LOCAL SETS

of the

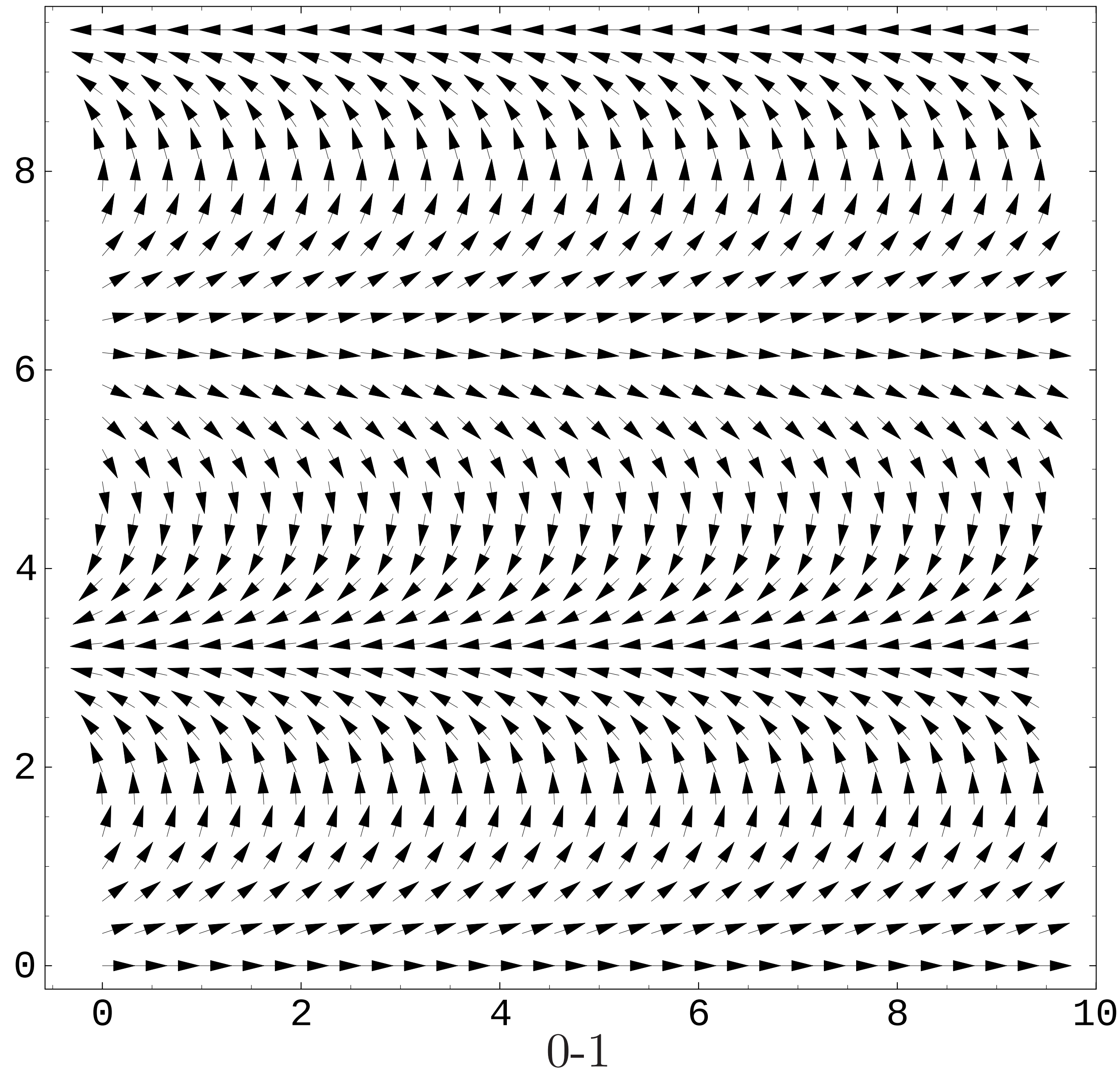
GAUSSIAN FREE FIELD

PART THREE

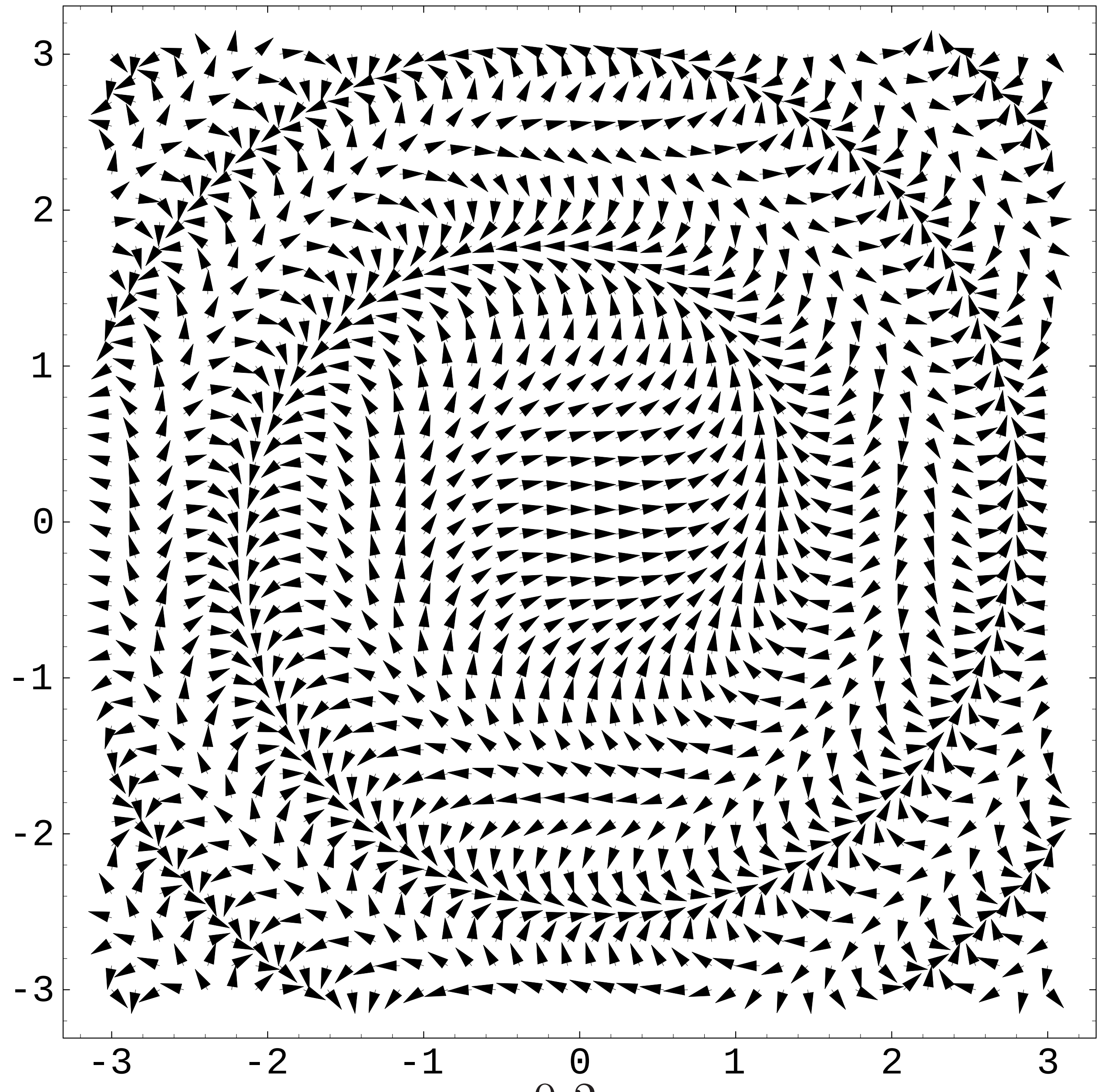
Scott Sheffield

based on work with Schramm; Schramm and Wilson; and Werner

Vector Field e^{ih} where $h(x, y) = \pi/2 - y$



Vector Field e^{ih} where $h(x,y) = x^2 + y^2$



0-2

Altimeter compass geometry

A ray in the altimeter compass geometry is a flow line of $e^{2\pi i(\alpha+h/\chi)}$ for some α.
Now let's modify our sense of direction. Call the direction $e^{2\pi i(\alpha+h/\chi)}$

1. East if $\alpha = 0$.
2. North if $\alpha = .25$.
3. West if $\alpha = .5$.
4. South if $\alpha = .75$.

If $h = 0$, then the rays of the AC geometry are those of ordinary Euclidean geometry. More generally, if h is Lipschitz, then the flow line of $e^{2\pi i(\alpha+h/\chi)}$ starting at a given point exists and is uniquely defined.

Conformal maps of AC geometries

Let h be defined on a domain D. Let g be a conformal map from D to D'. Fix $\chi > 0$. Then the AC geometry of (D, h) is the same as that of

$$(D', h + (\chi/2\pi) \arg g').$$

This implies in particular that if h is harmonic, then the rays are locally the images (under a conformal map) of the rays in a Euclidean geometry. To see this, let $\tilde{h}$ be an analytic function whose imaginary part is h, and let g be a map whose derivative is $e^{\tilde{h}}$.

AC geometries as affine connections

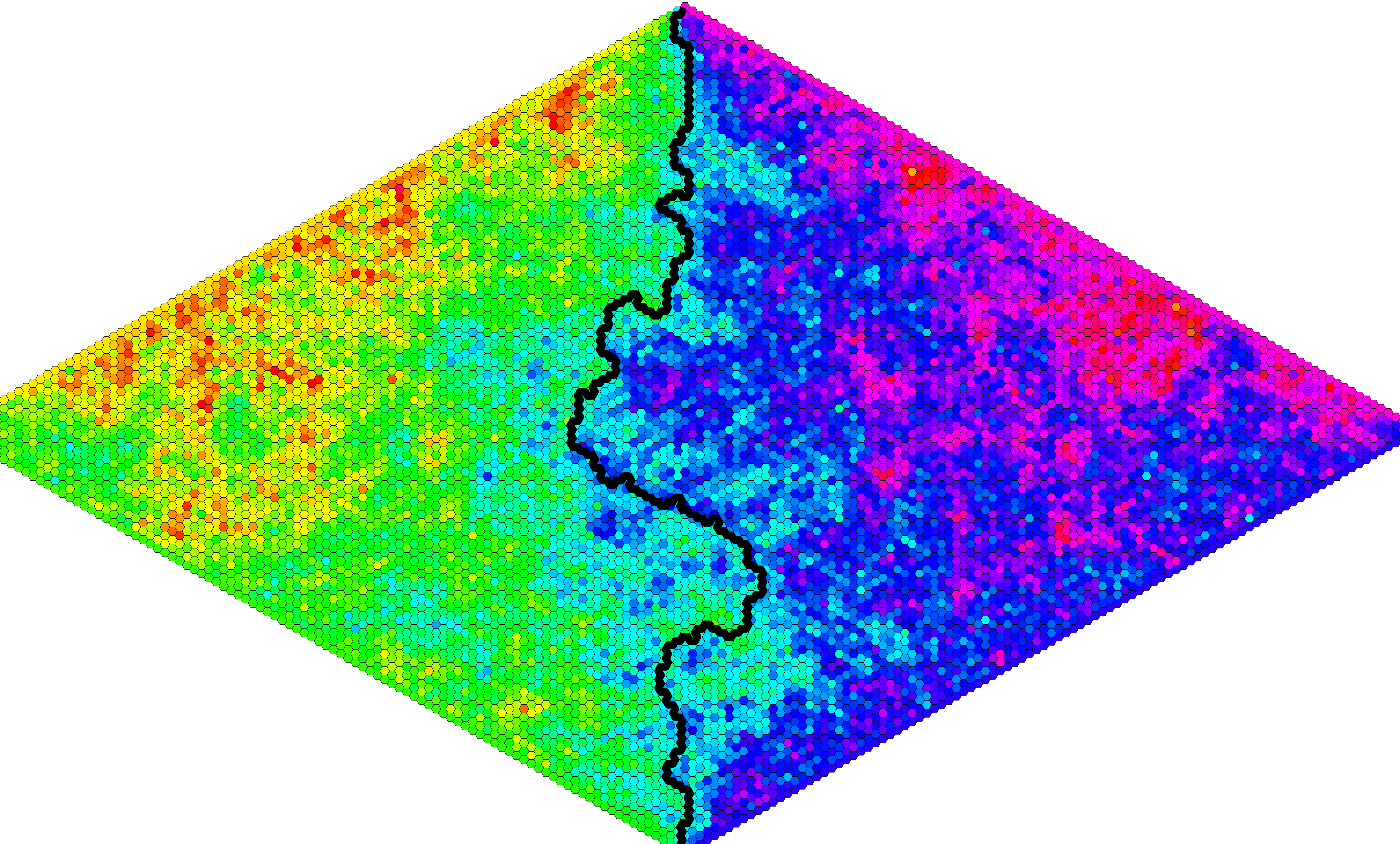
For a general h, we may view these paths as autoparallels of an affine connection whose holonomy group consists entirely of dilations. We can interpret an AC geometry as a non-metric geometry whose curvature is purely imaginary, namely an imaginary multiple of the charge density $-\Delta h$.

AC geometry of the GFF

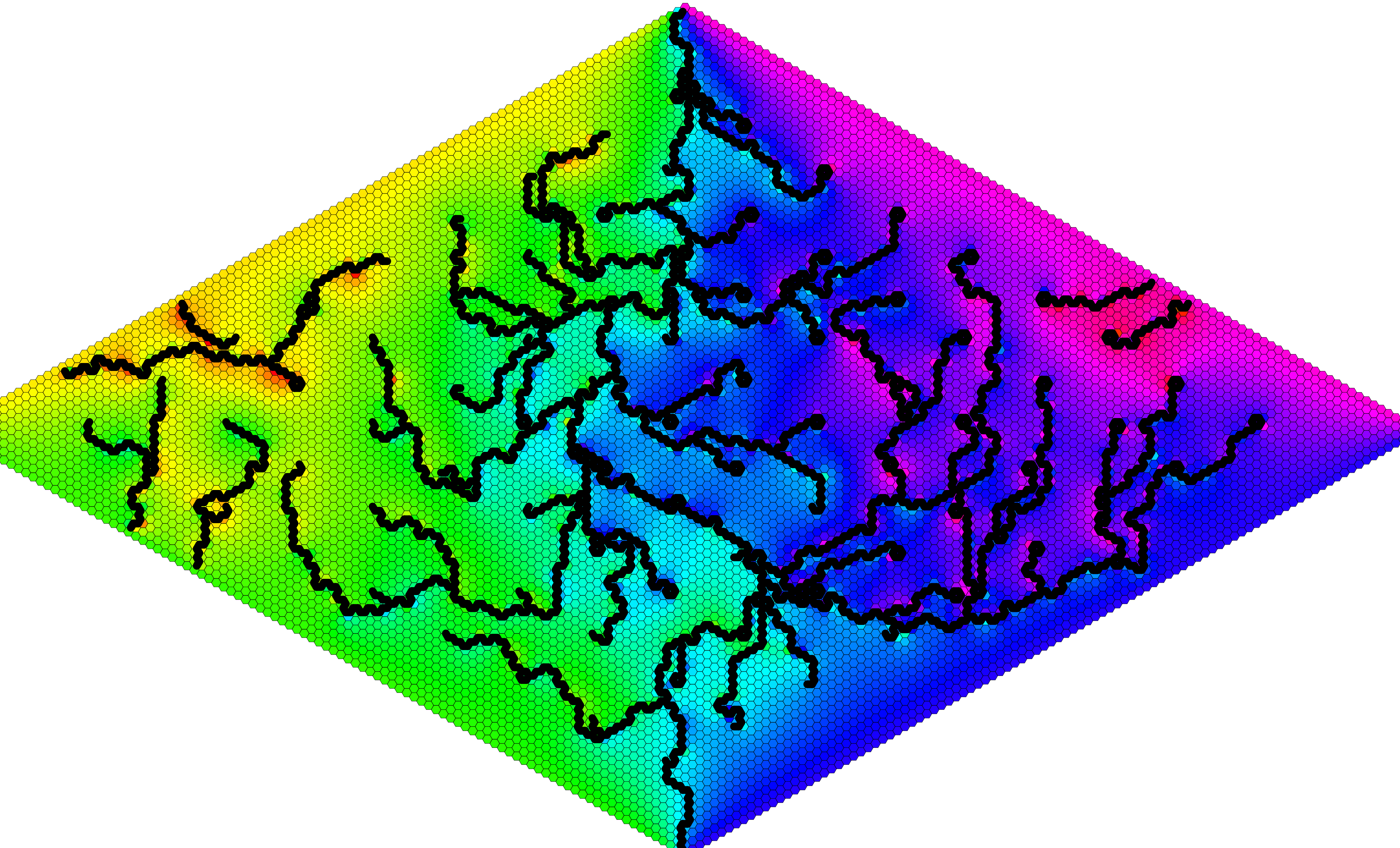
Question: Is there a natural way to define "flow lines" of $e^{ih/\chi}$ when χ is a constant and h is the continuous Gaussian free field (and $-\Delta h$ is a Coulomb gas)?

Answer: Yes, using the coupling between SLE_κ and the GFF given earlier. The flow lines are forms of SLE_κ where $0 < \kappa < 4$ and $\chi = \frac{4-\kappa}{2\pi}\lambda$. There is a constant "height gap" between one side of the flow line and the other. We may view this gap as an "angle gap." In radians, the gap is $\frac{\kappa\pi}{4-\kappa}$, i.e., $\frac{\kappa}{2(4-\kappa)}$ revolutions.

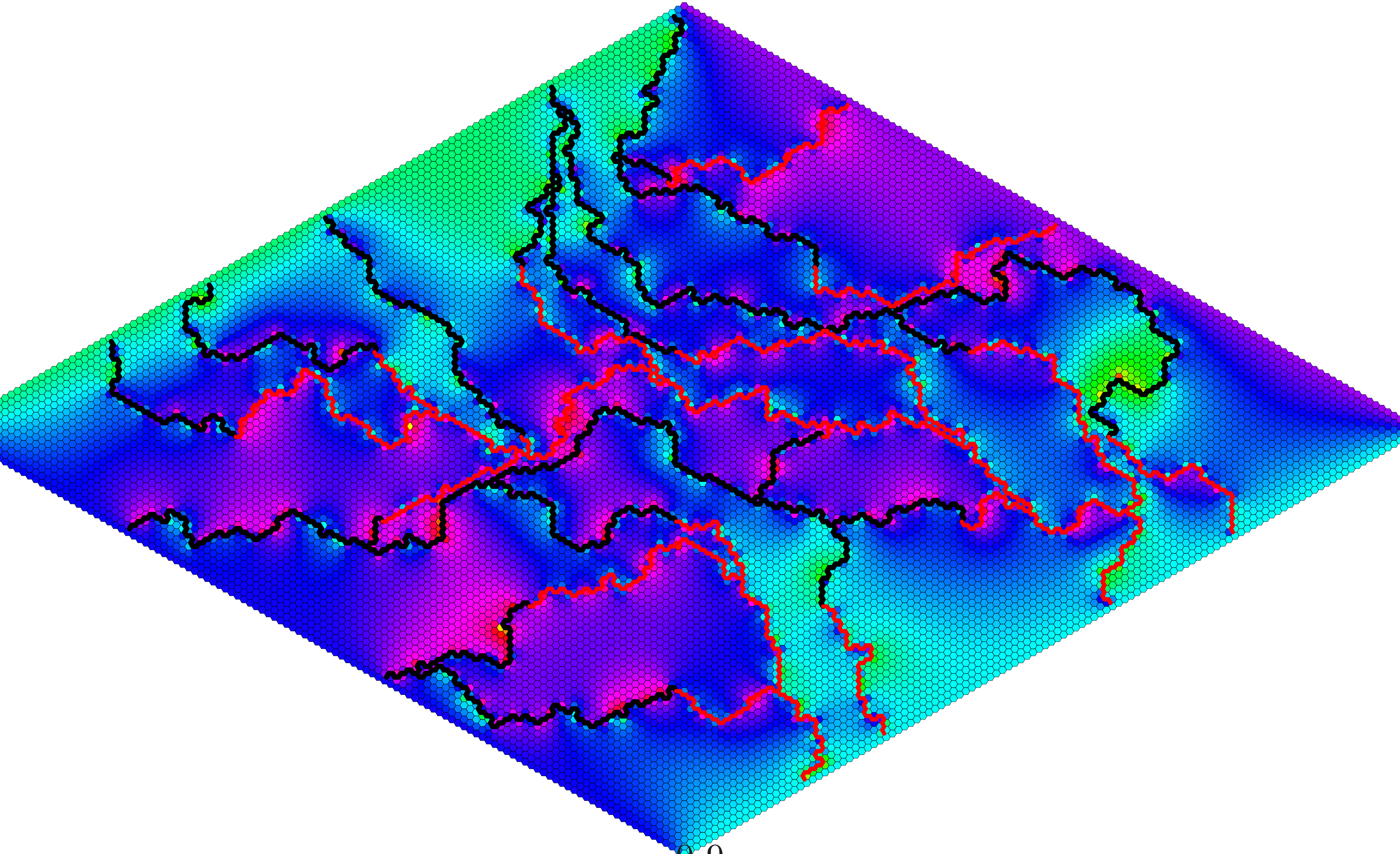
Discretized north-going line, $\kappa = .7$



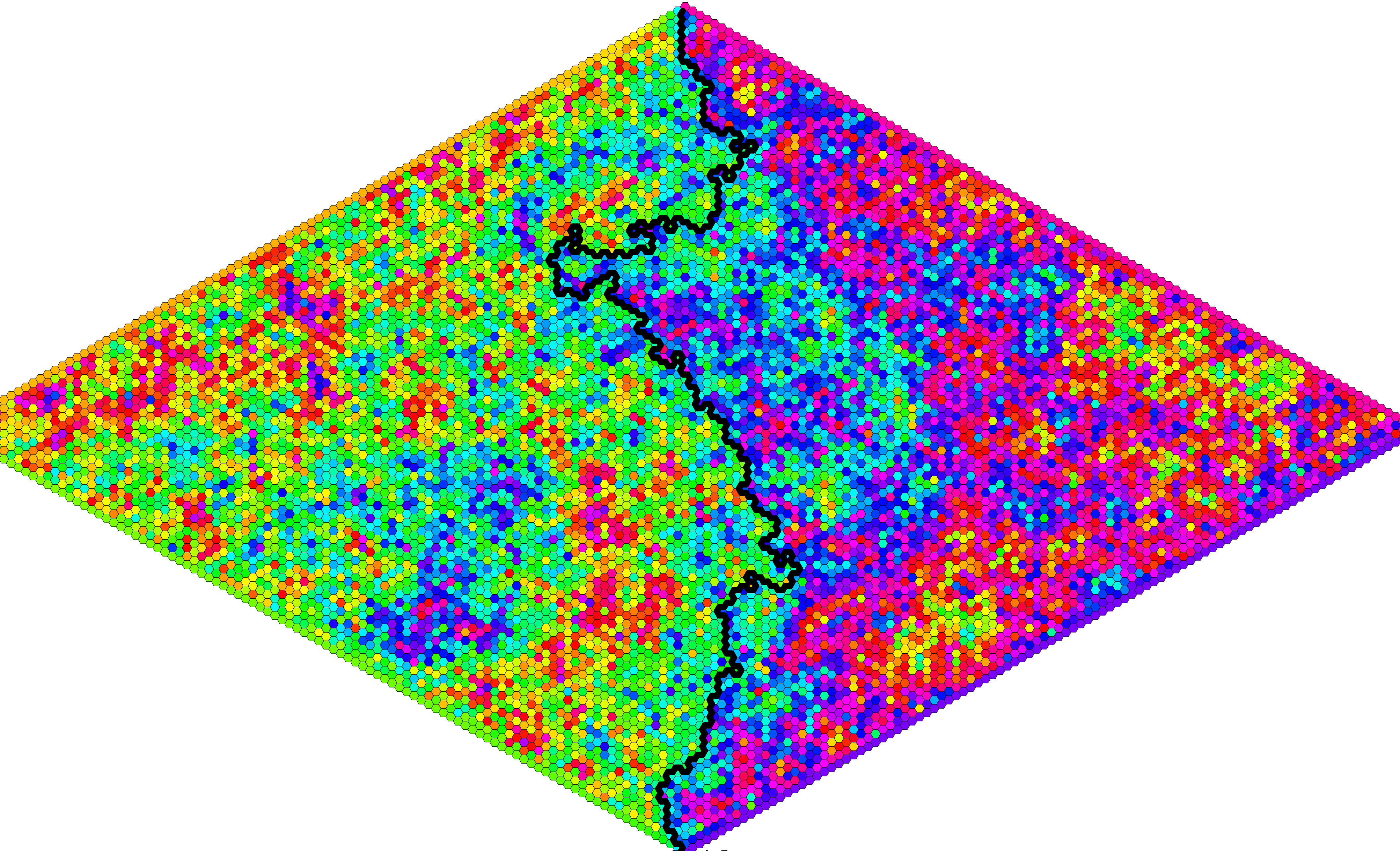
Discretized north-going tree, $\kappa = .7$



East/west-going trees, $\kappa = .7$



North-going line, $\kappa = 2$



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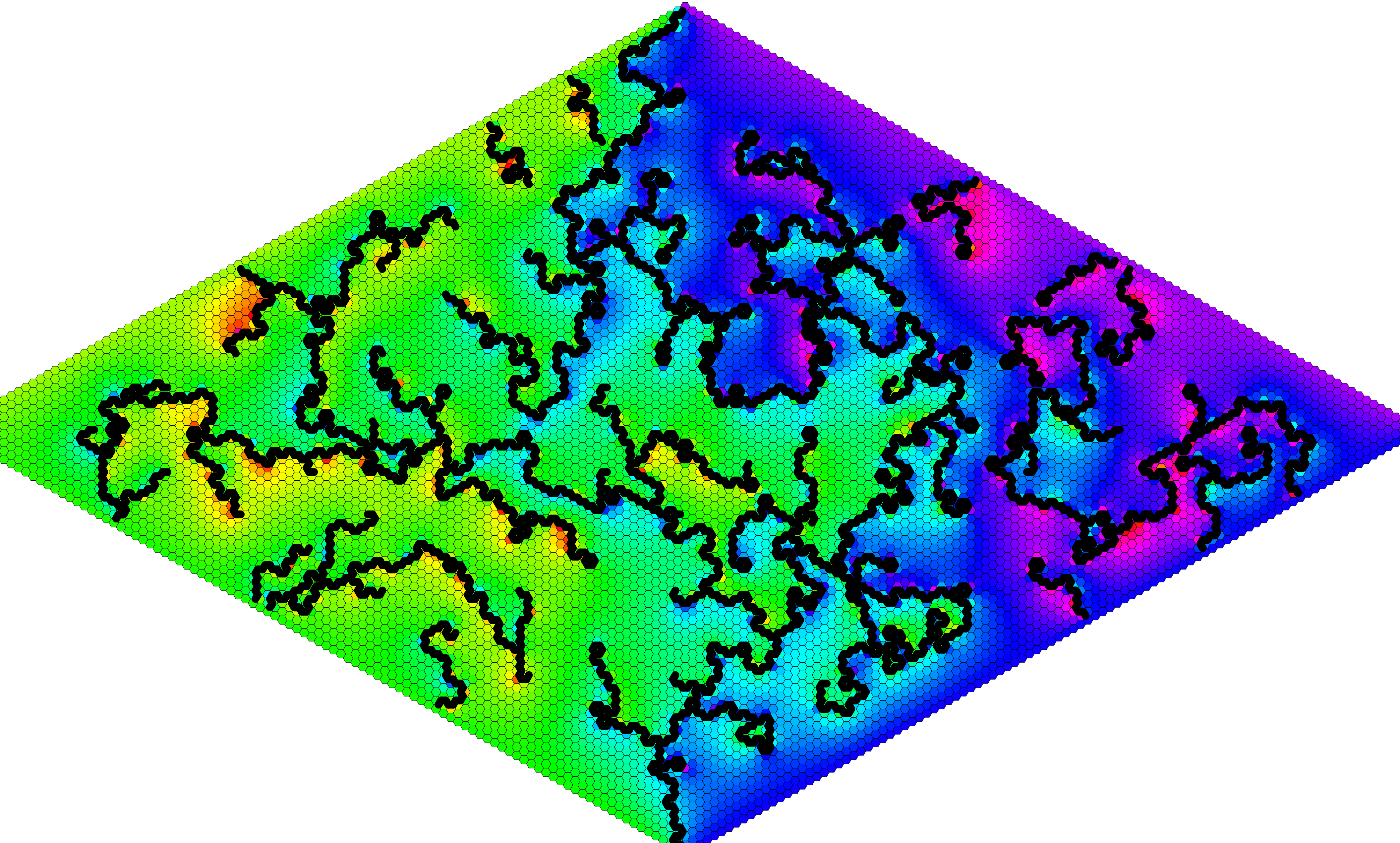
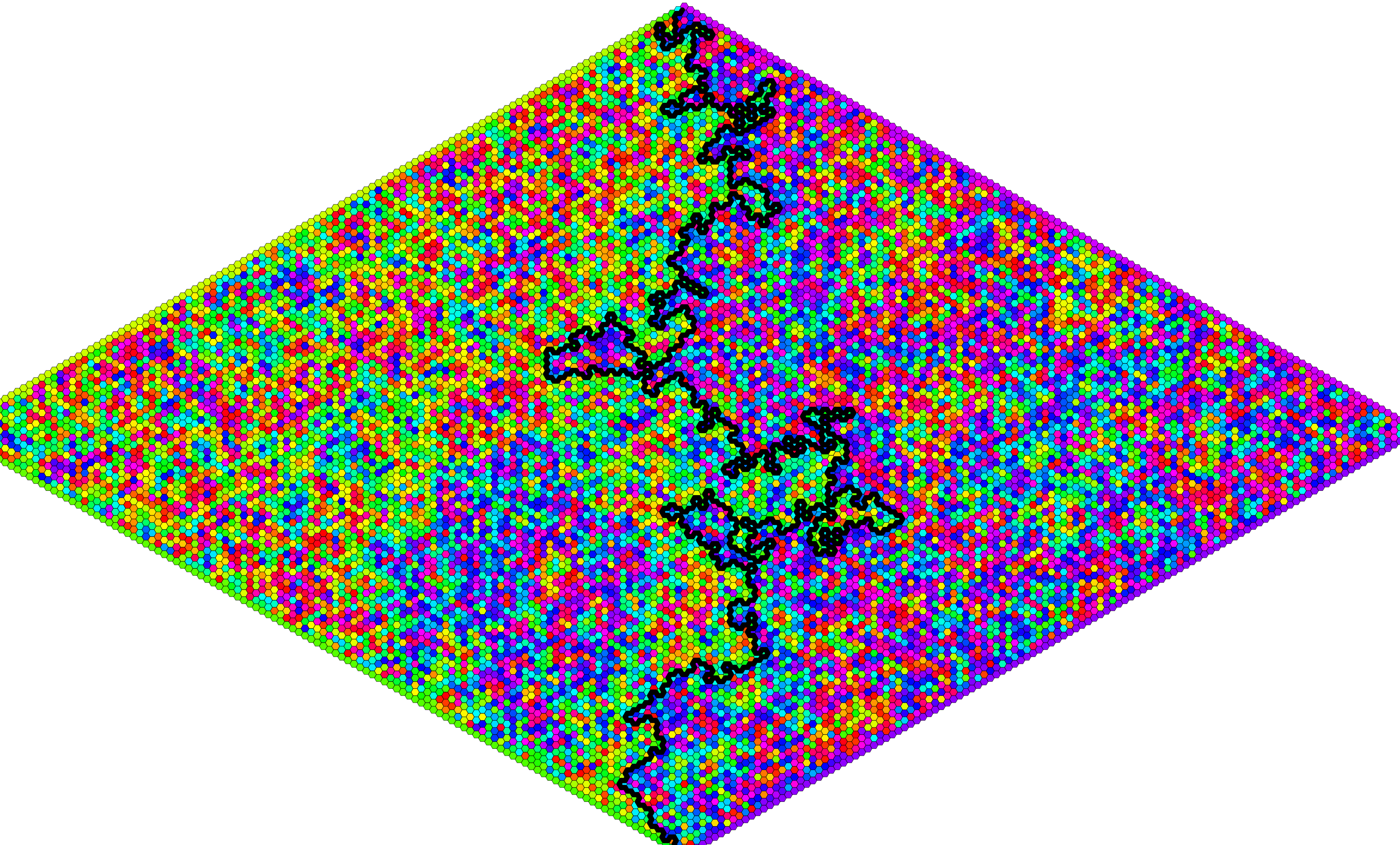
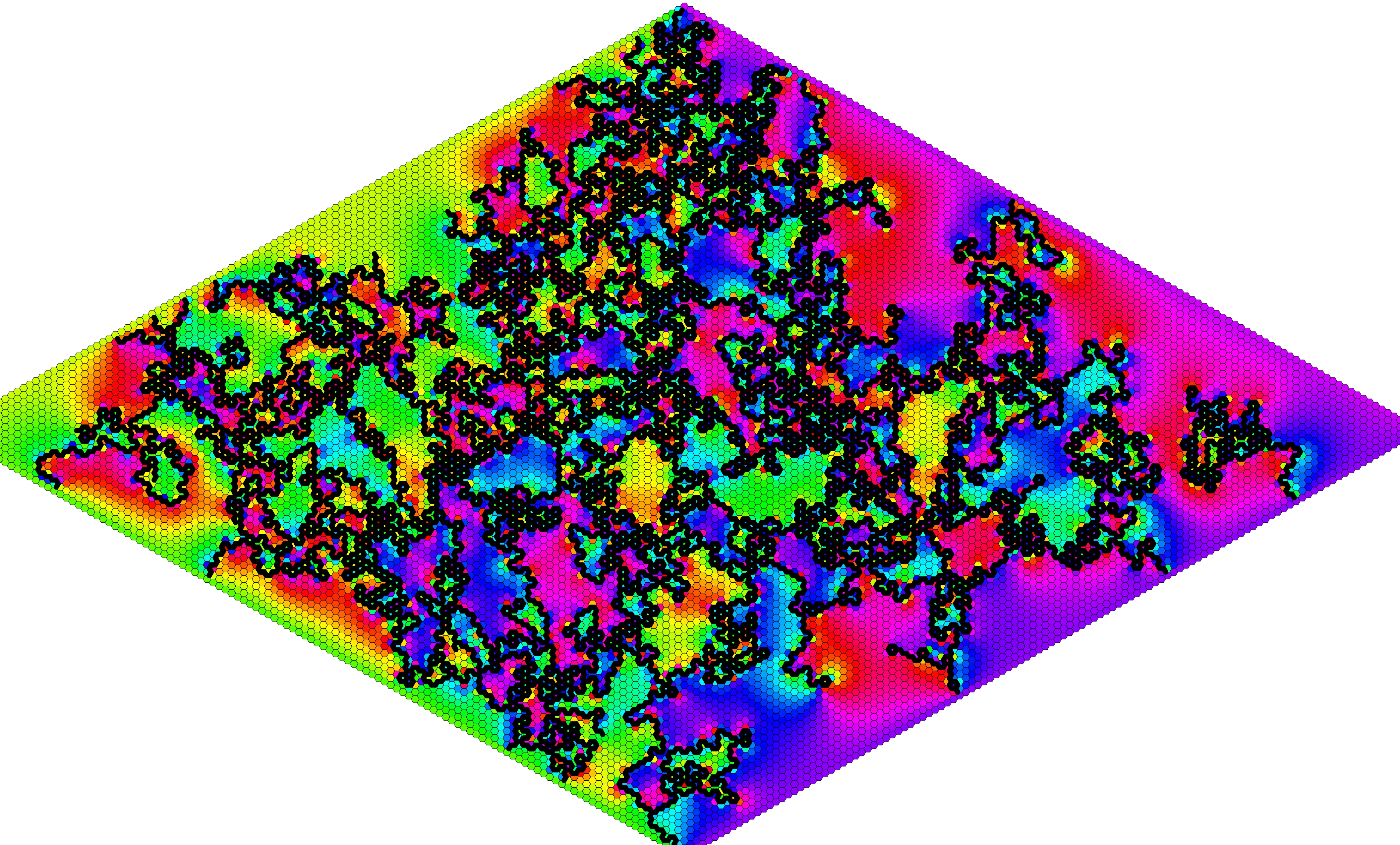


North-going line, $\kappa = 3.5$



North-going tree, $\kappa = 3.5$



Recall: Important martingale of SLE

Observe:

$$d[\log f_t(z) + \frac{4-\kappa}{4} \log g_t'(z)] = -\sqrt{\kappa} f_t(z)^{-1} dB_t.$$

Thus, for any fixed value of z, the following linear combination of the angle and the winding number is a martingale:

$$h_t(z) = -\frac{2\lambda}{\pi} \arg(f_t(z)) - \chi \arg f_t'(z) + \lambda$$

where $\lambda := \lambda(\kappa) := \sqrt{\frac{\pi}{2\kappa}}$ and $\chi := \chi(\kappa) := (4-\kappa)\lambda$.

We chose λ and χ in such a way that makes $dh_t(z)$ (which is a multiple of $\mathrm{Im}(f_t(z)^{-1})dB_t$) independent of κ.

Contour lines: local and deterministic

THEOREM: In the couplings (h, γ) of the free field h and an SLE_κ, as described above, the random set $\gamma([0,\infty])$ is a local set. In fact, for any stopping time T, the set $\gamma([0,T])$ is local. Moreover, these local sets are deterministic functions of h.

AC geometry of the GFF

The flow lines are forms of SLE_κ where $0 < \kappa < 4$ and $\chi = \frac{4-\kappa}{2\pi}\lambda$. There is a constant "height gap" between one side of the flow line and the other. We may view this gap as an "angle gap." In radians, the gap is $\frac{\kappa\pi}{4-\kappa}$, i.e., $\frac{\kappa}{2(4-\kappa)}$ revolutions.

Facts about AC lines of GFF

The following are derived from known facts about $\text{SLE}_{\kappa,\rho}$ processes.

1. Rays of differing α values may cross if and only if corresponding Euclidean lines would cross, in which case they may only cross once.
2. Two vertical paths angled away from each other may bounce off each other (but not cross) if and only if the angle difference between them is less than the height gap.
3. A ray may complete a full revolution and hit itself if and only if $2\lambda' > \chi$. It may complete $k/2$ revolutions and hit itself if $\kappa > 4k/(k+1)$. When $k = (1, 2, 3, 4 \ldots)$ the critical κ is $(2, 8/3, 3, 16/5, \ldots)$.

Duality

For $\kappa < 4$, the "outer boundary" of the east-going tree (with east-west paths as boundary conditions) is space-filling $\mathrm{SLE}_{\frac{16}{\kappa}}$. The main difficulty in proving this is to show that this tree and its boundary are well-defined.

Some special κ values explained

The value $\kappa = 2$ (height gap equals half turn) is the only one for which the north-going tree (when the boundary values are those we would have if there were a single north-going path wrapped around the domain) has the following Markov property: given the north-going ray γ_z from an interior point z to the boundary, the conditional north tree has the same law as the original tree, conformally mapped to the new domain $D \backslash \gamma_z$.

The **UST** has a discrete version of this property. Similarly, consider

$\kappa = 8/3$ and $\kappa' = 6$ (height gap equals full turn): when a west and east going path start from a point and wrap around and hit one another, there is no height gap between them at the hitting place; this implies a certain "endpoint invariance" property for SLE_6.

Percolation has a discrete version of this property.