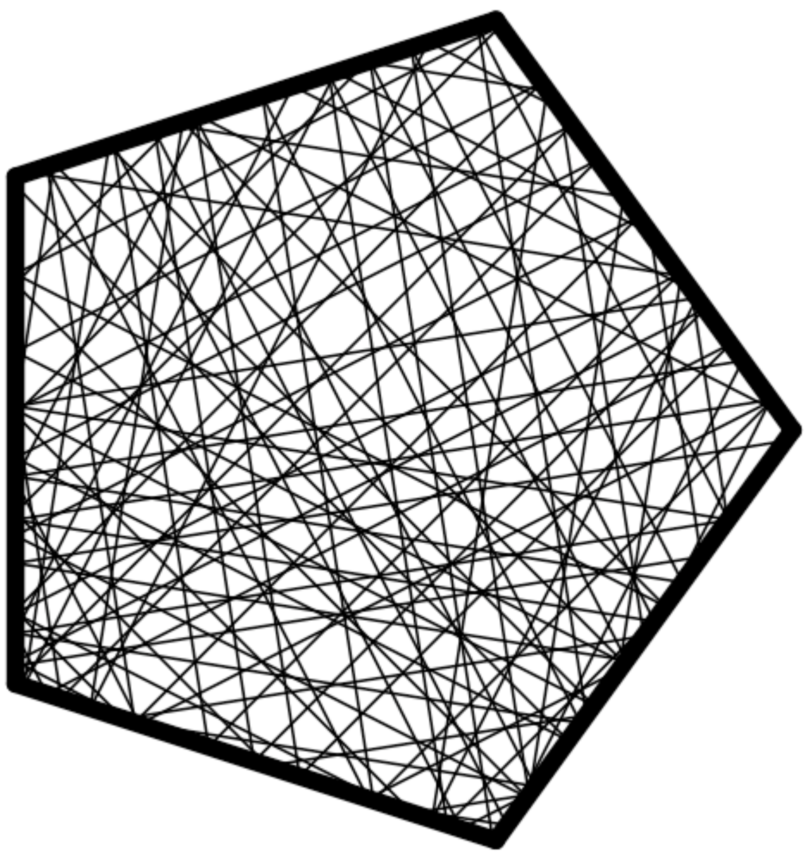
Meditations on results of Calta and McMullen

John Smillie

January 8, 2006

Joint work with Kariane Calta.

Billiards in planar billiard tables provide appealing examples in dynamical systems.



The article:

Zemljakov-Katok "Topological transitivity of billiards in polygons" Mat. Zametki 18 n. 2 291-300 1975

relates the billiard flow to translation surfaces.

A translation surface is a compact surface which is obtained by gluing together polygons in the plane so that the gluing maps are restrictions of translations.

We obtain singular points when the sum of the angles at a point is greater than 2π.

An *affine diffeomorphism* of a translation surface is a diffeomorphism for which the derivative is constant.

Two translation surfaces are *affinely equivalent* if there is an affine diffeomorphism between them.

Let $Aff(S)$ be the group of orientation preserving affine automorphisms of S.

$$d : Aff(S) \to SL(2, \mathbb{R})$$

The image of d is the *Veech group*, $V(S)$.

Translation surfaces arise in topology.

An affine automorphism $f : S \to S$ for which df is hyperbolic is a pseudo-Anosov diffeomorphism.

Thurston shows that every pseudo-Anosov homeomorphism of a surface can be represented as an affine diffeomorphism of some translation structure.

Translation structures arise in complex analysis.

A translation structure on a surface gives rise to a conformal structure.

If we start with a surface with a conformal then the compatible translation structures are represented by holomorphic abelian differentials.

The collection of translation structures on a given surface with a prescribed collection of singular points forms a topological space called a *stratum*.

Notation for strata:

$\mathcal{H}(n_1, \ldots n_j)$ where the ns correspond to excess angle at singular points

Examples:

$\mathcal{H}()$, translation structures on the torus (with no singular points)

$\mathcal{H}(2)$, translation structures on surfaces of genus two with one singular point

$\mathcal{H}(1,1)$, translation structures on surfaces of genus two with two singular points

There is a natural action of $SL(2,\mathbb{R})$ on a stratum so that the orbits of this action are the affine equivalence classes.

Given S and $\alpha \in SL(2,\mathbb{R})$ there is a translation structure αS and an affine map $f : S \to \alpha S$ with $Df = \alpha$.

The fact that any two translation structures on the torus of the same area are affinely equivalent means that the $SL(2,\mathbb{R})$ action on $\mathcal{H}()$ is transitive.

In particular $\mathcal{H}() = SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ where $SL(2,\mathbb{Z}) = Aff(T^2)$.

Renormalization principle

Natural questions about the dynamics and geometry of S translate into questions about the behavior of the $SL(2,\mathbb{R})$ orbit of S in its stratum.

Example: The Masur criterion for unique ergodicity of the vertical flow in S involves the behavior of $g_t(S)$ where:

$$g_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$$

Why are billiard trajectories in the square easier to understand than billiard trajectories in most other rational polygons?

Because the action of the action of $SL(2,\mathbb{R})$ on $\mathcal{H}() = SL(2,\mathbb{R})/Aff(T^2)$ is understood.

The key point is that $Aff(T^2) = SL(2,\mathbb{Z})$ maps to a *lattice* in $SL(2,\mathbb{R})$.

For which translation structures T is the Veech group a lattice in $SL(2, \mathbb{R})$?

Such T are called *lattice* surfaces.

Gutkin studied translation structures T for which $V(T) \subset SL(2, \mathbb{Z})$.

Veech discovered examples of translation structures T for which $V(T)$ is a lattice contained in $SL(2, K) \subset SL(2, \mathbb{R})$ for certain number fields K.

Lattice surfaces are examples of *exceptional* surfaces in that their $SL(2,\mathbb{R})$ orbits are not dense in their stratum component.

Properties of exceptional surfaces are presumably not the same as those of other surfaces in their stratum component.

If we want to understand properties of one particular billiard table "almost everywhere" results are not so useful.

It is an important and interesting question to find all exceptional surfaces.

The answer is not known in general but Calta and McMullen answer this question in genus two.

McMullen gives an answer in terms of real multiplication on Jacobians of surfaces. McMullen's condition makes sense in higher genus and degree but the corresponding set is not $SL(2, \mathbb{R})$ invariant in these cases.

Calta gives an answer in terms of Property X. Calta's condition makes sense in higher genus and degree and the corresponding set *is* $SL(2, \mathbb{R})$ invariant in these cases.

We begin by describing Property X.

The Sah-Arnoux-Fathi (SAF) invariant is of an interval exchange transformation takes its values in $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$.

If the interval exchange takes intervals of lengths $\ell_1, \ldots \ell_n$ and translates them by distances $t_1, \ldots t_n$ then the value of the invariant is

$$\sum_{j=1}^{n} \ell_j \wedge t_j.$$

If the interval exchange is periodic then the invariant is zero.

If S has a cylinder decomposition in a particular direction then the SAF invariant in that direction is zero.

We say that a direction is algebraically periodic if the SAF invariant in that direction vanishes.

Unlike the property of having an actual cylinder decomposition the property of being algebraically periodic is *scissors congruence invariant*.

Let us say that two translation structures S and T are *scissors congruent* if there is a polygonal decomposition of S so into polygons $P_1 \dots P_n$ so that we can reassemble them to create T.

We insist that edges match with entire edges when we reassemble.

Consider the result of gluing two isogenous tori together along a slit.

Theorem 1. *If S has three algebraically periodic directions then it has infinitely many.*

If the directions with slope 0, 1 and ∞ are algebraically periodic then there is a field K so that the collection of algebraically periodic directions are exactly those with slopes in K.

K is a number field with $\deg(K) \leq genus(S)$.

If S has three algebraically periodic directions we call S *algebraically periodic* and we call K its *periodic direction field*.

Genus two situation.

There are two genus two strata: $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$.

In the first case all algebraically periodic surfaces are lattice surfaces. The periodic direction field can either be the rationals or a quadratic field.

In the second case the algebraically periodic surfaces give closed, proper, $SL(2,\mathbb{R})$ invariant submanifolds of the strata. All exceptional surfaces are contained in these closed subets.

Here is a family of examples in higher genus.

Theorem 2. *If a translation surface S has an orientation preserving pseudo-Anosov automorphism with expansion constant λ then S is completely algebraically periodic and its periodic direction field is $\mathbb{Q}[\lambda + \lambda^{-1}]$.*

The field $\mathbb{Q}[\lambda + \lambda^{-1}]$ is called the trace field of the pseudo-Anosov. Gutkin-Judge and Kenyon-Smillie show that for an S with a pseudo-Anosov automorphism the trace field is the *holonomy field* of the surface. This means that we can assume that all coordinates of all saddle connections lie in this field.

Corollary 3. *If S is a lattice surface then S is algebraically periodic.*

What we see is a connection between exceptional surfaces and algebraically periodic surfaces in the case of lattice surfaces and in genus two.

The remainder of the talk is about algebraically periodic surfaces.

We hope to be able to announce some results about the connection between the two properties soon.

It is not known which fields can be obtained as trace fields of pseudo-Anosovs but we have:

Theorem 4. *Every number field is the algebraic periodic direction field for some translation surface.*

We can be quite explicit in the construction of these examples. We can also realize them as translation structures coming from billiard tables. (L-shaped tables are a special case.)

Let $K \subset \mathbb{R}$ be a number field. We can write $K = \mathbb{Q}(\lambda)$ where $\lambda > 0$ is an algebraic number with minimal polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Let

$$q(x) = \frac{p(x)}{(x-\lambda)} = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0.$$

Let $\alpha_j = \lambda_{j-1}$ and let $\beta_j = \frac{b_{j-1}}{q'(\lambda)}$.

Construct rectangles R_j with height α_j and widths $|\beta_j|$ for $j = 1 \ldots n$.

Glue together those rectangles R_j for which $\beta_j > 0$. Excise rectangles R_j with $\beta_j < 0$.

The result is a planar billiard table whose translation surface has the required property.

How do we recognize algebraically periodic surfaces and their direction fields?

Consider a translation surface S with a triangulation dividing the surface into triangles Δ_j. Say that v_j and w_j are two of the sides of Δ_j. Say that $v_j = \begin{bmatrix} a_j \\ c_j \end{bmatrix}$ and that $w_j = \begin{bmatrix} b_j \\ d_j \end{bmatrix}$.

Let us assume that all these numbers are contained in a number field L which we can assume to be Galois.

Theorem 5. *S is algebraically periodic if and only if the following equations hold for any $\sigma \in Gal(L)$.*

$$\sum_j a_j\sigma(d_j) - c_j\sigma(b_j) = \sum_j -b_j\sigma(c_j) + d_j\sigma(a_j) \quad (1)$$

$$\sum_j b_j\sigma(d_j) - d_j\sigma(b_j) = \sum_j -c_j\sigma(a_j) - a_j\sigma(c_j) = 0 \quad (2)$$

The direction field K is the subfield of L fixed by the collection of σ's for which $\sum_j a_j\sigma(d_j) - c_j\sigma(b_j) \neq 0$.

The *homological affine group* of T, HAG(T), is a subgroup of $SL(2,\mathbb{R})$ that contains the Veech group but is easier to calculate.

The homological affine group consists of matrices $\alpha \in SL(2,\mathbb{R})$ for which there is an A such that:

$$\begin{array}{ccc} H_1(S,\mathbb{Q}) & \xrightarrow{h} & \mathbb{R}^2 \\ \downarrow A & & \downarrow \alpha \\ H_1(S,\mathbb{Q}) & \xrightarrow{h} & \mathbb{R}^2 \end{array}$$

where h is the holonomy homorphism and A preserves the intersection number of homology classes.

The periodic direction field of S is contained in the holonomy field of S. If these fields are equal we say that S is *completely algebraically periodic* or that S satisfies *Property X*.

Connection with McMullen's approach:

Theorem 6. *If S is completely algebraically periodic then $HAG(S) = SL(2, K)$ where K is the periodic direction field.*

All real fields of degree 2 are totally real. For fields of higher degree this need not be the case.

If S has a parabolic automorphism then S has a special type of cylinder decomposition. In particular S has an algebraically periodic direction.

Theorem 7. *If S has a parabolic automorphism and a second algebraically periodic direction then S is algebraically periodic and the periodic direction field is totally real.*

Related result of Hubert-Lanneau.

Theorem 8. *If S is a connected sum of isogenous tori then S is algebraically periodic and the periodic direction field is totally real. Conversely if K is totally real then there is a connected sum of isogenous tori with direction field K.*

All results come from an analysis of the J invariant:

$$J(S) \in \mathbb{R}^2 \wedge_{\mathbb{Q}} \mathbb{R}^2.$$

In terms of a triangulation we have:

$$J(S) = \sum_j \begin{bmatrix} a_j \\ c_j \end{bmatrix} \wedge \begin{bmatrix} b_j \\ d_j \end{bmatrix}$$

- J is independent of the triangulation.

- J is a scissors congruence invariant.

- J is additive for connected sums.

- J determines the algebraically periodic directions.

- In interesting cases $J(S) \in L^2 \wedge_{\mathbb{Q}} L^2$ for some number field L.

- $SL(2, L)$ acts on L^2 and hence on $L^2 \wedge_{\mathbb{Q}} L^2$.

- We define a $Iso(J) \subset SL(2, L)$ consisting of matrices that preserve J.

- We analyze $L^2 \wedge_{\mathbb{Q}} L^2$ as a $SL(2, L)$ module using Galois automorphisms or field embeddings.

  This gives $\mathbb{Q}$ linear equations that determine $Iso(J)$.

- $HAG(S) \subset Iso(J(S))$

Where does total reality come from?

For distinct field embeddings we have

$$\sum_j \sigma(a_j)\tau(d_j) - \sigma(c_j)\tau(b_j) = 0$$

If K has a complex embedding σ then take $\tau = \bar{\sigma}$.

Geometric hypotheses mean that J has the form

$$J(S) = \sum_j \begin{bmatrix} a_j \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ a_j \end{bmatrix}$$

with $\sum a_j = 1$.

Putting these together we get:

$$\sum_j \sigma(a_j)\bar{\sigma}(a_j) = \sum_j |\sigma(a_j)|^2 = 0$$

so all $\sigma(a_j)$ are zero hence all a_j are zero. But this contradicts $\sum a_j = 1$.