27/10-08 Göteborg

From volumes of line bundles to equidistribution

Robert Berman

Notation:

- X is a compact complex manifold ($\dim X = n$)

- $L \to X$ is a holomorphic line bundle

A *smooth metric* on L is written (locally) as $h = e^{-2\phi}$. The additive object ϕ est is called a weight. The (normalized) *curvature* is then $dd^c\phi (= \frac{i}{4\pi}\partial\bar{\partial}\phi)$:

$$[dd^c\phi] = c_1(L) \in H^2(X, \mathbb{Z}).$$

A *singular* weight ψ may attain $-\infty$ and ψ is called *psh weight* if the curvature $dd^c\psi$ is a positive current.

Recall: the "algebraic" volume

$$\mathrm{Vol}(L) :=:= \limsup_{k\to\infty} k^{-n} \dim H^0(X, kL) n!$$

and L is called big if $\mathrm{Vol}(L) > 0$.

$$L \text{ ample (nef)} \Rightarrow \mathrm{Vol}(L) = c_1(L)^n$$

where L is ample $\exists\, \phi$ *smooth* and *spsh. If* L *is merely big then*

$$\mathrm{Vol}(L) = \int_{X-``\{\psi=-\infty\}"} (dd^c\psi)^n := \int_X \mathrm{MA}(\psi),$$

where ψ is any weight on L with *minimal* singularities [Boucksom'02]. Such a weight may be obtain from a smooth weight ϕ by

$$P\phi := \sup\{\psi \;\; \psi\,\mathrm{psh},\, \psi \leq \phi\}.$$

The Monge-Ampere measure $\mathrm{MA}(P\phi)$ is called the *equilibrium measure* of (X, ϕ).

The "metric" (weighted) volume

$$\phi \mapsto \mathcal{B}_\infty(\phi) := \left\{ s \in H^0(L) : \ \ \|se^{-\phi}\|_{L^\infty(X)} \leq 1 \right\}$$

(also have $\mathcal{B}_2(\phi)$ w.r.t. $L^2(X, \omega_n)$)

$$\mathrm{Vol}(\phi, \phi') := \limsup_{k \to \infty} k^{-(n+1)} \log \frac{\mathrm{Vol}\mathcal{B}(k\phi)}{\mathrm{Vol}\mathcal{B}(k\phi')}(n+1)!$$

$\Rightarrow \mathrm{Vol}(\phi, \phi + 1) = \mathrm{Vol}(L)$

Thm [B-B'08]: Let ϕ and ϕ be, say, continuous. Then

$$\mathrm{Vol}(\phi, \phi') = \mathcal{E}(P\phi, P\phi')$$

$$\mathcal{E}(\psi, \psi') := \sum_{j=0}^{n} \int_{X - \text{"}\{\psi, \psi' = -\infty\}\text{"}} (\psi - \psi')(dd^c\psi) \wedge (dd^c\psi')^{n-j}/(n+1)$$

$\Rightarrow \mathcal{E}(\psi, \psi + 1) = \mathrm{Vol}(L)$

(secondary class of *Bott-Chern*). [Bismut-Gillet-Soule]

Fundamental property: $d\mathcal{E}_\psi = \mathrm{MA}(\psi)$ [Aubin-Mabuchi]

Thm [B-B'08] $\phi \mapsto \mathcal{E} \circ P$ is

- differentiable $(d(\mathcal{E} \circ P)_\phi = \mathrm{MA}(P\phi))$ (not $\mathcal{C}^2$!)
- concave

Examples

Let $(X, L) = (\mathbb{P}^1, \mathcal{O}(1))$.

$\mathbb{P}^1 \supseteq \mathbb{C}^1_z$: $\phi(z)$ function on $\mathbb{C}$. $\phi'(z) := \log\max\{1, |z|\}$

$$(\text{Vol'}\mathcal{B}(k\phi))^{-k} \sim \sup_{(z_0,...,z_k)\in\mathbb{C}^{k+1}} \prod_{i<j} (|z_i - z_j|\, e^{-\phi(z_i)} e^{-\phi(z_j)})$$

($\rightarrow$transfinite diameter of $(\mathbb{C}, \phi)$).

$$k^{-2} \log \text{Vol'}\mathcal{B}(k\phi) \sim \inf_{(z_1,...,z_k)\in\mathbb{C}^k} E_\phi(z_1, ..., z_k)$$

$$E_\phi(z_1, ..., z_k) = -\frac{1}{2} \sum_{i\neq j} \log|z_i - z_j|\, k^{-2} + \sum_i \phi(z_i))$$

Hence, $k^{-2} \log \text{Vol'}\mathcal{B}(k\phi)$ converges towards the weighted logarithmic energy of the equilibrium measure of $(\mathbb{C}, \phi)$.

If $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$ on obtains a formula for the (weighted) transfinite diameter of Leja ('59). [Unweighted case: Rumely'06, Rumely-DeMarco'06].

The "arithmetic" volume

Let $L \to X$ be defined over $\mathbb{Z}$ (so that X is an arithmetic variety) and ϕ weight on $L \to X(\mathbb{C})$ (i.e. "at infinity")

$$\mathrm{Vol}_{\mathbb{Z}}(L,\phi) := \limsup_{k\to\infty} k^{-(n+1)} \log \frac{\mathrm{Vol}\mathcal{B}(X(\mathbb{C}), k\phi)}{\mathrm{Vol}\Lambda_{\mathbb{Z}}}$$

where $\Lambda_{\mathbb{Z}}$ is the lattice of sections defined over $\mathbb{Z}$. [Gillet-Soule, Abbes-Bouche, Rumely-Lau-Varley]

"Trivial" observation:

$$\mathrm{Vol}_{\mathbb{Z}}(L,\phi) - \mathrm{Vol}_{\mathbb{Z}}(L,\phi') = \mathrm{Vol}(\phi,\phi')$$

the weighted volume for $X(\mathbb{C})$!

This leads to a generalization to big line bundles of the equidistribution thm. of Yuan'06 [Szpiro-Ullmo-Zhang'97] for a sequence of points (x_i) whose heights tend to zero:

The Galois orbit of x_i on $X(\mathbb{C})$ becomes equidistributed on the equilibrium measure $\mathrm{MA}(P\phi)$ (generically).

Pf. of Vol$=\mathcal{E}\circ P$: enter the hero

The Bergman measure: $\beta_{(k\phi,\omega_n)} = \sum_j |s_j|^2 e^{-2\phi}\omega_n$ where (s_j) base ON in $H^0(kL)$ for $\|se^{-\phi}\|_{L^2(X,\omega_n)}$.

$$d_\phi(k^{-(n+1)}\log\frac{\mathrm{Vol}\mathcal{B}_2(k\phi)}{\mathrm{Vol}\mathcal{B}_2(k\phi')}) = k^{-n}\beta_{(k\phi,\omega_n)}$$

$$\rightarrow \mathrm{MA}(P\phi)) \ \ (\text{B.'07})$$

$$d_\phi(\mathcal{E}\circ P) = \mathrm{MA}(P\phi) \ \ (\text{B-B})$$

In particular,

$$k^{-(n+1)}\log\frac{\mathrm{Vol}\mathcal{B}_2(k\phi)}{\mathrm{Vol}\mathcal{B}_2(k\phi')} \rightarrow (\mathcal{E}\circ P)(\phi,\phi')$$

Finally, use $\mathrm{Vol}\mathcal{B}_2(k\phi) \sim \mathrm{Vol}\mathcal{B}_\infty(k\phi)$.

Equidistribution

$$\mathcal{L}(k\phi, \mu_k) := k^{-(n+1)} \log \mathrm{Vol}\mathcal{B}_2(k\phi, \mu_k)$$

$$= -k^{-(n+1)} \log \det \left(\int_X s_i \bar{s}_j e^{-k2\phi} \mu_k \right)_{i,j}$$

Thm (B-B-W)'08): Fix ϕ and suppose that (μ_k) is an asymptotic minimizing sequence of $\mathcal{L}(k\phi, \cdot)$:

$$\mathcal{L}(k\phi, \mu_k) \to \inf_{\mu_k} \mathcal{L}(k\phi, \mu_k) = (\mathcal{E} \circ P)(\phi).$$

Then $\beta_{(k\phi, \mu_k)} \to MA(P\phi)$ weakly.

Pf: Let $f_k(t) := \mathcal{L}(k(\phi + tu), \mu_k)$ and $f(t) := (\mathcal{E} \circ P)(\phi + tu)$. Then f_k and f are concave and $\mathcal{C}^1$ and

$$f_k(0) \to f(0)$$

$$\inf_k f_k(t) \geq f(t)$$

Hence,

$$\frac{df_k(0)}{dt} \to \frac{df(0)}{dt},$$

Q.E.D.

Implies equidistribution in “different” situations:

- (conjecture of Leja, Siciak,...) If $\mu_k = \frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{x_i}$ where $(x_1, ..., x_{N_k})$ maximizes

  $$X^k \ni (x_1, ..., x_{N_k}) \mapsto \left|\det(s_i(x_j)e^{-k\phi(x_j)})\right|^2$$

  (since $\mu_k = \beta_k$ in this discrete case!)
- (conjecture of Bloom-Levenberg,...). If E is a, say smooth, domain in $\mathbb{C}^n$ (or $\mathbb{R}^n$) and $\mu_k \equiv 1_E * Lesbesgue\, measure$. Then get equidistribution of the Bergman measures β_k.

Probabilistic aspects [B.]

Consider the "configuration space" X^{N_k} of N_k particles with the probability measure whose density is

$$\rho(x_1, ..., x_{N_k}) := \left|\det(s_i(x_j))\right|^2_{k\phi} / N_k!$$

Get a random measure (determinatal point process)

$$(x_1, ..., x_{N_k}) \mapsto \frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{x_i} \tag{0.1}$$

which (1) converges in probability to the equil.measure $MA(P\phi)$ (2) satifies a CLT in the "bulk":

for any u supported in the pseuodo-interiour of the support of the equil.measure $MA(P\phi)$

$$\frac{1}{k^{n-1}}(u(x_1) - \mathbb{E}u(x_1)) + ... + (u(x_{N_k}) - \mathbb{E}u(x_{N_k})\}$$

converges in distribution to a centered Gaussian with variance $\|du\|^2_{dd^c\phi}$.