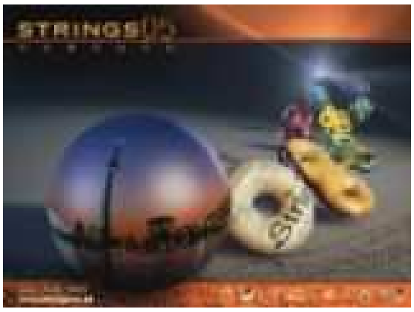
STRINGS05 Toronto July 11-16, 2005



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New results in AdS/CFT

or

Sasaki-Einstein metrics, toric quivers, and Z-minimisation

Work in collaboration with: J. Sparks;
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A. Hanany, D. Vegh, B. Wecht; S. -T. Yau

Plan of the talk

- Why are Sasaki–Einstein metrics relevant?

- $Y^{p,q}$: metrics, toric singularities, and quivers

- Testing AdS/CFT with a-maximisation

- The geometric dual: Z-minimisation

- AdS/CFT duals from toric geometry

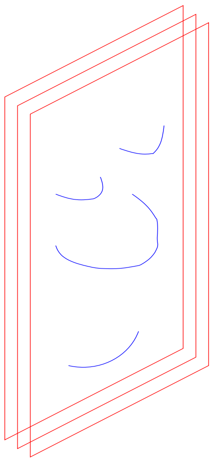
- Conclusions

AdS/CFT correspondence

D-branes in String Theory allows us to construct and study supersymmetric gauge theories.

N parallel D3-branes in $\mathbb{R}^{1,9}$ space-time:

N D3–branes



World-volume gauge theory: $\mathcal{N}=4$ $SU(N)$ super–Yang–Mills in 4d.

This theory is conformal: $\beta \equiv 0$

Supergravity description: a large number N of D3-branes back-react on the geometry, curving space-time.

In the "near-horizon" the geometry becomes $\text{AdS}_5 \times S^5$

This is a solution of Type IIB supergravity, preserving 32 supersymmetries:

$$ds^2 = ds^2(AdS_5) + ds^2(S^5) \qquad F^5_{RR} = N(\text{vol}(AdS_5) + \text{vol}(S^5))$$

AdS/CFT [Maldacena]

String theory on $AdS_5 \times S^5$ is dual to $\mathcal{N}=4$ SYM theory.

Branes at Calabi–Yau singularities

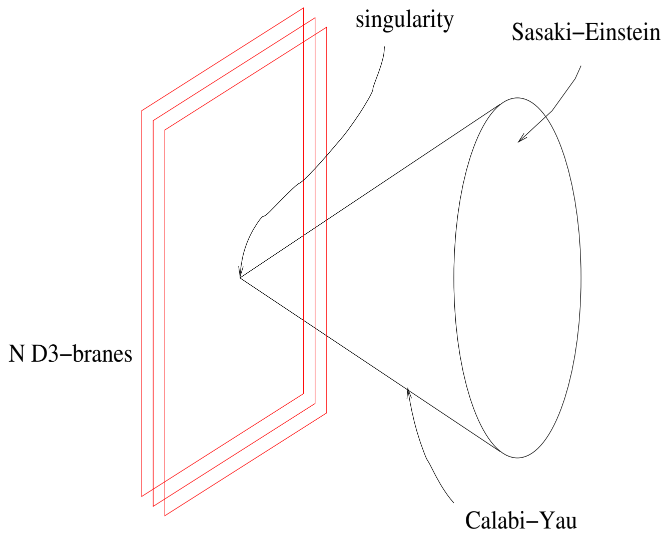
We are interested in replacing $\mathcal{N}=4$ SYM with different 4d supersymmetric conformal field theories (SCFT).

$\mathcal{N}=1$ supersymmetric gauge theories can be obtained from D3-branes, replacing $\mathbb{R}^{1,9}$ with $\mathbb{R}^{1,3} \times$ Calabi–Yau.

To obtain an AdS_5 factor in the metric, we consider cones:

$$ds^2(\mathbb{CY}) = dr^2 + r^2 ds^2(Y_5)$$

then, we place N D3 branes transverse to the CY cone



The geometry, after back-reaction of the branes, is now $AdS_5 \times Y_5$. (This is a smooth geometry).

AdS/CFT: $AdS_5 \times Y_5$ is dual to an $\mathcal{N}=1$ SCFT

The compact five-manifolds Y_5 are Sasaki–Einstein – here we take this as a definition.

Sasaki–Einstein manifolds admit Killing spinors ϵ:

$$\nabla_m \epsilon = \frac{i}{2} \gamma_m \epsilon$$

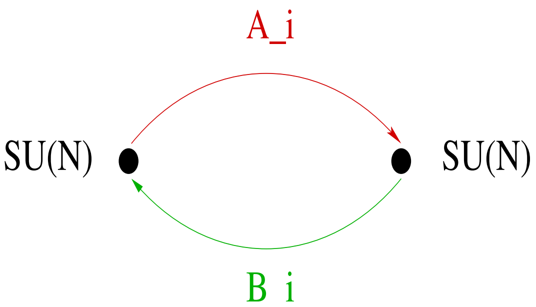
$\rightarrow$ supersymmetry is preserved in Type IIB supergravity.

AdS/CFT with the conifold

The first non-orbifold example of AdS/CFT dual pair was discovered by [Klebanov,Witten]

The Calabi–Yau is the conifold, whose Sasaki–Einstein base is the $T^{1,1}$ metric [Romans]

The dual to $AdS_5 \times T^{1,1}$ is a $\mathcal{N}=1$ $SU(N)\times SU(N)$ quiver gauge theory, with bi-fundamental fields: A_i, B_i $i=1,2$



quiver representation of the gauge theory

$$U(1) \text{ isometry of } T^{1,1} \quad \rightarrow \quad U(1)_R \ R\text{–symmetry}$$

$$SU(2)\times SU(2) \text{ isometry of } T^{1,1} \quad \rightarrow \quad \text{flavour symmetry}$$

Central charge in IR $a = \frac{\pi^3 N^2}{4\,\mathrm{vol}(T^{1,1})}$ [Henningson,Skenderis]

- AdS/CFT allows us to study supersymmetric gauge theories in terms of Sasaki–Einstein geometry.

- It would be nice to have more Sasaki–Einstein metrics.

The $Y^{p,q}$ metrics [Gauntlett,DM,Sparks,Waldram]

In Feb '04 we found an infinite family of 5d Sasaki–Einstein metrics, labeled by 2 integers p and q:

$$\begin{aligned} ds^2(Y^{p,q}) &= \frac{1-y}{6}(d\theta^2+\sin^2\theta d\phi^2)+\frac{1}{w(y)v(y)}dy^2 \\ &+ \frac{v(y)}{9}[d\psi-\cos\theta d\phi]^2+w(y)\,[d\alpha+A]^2 \end{aligned}$$

The functions $w(y,b)$, $v(y,b)$ depend on p and q through the parameter

$$b=\tfrac{1}{2}-\tfrac{p^2-3q^2}{4p^3}\sqrt{4p^2-3q^2}$$

The metrics have $SU(2)\times U(1)\times U(1)$ isometry.

And the topology of $S^2\times S^3$.

The volume is $\mathrm{vol}(Y^{p,q})=\frac{q^2[2p+(4p^2-3q^2)^{1/2}]}{3p^2[3q^2-2p^2+p(4p^2-3q^2)^{1/2}]}\pi^3$

These were the first examples of infinite families of explicit Sasaki–Einstein metrics.

• Many new tests of the AdS/CFT correspondence can be done... provided one finds the dual gauge theories.

Toric Calabi–Yau cones $C(Y^{p,q})$

Q: How can the dual gauge theory be determined?

A: The best way is by studying the geometry of the cone.

By definition of Sasaki–Einstein, the metric cones

$$ds^2(C(Y^{p,q})) = dr^2 + r^2 ds^2(Y^{p,q})$$

are (non compact) Calabi–Yau spaces.

Calabi-Yau's can be characterized in terms of ($SU(3)$-invariant) forms. These are the Kähler 2-form J and the holomorphic $(3,0)$ form Ω

$$\text{Calabi–Yau} \quad \equiv \quad \{dJ = d\Omega = 0\}$$

Two important properties:

- There is a $T^3 \simeq U(1)^3$ symmetry preserving J, Ω, g_{mn}

$$T^3 \subset U(1)^2 \times SU(2) \quad \text{isometry}$$

- There is a Killing vector K (Reeb) obtained as

$$K = J \cdot (r\frac{\partial}{\partial r}) \quad \text{or equivalently} \quad K_m = \bar{\epsilon}\gamma_m\epsilon$$

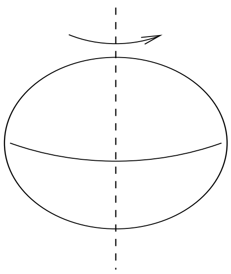
$\Rightarrow$ The Calabi–Yau's $C(Y^{p,q})$ are toric, with $K \in T^3$.

Toric geometry and $Y^{p,q}$ [DM,Sparks]

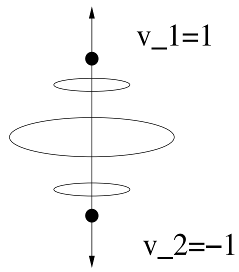
Toric geometry provides the link between the metrics and the gauge theories.

Toy example: symplectic toric 2-sphere S^2 $J = \sin\theta d\theta \wedge d\phi$

Here there is a $T^1 = U(1)$ symmetry, generated by $\partial/\partial\phi$:



can be represented as



$$S^2 = \{U(1) \text{ fibered over the interval } I\}.$$

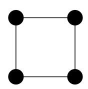
The map $\mu : S^2 \to I$ is called the moment map.

Toric geometry allows to describe complicated spaces in terms of their images under the moment map. These are simple convex polytopes, extending the case of S^2 above.

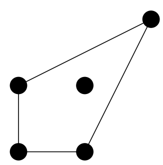
For toric Calabi–Yau, this information is encoded in diagrams living on $\mathbb{Z}^2$, called "toric diagrams".

The toric diagrams for the $Y^{p,q}$ Calabi–Yau singularities have four vertices. The location of the vertices depend on p, q. For example:

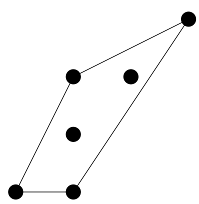
$Y^{1,0}$ (conifold) :



$Y^{2,1}$ (del Pezzo 1) :



$Y^{3,1}$:



The dual "toric quiver" gauge theories

[Benvenuti,Franco,Hanany,DM,Sparks]

One can now address the issue of obtaining the gauge theory duals to the geometry.

Which supersymmetric gauge theories are expected to be the dual of a given toric Calabi–Yau space $\mathcal{M}$ (or equivalently, Sasaki–Einstein)?

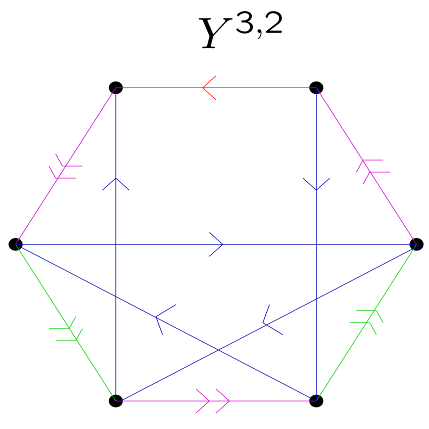
1) the moduli space of the theory is $\mathcal{M}$

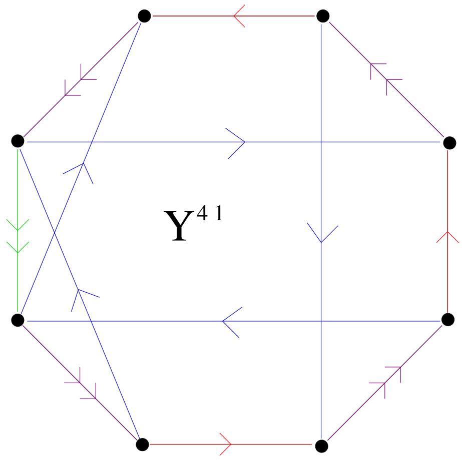
2) the theory should flow to a conformal fixed point (AdS_5)

We should look at quiver gauge theories of a particular kind, named toric.

- There are constraints from the geometry. For instance

$$\text{area toric diagram} = 2p \quad \Rightarrow \quad SU(N)^{2p} \text{ theory}$$





4 kinds of bifundamental fields:

p U doublets, q V doublets, $p-q$ Z singlets, $p+q$ Y singlets

$$\text{superpotential} \quad W = \sum^{2q} UVY + \sum^{p-q} ZUYU$$

a-maximisation as a test of AdS/CFT

Geometry:

central charge of SCFT: $a_{\text{geom}}[p,q] = \frac{\pi^3}{4\text{vol}(Y)} N^2$

R-charges of bifundamentals: $R^a_{\text{geom}}[p,q] = \frac{\pi \text{vol}(\Sigma_a)}{3\text{vol}(Y)}$

Field theory:

Any $4d$ $\mathcal{N}=1$ SCFT has a $U(1)_R$ R-symmetry.

Exact R-charges determine central charges [Anselmi et al.]

$a = \frac{3}{32}(3\text{Tr}R^3 - \text{Tr}R)$ $\quad\quad$ $< T^\mu_\mu >= c(\text{Weyl})^2 + a(\text{Euler})^2$

Problem: the full symmetry group of a SCFT may contain additional global flavour symmetries. The abelian part – F_I – can mix with $U(1)_R$.

Resolution [Intriligator,Wecht]: consider a "trial" R-symmetry

$$R_{\text{trial}} = R_0 + \sum_I s^I F_I$$

• The exact R-charges are those that maximise the central charge a_{trial} as a function of the s^I. And $a = a_{\text{max}}$. [NB: variational problem]

Applying a-maximisation to the $Y^{p,q}$ quiver gauge theories one finds analytic expressions for $a[p,q]$ and $R[p,q]$.

• These agree with the values $a_{\text{geom}}, R_{\text{geom}}$ obtained from the $Y^{p,q}$ metric. [Bertolini,Bigazzi,Cotrone] for $Y^{2,1} \simeq dP_1$, then [BFHMS] in general.

The dual of a-maximisation: Z-minimisation

[DM,Sparks,Yau]

central charge a, R charges $\Big\}$ $\overset{\text{AdS/CFT}}{\longleftrightarrow}$ $\Big\{$ total volume vol(Y), volume susy submanifolds

Is it possible to extract these charges from the geometry, if we don't know the metric?

Hints from a-maximisation: 1) extremal problem 2) exact $U(1)_R$ R-symmetry $\Leftrightarrow$ $U(1)$ isometry (Reeb vector)

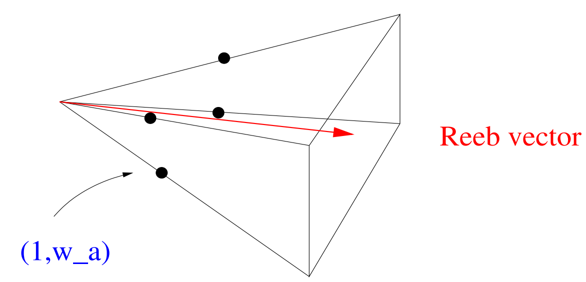
• Start with an arbitrary toric CY singularity: we are given a set of integral vectors $w_a \in \mathbb{Z}^2$, and a "trial" Reeb vector

$$K = b_i \frac{\partial}{\partial \phi_i} \in U(1)^3 \simeq U(1)_0 \times U(1)_{F_1} \times U(2)_{F_2}$$

$K \to b = b_0 + \sum_{i=1}^{2} b_i e_i \qquad e_i$ generators of $U(1)_{F_1} \times U(2)_{F_2}$

Construct polytope

$\Delta_b[w_1, \ldots, w_d, b] =$



Einstein–Hilbert action $\longrightarrow$ $Z[w_1, \ldots, w_d; b] \sim \text{vol}(\Delta_b)$

$$\text{The metric is Sasaki–Einstein} \Leftrightarrow \frac{\partial Z}{\partial b_i} = 0$$

Once the "exact" Reeb vector is determined, the relevant volumes can be computed. E.g.

$$\text{vol}(Y) \sim Z[b = b_{\text{min}}]$$

$\Rightarrow$ The central charge a and the R-charges R are computed using only the data defining the toric singularity.

Gauge theories from toric geometry [FHMSVW]

How much information on the gauge theory can we extract from the (toric) geometry, without using the metric?

• All the bifundamental fields, with their multiplicities, R-charges, baryonic charges, and flavour charges can be extracted from the toric data, i.e. the vectors $w_i \in \mathbb{Z}^2$.

In the toric case, we can now perform arbitrarily many checks of AdS/CFT, without the metrics.

E.g. "$L^{a,b,c,d}$": most general toric singularity with four external points in the diagram:

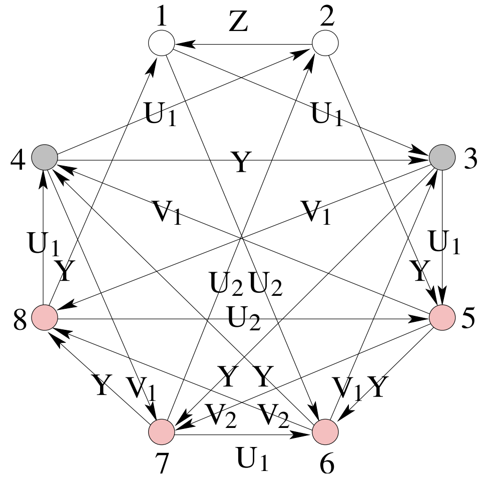
Field	$U(1)_R$	number	$U(1)_B$	$U(1)_{F_1}$
Y	R_1	b	a	1
U_1	R_2	d	$-c$	0
Z	R_3	a	b	0
U_2	R_4	c	$-d$	-1
V_1	$R_3 + R_4$	$b - c$	$c - a$	-1
V_2	$R_2 + R_3$	$c - a$	$b - c$	0

(See also [Benvenuti,Kruczenski], [Butti,Forcella,Zaffaroni])

The R-charges R_a can be computed using Z-minimisation, or the explicit Sasaki–Einstein metrics [Cvetic,Lu,Page,Pope] (See also [DM,Sparks]).

Details of quiver obtained using brane tilings → A. Hanany's talk.

Example of the $L^{1,7,4}$ quiver:



Conclusions and outlook

• Sasaki–Einstein metrics $Y^{p,q} \to$ new tests of AdS/CFT (before, only S^5 and $T^{1,1}$ were known).

• New infinite families of quiver gauge theories (before, a few examples were known, e.g. del Pezzo's).

• Progress towards a 1-1 correspondence between toric singularities and $\mathcal{N}=1$ quiver gauge theories. E. g. brane tilings.

• Z-minimisation $\to$ compute volumes of Sasaki–Einstein metrics $\Rightarrow$ charges in the dual SCFT.

• A lot of information on the dual gauge theory can be obtained from the geometry, without the explicit metric. I.e. all bifundamentals with their global quantum numbers.

• More information can be obtained from the toric singularity $\to$ BPS spectrum of gauge invariant operators. [in progress]

• Better understanding of relation between a-maximisation and Z-minimisation achieved recently. [Butti,Zaffaroni], [Barnes,Gorbatov,Intriligator,Wright] (to appear).

• Interesting to explore emerging connections with seemingly unrelated subjects: e.g. Calabi–Yau crystals.