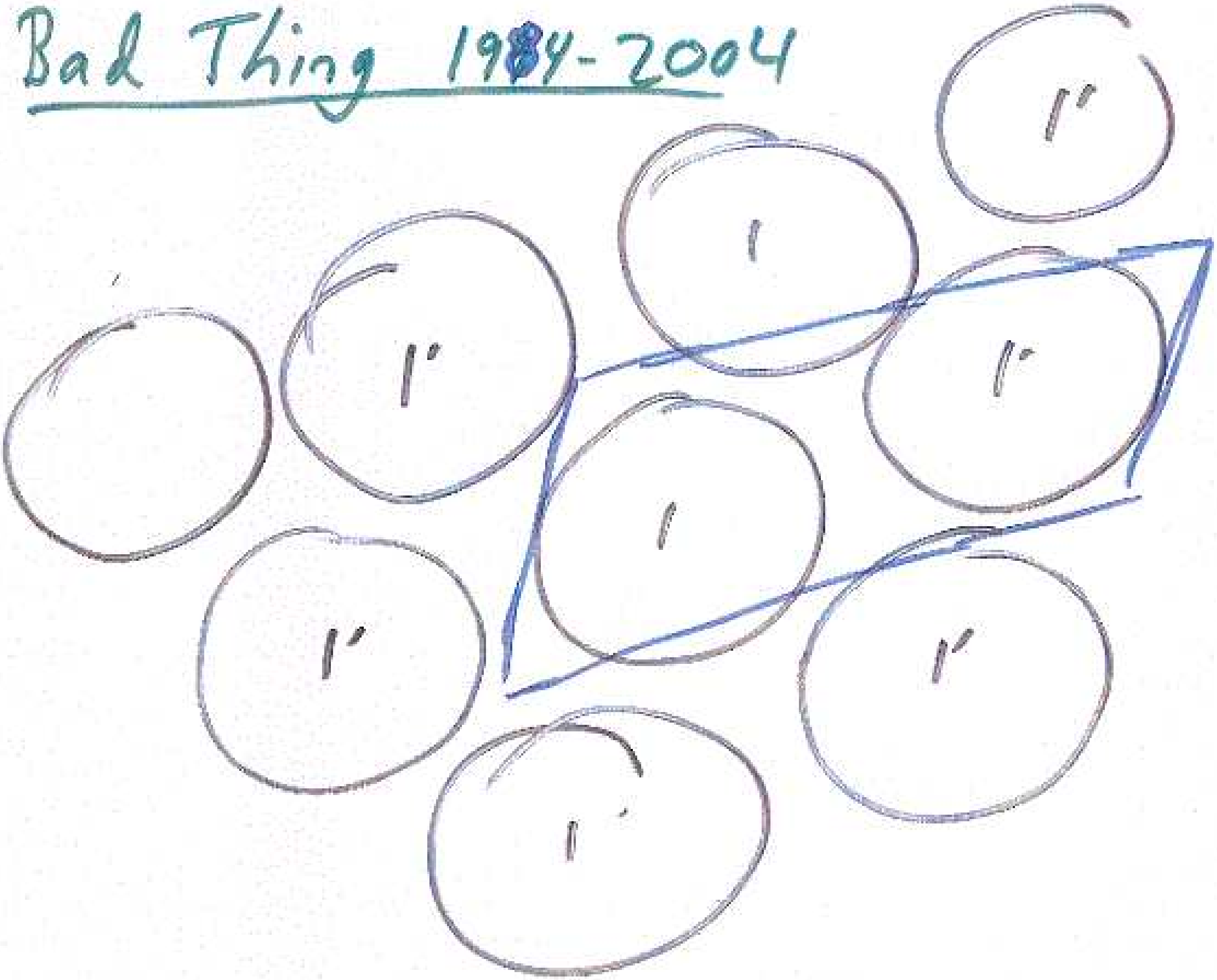
F

Tube density [Prez $r \leq 7.1$]

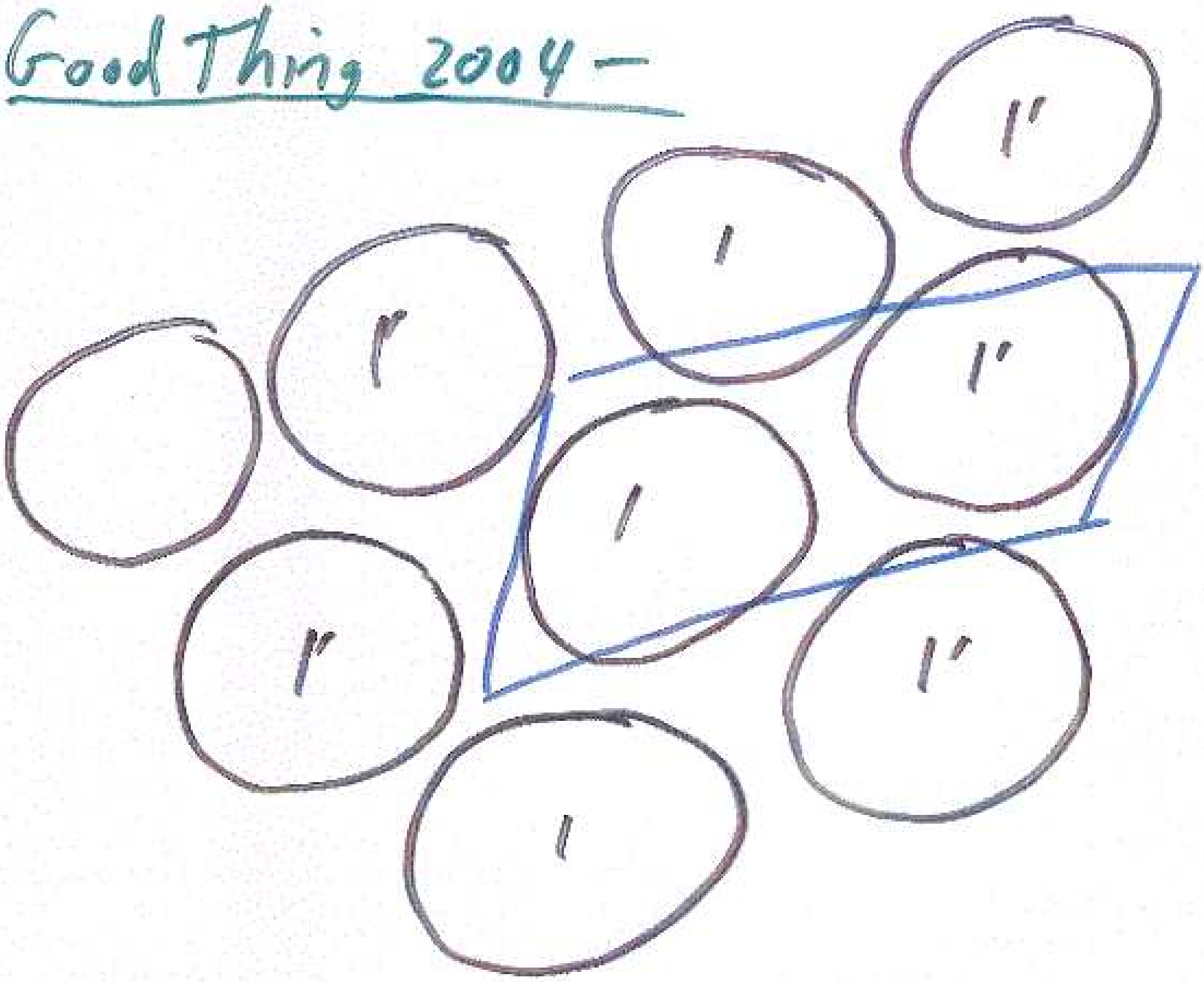
[Marshall-Martin $r \geq 7.1$]



Bad Thing 1984-2004



Good Thing 2004 –



TTS

Theorem* (G - Meyerhoff - Milley)

If N is complete, 1-cusped and $Vol(N) \leq 2.7$ then N is one of

m003
m004
m006
m007
m009
m010

The first 6 cusped orientable mflds in the snappea Census

* subject to checking snappea output

Motivating Philosophy

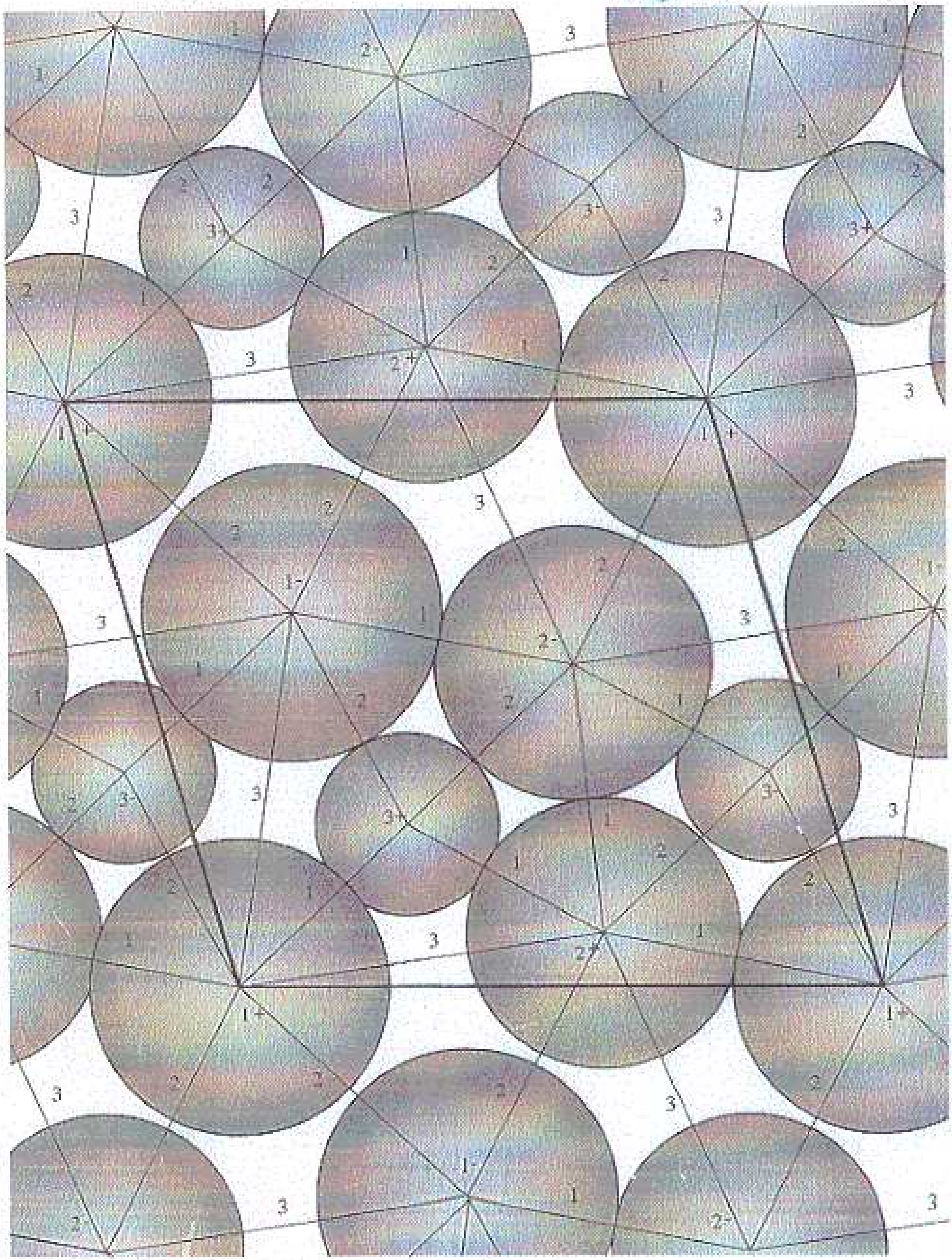
Let N low volume hyp 3-manif.
T torus bounding a
maximal cusp V or
maximal tube V about shortest geod.

slowly expand T -
in usual way obtain
a handle str in $M - \mathring{V}$
with 1, 2, 3 handles.

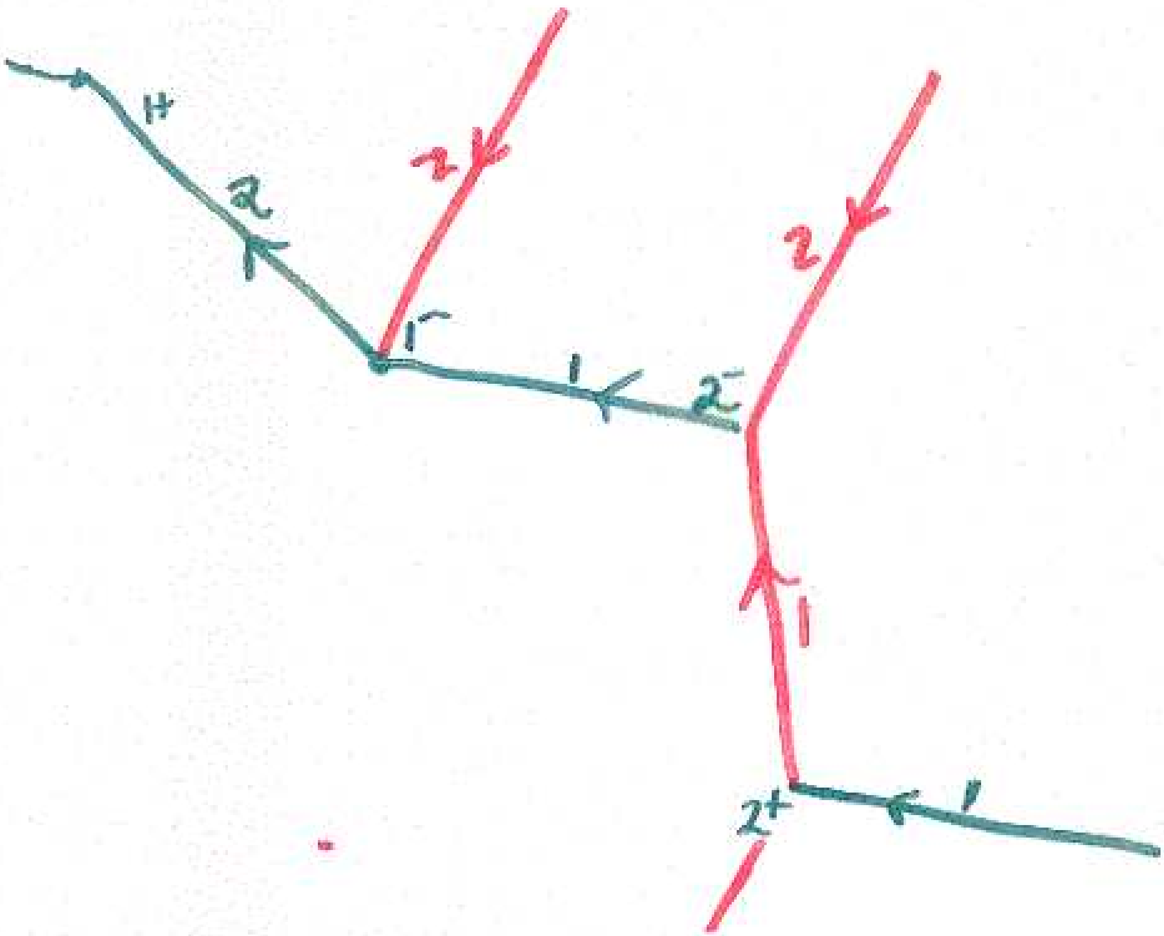
We expect to find among
the 1, 2 handles a mom ≤ 4
structure, i.e. a particular
submanifold $M \subset N$ s.t.
$N - \mathring{M}$ = solid tori & cusps.

M011

Vol = 2.781 Cusp Area = 4.270
Cusp Vol = 2.135



$\sqrt{16}$



H
2
2
2
1⁻
1
2⁻
1
2⁺
1

Mom-n Structure (M, T, Δ)

M: Compact ∂-manifold
∂M = union of tori

T: Component of ∂M

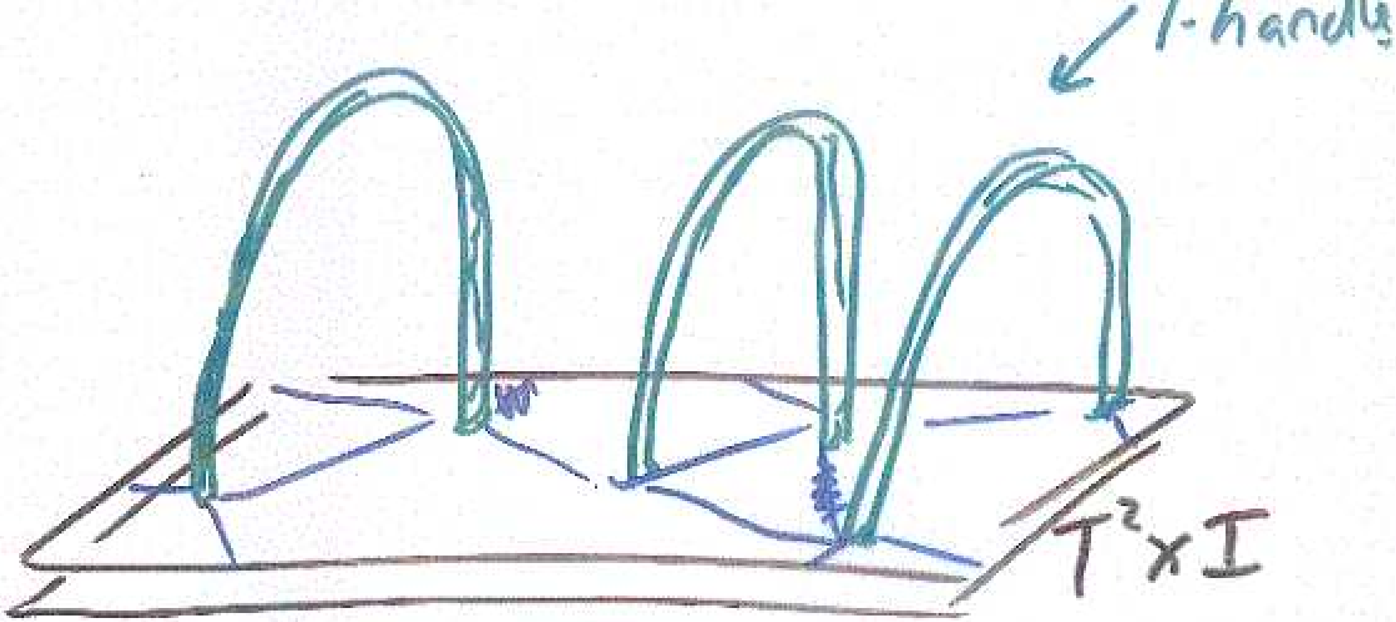
Δ: Handle str. on M
Start with $T \times I$
attach n 1-handles
" " 2-handles

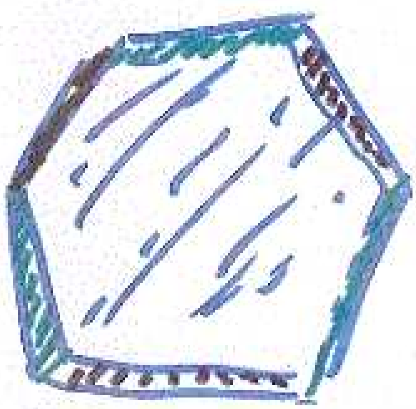
to $T \times 1$ side.

Valence (2-handles) = 3

There are 3 hyp Mom-2's
18 (new) hyp Mom 3's *
Conj. 117 (new) hyp Mom 4's

A Mom-n manifold has Matveev Complexity $\leq 2n$





Valence-3
2-handle

Internal Mom-n Structure

$(M, T, \Delta) \subset N_{hyp}$ s.t.

$i_{*}(\pi_1(M))$ not abelian

and

each comp of ∂M bounds a

Cusp or

Solid torus

$\therefore$ $N = M$ or is a filling of M

Mom Criterian If

$f: M \rightarrow N$ embedding $n \leq 4$

M mom-n, N hyp, $f_{*}(\pi_1(M))$ not abelian then N has an internal $\underset{hyp}{\wedge}$ Mom-n Str.

<u>Definition</u> Let N^3 be a 1-cusped mfld, H_∞ maximal cusp $\{H_i\}$ $\pi_1(N)$-translates of H_∞.

Partition $\{H_i\}$ into <u>orthoclasses</u> $\Theta(1), \Theta(2), \Theta(3)$ by

$H_i \sim H_j$ if H_i is a $\mathbb{Z}\oplus\mathbb{Z}$ translate of H_j or H_j^* (an Adams ball to H_j)

Order the classes so that

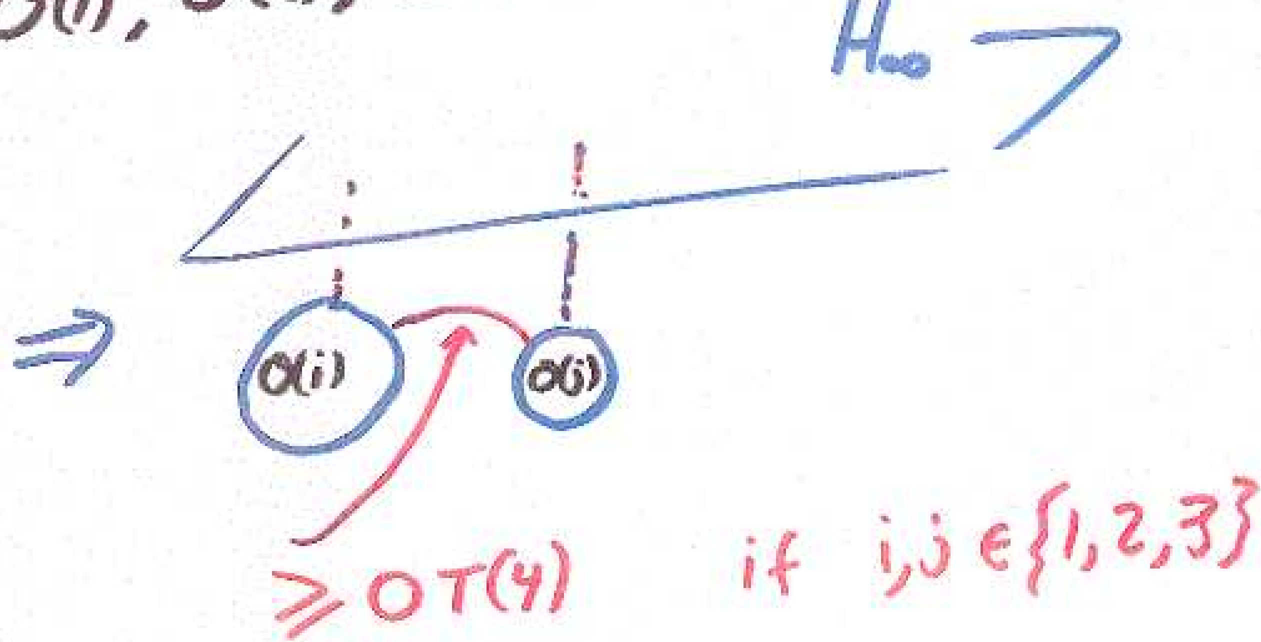
$OT(1) \leq OT(2) \leq \cdots$

where $OT(i) = d_{\mathbb{H}^3}(H_i, H_\infty)$

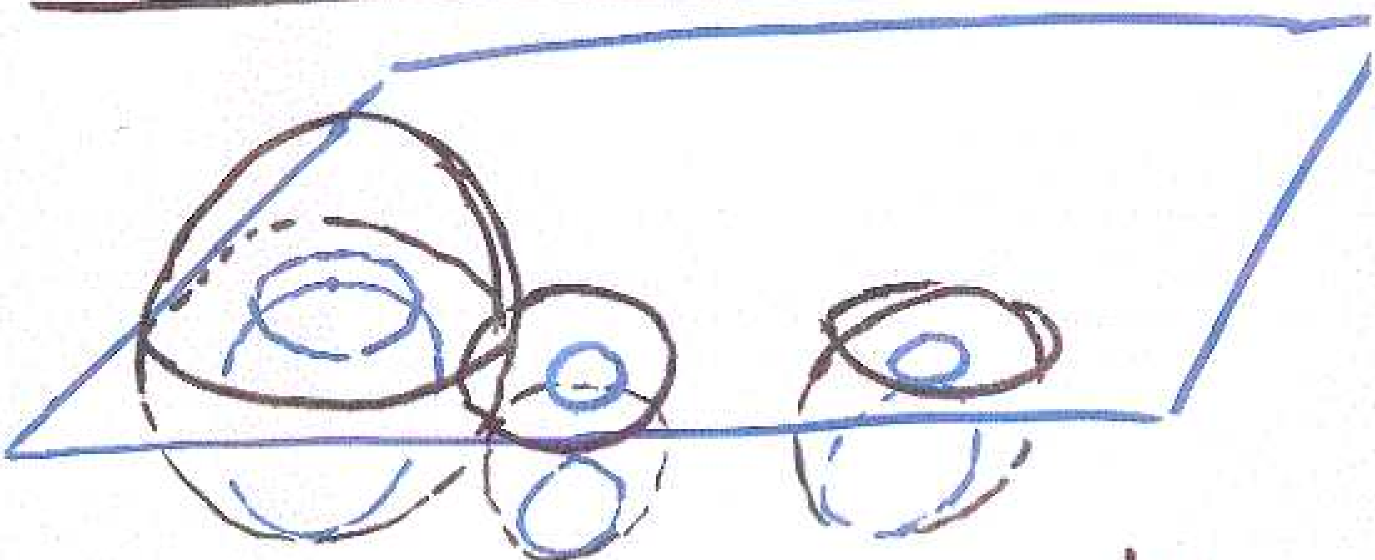
<u>Note</u> $OT(1) = 0$

How to estimate Vol(N) if there is no Mom-3 structure involving $O(1), O(2), O($

<u>Very special Case</u> There are no 2-handles involving $O(1), O(2), O(3)$.



Estimating the area of the maximal torus T_∞



Expand each $Q(i)$ $i \leq 3$ by $\frac{OT(4)}{2}$. Project to T_∞. Calculate area of these projections.

How to Estimate $Vol(N)$

- Expand each H_i and H_∞ by $OT(4)/2$

- Compute Vol(expanded H_∞) using area estimate

- Subtract intersections of expanded horoballs.

Among Cusped census manifolds

First Mom-3 *

m038 Vol(m038) = 3.18

First Mom-4 *

m206 Vol(m206) = 4.06

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m125
m129
m203

Hyperbolic Mom-2's

* Based on quick visual observation

F.

Ortholength spectrum
(associated to translates of a shortest geodesic)
for the known closed small volume hyperbolic 3-manifolds

Manifold	Real Length of Shortest Geod.	r[length]	ReO(1)	ReO(2)	ReO(3)	ReO(4)	O(1) basing
Vol 1	.585	.6659*	1.159	1.220	1.996	1.996	$L/2+\pi$
Vol 2	.578	.6694*	1.151	1.301	2.046	2.046	L/2
Vol 3	.831	.5696*	.8314	1.317	1.420	1.420	0
Vol 4	.575	.6709*	1.240	1.539	2.008	2.166	L/2
Vol 5	.480	.7253*	1.441	1.562	2.329	2.459	0
Vol 6	.366	.8129	1.722	1.782	2.736	2.869	L/2
Vol 7	.794	.5814*	1.156	1.274	1.445	2.043	L/2
Vol 8	.365	.8138	1.708	1.821	2.704	2.780	0
Vol 9	.352	.8261	1.771	1.814	2.761	2.948	$L/2+\pi$
Vol 10	.362	.8166	1.710	1.864	2.696	2.728	L/2
Vol 11	.357	.8213	1.759	1.850	2.712	2.890	L/2