On Symmetric Stable Processes

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Symmetric α-Stable Process

is a Lévy process Z on $\mathbf{R}^n$ with

$$\mathbf{E}\, e^{i\xi\cdot(Z_t - Z_0)} = e^{-t|\xi|^\alpha},$$

where $\alpha \in (0, 2)$. Unlike Brownian motion, $t \mapsto Z_t(\omega)$ is discontinuous and Z_t has heavy tails:

$$\mathbf{P}(|Z_t| > \lambda) \approx \lambda^{-\alpha}.$$

- $\mathbf{E}|Z_t|^p < \infty$ if and only if $p < \alpha$.
- Self-similarity: $\{\lambda^{-1/\alpha}(Z_{\lambda t} - Z_0),\ t \geq 0\}$ has the same distribution as $\{Z_t - Z_0,\ t \geq 0\}$.

Generator

- Generator: $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$.

$$
\begin{aligned}
& (\Delta)^{\alpha/2} f(x) \\
= \quad & \int_{\mathbf{R}^n} e^{-i\xi\cdot x} \, |\xi|^\alpha \, \widehat{f}(\xi) \, d\xi \\
= \quad & \lim_{\delta \downarrow 0} \int_{|y-x|>\delta} (f(y) - f(x)) \frac{c(n,\alpha)}{|x-y|^{n+\alpha}} dy \\
= \quad & c(n,\alpha) \int_{\mathbf{R}^n} \Big(f(y) - f(x) - \nabla f(x) \cdot (y-x) 1_{\{|y-x|\le 1\}} \Big) \\
& \qquad \frac{c(n,\alpha)}{|x-y|^{n+\alpha}} dy.
\end{aligned}
$$

Construction of Stable Processes

Let B be BM($\mathbf{R}^n$) and Y be an independent $(\alpha/2)$-subordinator on $\mathbf{R}^+$:

$$\mathbf{E}\, e^{-\lambda Y_t} = e^{-t\lambda^{\alpha/2}}.$$

Then $Z_t := B_{Y_t}$ is a symmetric α-stable process on $\mathbf{R}^n$.

In contrast to the Brownian motion, the coordinate processes of symmetric α-stable process in $\mathbf{R}^n$ are 1-dimensional symmetric α-stable processes but are not independent each other.

Transience and Recurrence

When $n = 1$, symmetric α-stable process Z is pointwise recurrent when $\alpha > 1$, neighborhood recurrent when $\alpha = 1$, and is transient when $\alpha < 1$.

When $n \geq 2$, Z is always transient.

- Transition density function $p(t, x, y)$ estimate:

$$p(t, x, y) \asymp \min \left\{ t^{-n/\alpha}, \; \frac{t}{|x - y|^{n+\alpha}} \right\}$$

Here $f \asymp g$ means f/g is bounded between two positive constants.

Dirichlet from

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of Z is given by $\mathcal{F} = \mathrm{Dom}(\sqrt{-\mathcal{L}})$ and

$$\begin{aligned}\mathcal{E}(u,v) &= (\sqrt{-\mathcal{L}}u, \sqrt{-\mathcal{L}v})_{L^2(\mathbf{R}^n,dx)} \\ &= \frac{c(n,\alpha)}{2}\int_{\mathbf{R}^n\times\mathbf{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}dxdy.\end{aligned}$$

In fact,

$$\mathcal{F} = \{u \in L^2(\mathbf{R}^n, dx) :\ \mathcal{E}(u,u) < \infty\} = W^{\alpha/2,2}(\mathbf{R}^n).$$

Stable process in an open subset

Let D be an open subset of $\mathbf{R}^n$ and define

$$\tau_D = \inf\{t > 0 : \; Z_t \notin D\}.$$

Typically, $Z_{\tau_D} \in \mathbf{R}^n \setminus \overline{D}$.

Green function: $\int_D G_D(x,y) f(y) dy = \mathbf{E}_x \int_0^{\tau_D} f(Z_s) ds$.

Green Function Estimates: C-Song, Kulczcki

$$G_D(x,y) \asymp \min \left\{ \frac{1}{|x-y|^{n-\alpha}}, \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{|x-y|^n} \right\},$$

where $\delta_D(x) = \mathrm{dist}(x, D^c)$.

Censored stable process

A censored α-stable process Y in a domain D can be obtained from a symmetric α-stable process Z by killing it at the time when it jumps outside D, and then starting an independent copy of Z from the point where the jump originated. The procedure is repeated for the new process, and then by induction infinitely (countably) many times.

Equivalent characterization: Y has Dirichlet form

$$\begin{aligned} \mathcal{E}(u,v) &= c\int_{D\times D} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dxdy, \\ \mathcal{F} &= \overline{C_c^\infty(D)}^{\sqrt{\mathcal{E}_1}} = W_0^{\alpha/2,2}(D). \end{aligned}$$

Boundary behavior

- Does the censored process approach the boundary of D in a finite time?

Example: $D = (0,1) \setminus K$, where K is the Cantor set in $(0,1)$.

Definition. A set $D \subset \mathbf{R}^n$ is called an n-set if there is $c > 0$ such that

$$|B(x,r)| \geq cr^n \qquad \text{for every } x \in D \text{ and } r < 1.$$

Let Cap_α denote the capacity induced by the symmetric α-stable process in $\mathbf{R}^n$, or equivalently, by Riesz potential kernel $|x-y|^{\alpha-n}$.

Boundary behavior (continued)

Theorem. (Bogdan-Burdzy-C.)
Let D be a bounded open n-set in $\mathbf{R}^n$. Then the censored α-stable process in D is recurrent if and only if $\mathrm{Cap}_\alpha(\partial D) = 0$. When $\mathrm{Cap}_\alpha(\partial D) > 0$,

$$\mathbf{P}_x(\zeta < \infty \text{ and } Y_{\zeta-} \in \partial D) = 1 \qquad \text{for every } x \in D.$$

Corollary. If the boundary of a bounded n-set D has (locally) finite and positive d-dimensional Hausdorff measure. Then Y is recurrent if and only if $\alpha \leq n - d$. When $\alpha > n - d$,

$$\mathbf{P}_x(\zeta < \infty \text{ and } Y_{\zeta-} \in \partial D) = 1 \qquad \text{for every } x \in D.$$

Reflected Stable process

Consider the reflected Dirichlet space $(\mathcal{E}, \mathcal{F}^{\text{ref}})$ of Y, where

$$\mathcal{F}^{\text{ref}} = \{u \in L^2(D, dx) : \mathcal{E}(u,u) < \infty\} = W^{\alpha/2,2}(D).$$

When D is an n-open set, $(\mathcal{E}, \mathcal{F}^{\text{ref}})$ is a regular Dirichlet space on $\overline{D}$ and so there is an associated Hunt process Y^* starting from quasi-every point in $\overline{D}$.

Fact: Y is a subprocess of Y^* killed upon hitting ∂D.

So the above boundary behavior question is equivalent to whether $Y = Y^*$ or not.

Implication: $Y = Y^*$ if and only if $W_0^{\alpha/2,2}(D) = W^{\alpha/2,2}(D)$, yielding new results on Besov spaces.

Boundary Harnack principle

Theorem. (Bogdan-Burdzy-C.)
BHP holds for censored α-stable processes in $C^{1,1}$-smooth open sets with $\alpha > 1$. The BHP asserts that harmonic functions that vanish on the same part of the boundary decay at the same rate, which is $\delta_D(x)^{\alpha-1}$.

Green function estimate. (C-Kim)
Let G_D denote the Green function of censored α-stable process in a bounded $C^{1,1}$-smooth open set D with $\alpha > 1$. Then

$$G_D(x,y) \asymp \min\left\{ \frac{1}{|x-y|^{n-\alpha}}, \frac{\delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1}}{|x-y|^{n+\alpha-2}} \right\}.$$

d-set

A subset $F \subset \mathbf{R}^n$ is called a d-set, where $0 < d \leq n$, if there is a positive Borel measure μ on F and $c_1 \geq 1$

$$c_1^{-1} r^d \leq \mu(B(x,r)) \leq c_1 r^d \qquad \text{for all } x \in F \text{ and } 0 < r \leq 1.$$

Such a measure μ is called a d-measure on F.

The notion of d-set arises in the theory of function spaces and in fractal geometry. Geometrically self-similar sets are typical examples of d-sets. For example, any Lipschitz domain in $\mathbf{R}^n$ with a uniform Lipschitz constant is an n-set, Sierpinski gasket and Sierpinski carpet in $\mathbf{R}^2$ are d-sets with $d = \log 3/\log 2$ and $d = \log 8/\log 3$, respectively.

Dirichlet form on d-set

Let $F \subset \mathbf{R}^n$ be a closed d-set with the property that

$$\mu(B(x,r)) \le c_1 r^d \quad \text{for every } x \in F \text{ and } r > 0.$$

Let

$$J(x,y) = \frac{c(x,y)}{|x-y|^{d+\alpha}} \qquad \text{for } x, y \in F,$$

where $c(x,y)$ is a symmetric function on $F \times F$ such that

$$c_2^{-1} \le c(x,y) \le c_2 \qquad \text{for } \mu\text{-a.e. } x, y \in F.$$

Define

$$\begin{aligned} \mathcal{E}(u,v) &= \int_{F\times F} (u(x)-u(y))^2 J(x,y)\mu(dx)\mu(dy), \\ \mathcal{F} &= \left\{ u \in L^2(F,\mu) : \mathcal{E}(u,u) < \infty \right\}. \end{aligned}$$

Neumann heat kernel estimates

Theorem. (C.-Kumagai)
For $0 < \alpha < 2$, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on F. There is a Feller process Y^* on F associated with it $(\mathcal{E}, \mathcal{F})$ on $L^2(F, \mu)$ and Y^* has a Hölder continuous transition density function $p(t, x, y)$. Consequently, Y^* can be refined to start from every point in F. Moreover,

$$p(t,x,y) \asymp \min\left\{ t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}} \right\}$$

for $t \leq 1$ and $x, y \in F$, with the multiplicative constants depending only on n, d, α, and the constants c_1 and c_2.

Uniform Hausdorff dimensional result

Assume that $\alpha \leq d$.

Theorem. (Benjamini-C.-Rohde)
(i) For every $x \in F$,

$$\mathbf{P}_x \left(\dim_H Y^*(E) = \alpha \dim_H E \ \text{ for all Borel sets } E \subset \mathbf{R}_+\right) = 1.$$

(ii) Assume F is the closure of a bounded open n-set in $\mathbf{R}^n$. Let $S(\omega) = \{t \geq 0 : \ Y^*(\omega) \in \partial D\}$ be boundary occupation time set. Then

$$\dim_H S(\omega) = \max \left\{ 1 - \frac{n - \dim_H \partial D}{\alpha}, 0 \right\} \qquad \mathbf{P}_x\text{-a.s.}$$

for every $x \in \overline{D}$.

(continued)

(iii) Under the condition of (ii) with $\alpha \leq n$,

$$\dim_H Y^*(S(\omega)) = \max\{\alpha + \dim_H \partial D - n,\ 0\} \qquad \mathbf{P}_x\text{-a.s.}$$

for every $x \in \overline{D}$.

Similar results hold for normally reflected Brownian motions.

Jump processes of mixed type

A typical example is the symmetric jump process with jumping intensity

$$J(x,y) = \int_{\alpha_1}^{\alpha_2} \frac{c(\alpha,x,y)}{|x-y|^{d+\alpha}}\, \nu(d\alpha),$$

where ν is a probability measure on $[\alpha_1, \alpha_2] \subset (0,2)$, and $c(\alpha,x,y)$ is a jointly measurable function that is symmetric in (x,y) and is bounded between two positive constants. Rewrite the above jumping intensity J as

$$J(x,y) = \frac{c(x,y)}{|x-y|^d \phi(|x-y|)} \qquad \text{for every } (x,y) \in F \times F$$

with $\phi(r) = 1/\int_{\alpha_1}^{\alpha_2} r^{-\alpha} \nu(d\alpha)$.

Heat kernel estimate

Theorem. (C.-Kumagai)
Assume that F is a closed global d-set F. Then there is a Feller process X associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ with $J(x,y)$ and X has a continuous transition density function $p(t,x,y)$ with respect to the measure μ satisfying the following two-sided estimate

$$p(t,x,y) \asymp \frac{1}{(\phi^{-1}(t))^d} \wedge \frac{t}{|x-y|^d \phi(|x-y))}$$

for all $t > 0$ and all $x, y \in F$, where ϕ^{-1} is the inverse function of ϕ.

Remarks:

- Such a result is new even when F is $\mathbf{R}^n$.
- Stability. Parabolic Harnack inequality.

THE END