Mini-workshop in complex dynamics
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Lecture 1:

Surface dynamics and the Riemann-Zariski space

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X compact Kähler surface

$f: X \dashrightarrow X$ meromorphic, dominant

<u>Want</u>: construct interesting invariant objects (currents, measures ...)

<u>First step</u>: find invariant cohomology classes.

<u>Problem</u>: $f^{n*} \neq f^{*n}$ on cohomology in general

<u>Approaches</u>:

1) Try to find bimeromorphic model $X' \not\cong X$ where problem is not present.
   E.g. $X' = X$ blown up finitely many times

2) Study dynamics on the Riemann-Zariski space of X: the "limit of all blowups of X"

Focus on 2^{nd} approach in this talk

Tool more interesting than result!

Cohomology classes on compact Kähler surfaces

X compact Kähler surface

$H^{1,1}(X) = \{\bar\partial\text{-closed } (1,1)\text{-forms}\} / \{\bar\partial\text{-exact } (1,1)\text{ forms}\}$

$H^{1,1}_{\mathbb{R}}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{R})$

$H^{1,1}_{\mathbb{Z}}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$

<u>Intersection product</u>:

$$H^{1,1}_{\mathbb{R}}(X) \times H^{1,1}_{\mathbb{R}}(X) \longrightarrow \mathbb{R}$$

$$(\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta =: (\alpha \cdot \beta)$$

<u>Hodge Index Thm</u>: Inters. form has index $(1, h^{1,1}(X) - 1)$

<u>Cor</u>: If $(\alpha \cdot \alpha) \geq 0$, $(\beta \cdot \beta) \geq 0$ then

$$(\alpha \cdot \beta)^2 \geq (\alpha \cdot \alpha)(\beta \cdot \beta)$$

with equality iff α, β proportional.

<u>Pf</u>: If α, β not proportional, int. form can't be pos. semidefinit on $\mathbb{R}\alpha + \mathbb{R}\beta$ $\square$

Positivity

T positive closed $(1,1)$-current on $X \Rightarrow$
T induces class $\{T\} \in H^{1,1}_{\mathbb{R}}(X)$

Special case: $T = [C]$ current of integration.

Def: $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ psef (pseudoeffective) if $\alpha = \{T\}$.

write $H^{1,1}_{psef}(X) \subseteq H^{1,1}_{\mathbb{R}}(X)$ psef cone

Def: $\alpha \geq \beta \Leftrightarrow \alpha - \beta$ psef.

NB: $\alpha \geq \beta$, $\beta \geq \alpha \Rightarrow \alpha = \beta$!

Def: $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ Kähler class if $\alpha = \{\omega\}$, ω K. form

Def: $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ nef if $(\alpha \cdot \beta) \geq 0 \quad \forall \beta$ psef

$\rightsquigarrow H^{1,1}_{nef}(X) \subseteq H^{1,1}_{\mathbb{R}}(X)$ nef cone

Prop: (i) nef $\Rightarrow$ psef
(ii) $H^{1,1}_{nef}$, $H^{1,1}_{psef}$ are dual cones
(iii) $H^{1,1}_{nef}(X) = \overline{H^{1,1}_{Kähler}(X)}$

Def: α big + nef if α nef and $(\alpha \cdot \alpha) > 0$.

Holomorphic mappings

$f : X \longrightarrow X$ holo, surjective

$f^* : H^{1,1}_{\mathbb{R}}(X) \circlearrowleft$ (forms)
$f_* : H^{1,1}_{\mathbb{R}}(X) \circlearrowleft$ (currents)
} preserve psef, nef classes

$(f_*\alpha \cdot \beta) = (\alpha \cdot f^*\beta)$

$f_* f^* \beta = \lambda_2 \cdot \beta$

$\lambda_2 =$ top. deg of f

$\lambda_1 :=$ spectral radius of f^* (or f_*) on $H^{1,1}_{\mathbb{R}}(X)$.

$\lambda_1^2 \geq \lambda_2$

__Thm A__ [Diller-Favre '01] There exist nef eigenclasses $\theta_*, \theta^* \in H^{1,1}_{\mathbb{R}}(X)$ for f_*, f^*:

$$f_*\theta_* = \lambda_1 \theta_* \ , \quad f^*\theta^* = \lambda_1 \theta^* .$$

__Thm B__ [D-F] If $\lambda_1^2 > \lambda_2$ then θ_*, θ^* are unique up to scaling. Moreover,

$$\begin{cases} \lambda_1^{-n} f^{*n} \alpha = \text{const} \cdot \theta^* + O\left(\left(\frac{\lambda_2}{\lambda_1^2}\right)^{n/2}\right) \\ \lambda_1^{-n} f_*^n \alpha = \text{const} \cdot \theta_* + O\left(\left(\frac{\lambda_2}{\lambda_1^2}\right)^{n/2}\right) \end{cases} \quad n \to \infty$$

for any $\alpha \in H^{1,1}_{\mathbb{R}}(X)$

In fact, Thm A, B reduce to linear algebra: a version of the Perron-Frobenius Thm

W = real vector space, $\dim W < \infty$
$C \subseteq W$ strict, closed convex cone w. interior $\neq \emptyset$
$S : W \to W$ linear map, $SC \subseteq C$, $\rho(S) = \lambda_1$

<u>Thm A'</u>: $\exists v_0 \in C$ s.t. $Sv_0 = \lambda_1 v_0$

For Thm B, assume W equipped with inner product of Minkowski type, and

$S : W \to W$, $T : W \to W$ $\quad SC \subseteq C$, $TC \subseteq C$
$(Sv \cdot w) = (v \cdot Tw)$ (adjoint) $\rho(S) = \rho(T) = \lambda_1$
$S \circ T = \lambda_2 \cdot id$ where $\lambda_2 < \lambda_1^2$

<u>Thm B'</u>: $\exists v_0, w_0 \in C$ s.t $Sv_0 = \lambda_1 v_0$, $Tw_0 = \lambda_1 w_0$,

$$\begin{aligned} \lambda_1^{-n} S^n v &= \text{const} \cdot v_0 + O((\lambda_2/\lambda_1^2)^{n/2}) \\ \lambda_1^{-n} T^n w &= \text{const} \cdot w_0 + O(\text{ — '' — }) \end{aligned} \quad n \to \infty$$

<u>Rem</u>: Used $f : X \to X$ holo:

$\cdot\ f_* f^* = \lambda_2 \cdot id$
$(\cdot\ f^{n*} = f^{*n}$ etc$)$

((Can adopt argument to $f : X \dashrightarrow X$ algebraically stable as in [DF]))

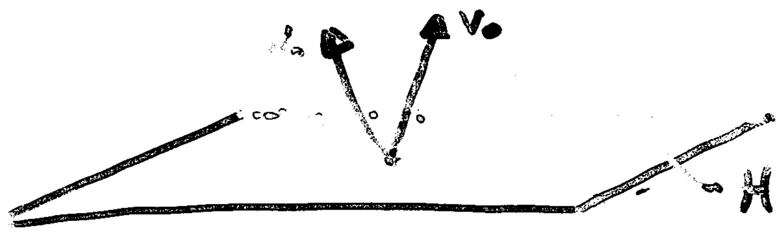
Steps in Proof of Thm B'

1. Know v_0, w_0 exist from Thm A'
2. $Sw_0 = \frac{\lambda_2}{\lambda_1} w_0 \neq \lambda_1 w_0$

   $\Rightarrow v_0, w_0$ not proportional
3. $\lambda_1^2 (w_0 \cdot w_0) = (Tw_0 \cdot Tw_0) = \lambda_2 (w_0 \cdot w_0)$

   $\Rightarrow (w_0 \cdot w_0) = 0$
4. Hodge $\Rightarrow (v_0 \cdot w_0) > 0$.
5. $\mathcal{H} := \{ v \in W \mid (v \cdot v_0) = (v \cdot w_0) = 0 \}$.

   $\Rightarrow W = \mathbb{R} v_0 \oplus \mathbb{R} w_0 \oplus \mathcal{H}$



$(\cdot)$ neg. definite on $\mathcal{H}$

$T\mathcal{H} \subset \mathcal{H}$

$(\cdots)$

Finding a good model

Perron-Frobenius argument works also if $f : X \dashrightarrow X$ is algebraically stable:

$$\text{curve} \xrightarrow{f} \text{pt} \xrightarrow{f} \circ \qquad \xrightarrow{f} \text{pt} \xrightarrow{f} \text{curve}$$

does not occur.

Q: Given $f : X \dashrightarrow X$, when can we make bimeromorphic change of coordinates to render f AS?

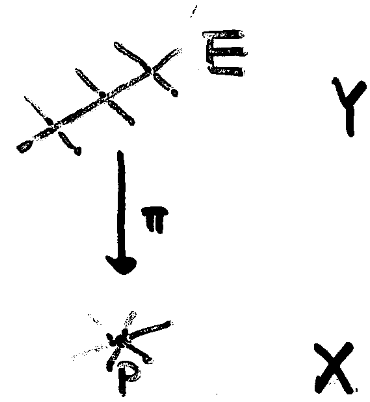
Difficult! Understood when:

1) f birational [DF]
2) f monomial [F]
3) f polynomial [FJ] (lecture 3)

In any case, can try to change X using blowups

Blowups I

$p \in X$ point
$\pi : Y \to X$ blowup of p
$E = \pi^{-1}(p)$ exceptional divisor
$\{E\} \in H^{1,1}_{\mathbb{R}}(Y)$ psef, not nef
$(\{E\}\cdot\{E\}) = -1$



$$\pi^* : H^{1,1}(X) \longrightarrow H^{1,1}(Y) \qquad \text{(forms)}$$
$$\pi_* : H^{1,1}(Y) \longrightarrow H^{1,1}(X) \qquad \text{(currents)}$$

"Projection formula" $\begin{cases} \pi_*\pi^* = \mathrm{id} \\ (\alpha\cdot\pi^*\beta)_Y = (\pi_*\alpha\cdot\beta)_X \\ (\pi^*\beta\cdot\pi^*\beta)_Y = (\beta\cdot\beta)_X \end{cases}$

$$H^{1,1}_{\mathbb{R}}(Y) \cong \pi^* H^{1,1}_{\mathbb{R}}(X) \oplus \mathbb{R}\{E\} \qquad \text{orthogonal}$$

<u>Convention 1</u>: identify $\beta \in H^{1,1}(X)$ and $\pi^*\beta \in H^{1,1}(Y)$

$$\longrightarrow \quad H^{1,1}_{\mathbb{R}}(Y) \cong H^{1,1}_{\mathbb{R}}(X) \oplus \mathbb{R}\{E\}.$$

<u>NB</u>: $\beta \in H^{1,1}(X) \begin{cases} \text{psef} \\ \text{nef} \\ \text{big+nef} \end{cases} \iff \beta \in H^{1,1}(Y) \begin{cases} \text{psef} \\ \text{nef} \\ \text{big+nef} \end{cases}$

Blowups II

$\pi : Y \longrightarrow X$ <u>composition of blowups</u>

$$Y = X_n \xrightarrow{\pi_n} X_{n-1} \longrightarrow \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X$$

$E_j = \pi_j^{-1}(p_j) \subseteq X_j$ exc. div.

$e_j := \{E_j\} \in H^{1,1}_{\mathbb{R}}(X_j)$

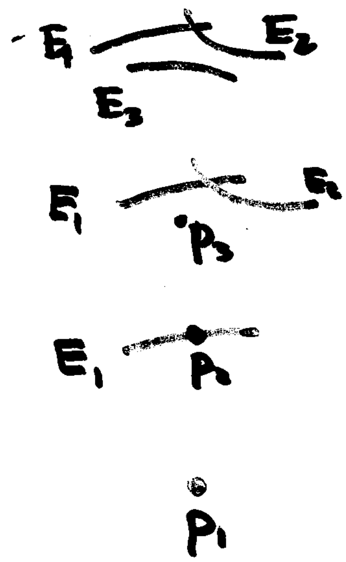
Using Convention 1:

$$H^{1,1}_{\mathbb{R}}(Y) \cong H^{1,1}(X) \oplus \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n \qquad \underline{\text{ON}}$$

<u>Convention 2</u>:

Identify $E_1 \subset X_1$
with its <u>strict</u> transform $\subset X_2$ etc

<u>NB</u>: $e_j \neq \{E_j\}$ in general.



$$\begin{array}{c} \vdots \\ \downarrow \\ X_3 \\ \downarrow \pi_3 \\ X_2 \\ \downarrow \pi_2 \\ X_1 \\ \downarrow \pi_1 \\ X_0 \\ \shortparallel \\ X \end{array}$$

Blowups III

Can use blowups to "improve" maps.

Consider $f : X \dashrightarrow Y$ mero, dominant

Fact 1: [Elimination of indeterminacy pts]

$$\begin{tikzcd} X_\pi \arrow[d, "\pi"'] \arrow[dr, "\text{holo}"] & \\ X \arrow[r, dashed, "f"] & Y \end{tikzcd}$$

$\exists \pi$ comp'n of blowups.

Fact 2: [Elimination of contracted curves]

$$\begin{tikzcd} & Y_\varpi \arrow[d, "\varpi"] \\ X \arrow[ur, squiggly, "\text{does not contract curves}"] \arrow[r, dashed, "f"] & Y \end{tikzcd}$$

$\exists \varpi$

However, blowing up does not necessarily simplify dynamics : same source, target!

Can create new indet pts + contracted curves

Draconian approach : blow up everything!

The Riemann-Zariski space

$\pi : X_\pi \longrightarrow X$ (composition of) blowups)

Any two blowups can be dominated by third

Def: $\mathfrak{X} := \varprojlim_{\pi} X_\pi$ RZ space

Can view $\mathfrak{X}$ as locally ringed space, but we don't need it!

$\pi' \geq \pi \iff \exists$ blowup $X_{\pi'} \xrightarrow{\mu} X_\pi \xrightarrow{\pi} X$

$$\begin{tikzcd} X_{\pi'} \arrow[r, "\mu"] \arrow[rr, bend right, "\pi'"'] & X_\pi \arrow[r, "\pi"] & X \end{tikzcd}$$

Convention : identify $H^{1,1}_{\mathbb{R}}(X_\pi)$ with
$\mu^* H^{1,1}_{\mathbb{R}}(X_\pi) \subset H^{1,1}_{\mathbb{R}}(X_{\pi'})$

$C(\mathfrak{X}) := \varinjlim_{\pi} H^{1,1}_{\mathbb{R}}(X_\pi)$ Cartier classes

(union of all $H^{1,1}_{\mathbb{R}}(X_\pi)$ by conv'n)

$W(\mathfrak{X}) := \varprojlim_{\pi} H^{1,1}_{\mathbb{R}}(X_\pi)$ Weil classes

What do these mean, concretely?

- A <u>Cartier class</u> on $\mathfrak{X}$ is a cohomology class on some X_π (use convention) : the class is determined in X_π.
- A <u>Weil class</u> $\alpha \in W(\mathfrak{X})$ is a collection of $\alpha_\pi \in H^{1,1}_{\mathbb{R}}(X_\pi)$, the incarnation of α on X_π, compatible by pushforward.

Can also describe Weil divisors as follows

<u>Def</u>: An exceptional prime is an exceptional prime divisor $E \subseteq X_\pi$ some $\pi : X_\pi \to X$. Identify strict transforms.

Then a Weil divisor on $\mathfrak{X}$ is given by:

- a class $\alpha_X \in H^{1,1}_{\mathbb{R}}(X)$
- a real valued function on {exc. primes}.

$$C(\mathfrak{X}) \hookrightarrow W(\mathfrak{X})$$ dense image in proj. limit topology.

<u>Rem</u>: Weil, Cartier classes introduced by Manin. Used for Cremona gp by Cantat
Also: Farey blowup of Hubbard-Papadopol

L^2-classes

Intersection form on $H^{1,1}_{\mathbb{R}}(X)$

$\longrightarrow$

Pairing $\quad W(\mathfrak{X}) \times C(\mathfrak{X}) \longrightarrow \mathbb{R}$

Restrict to $\quad C(\mathfrak{X}) \times C(\mathfrak{X}) \longrightarrow \mathbb{R}$

Non-degenerate, Minkowski type

<u>Def</u>: $L^2(\mathfrak{X}) :=$ completion of $C(\mathfrak{X})$

$C(\mathfrak{X}) \subset L^2(\mathfrak{X}) \subset W(\mathfrak{X})$

Can show: $L^2(\mathfrak{X}) \cong H^{1,1}_{\mathbb{R}}(X) \oplus \ell^2(D)$

$\qquad D = \{\text{exc. primes}\}.$

<u>Lemma</u>: If $\alpha \in W(\mathfrak{X})$ has incarnations $\alpha_\pi \in H^{1,1}_{\mathbb{R}}(X_\pi)$ then:

(1) $\quad \pi \longmapsto (\alpha_\pi^2)$ is decreasing

(2) $\quad \alpha \in L^2(\mathfrak{X}) \iff \inf_\pi (\alpha_\pi^2) > -\infty$.

Positivity on $\mathcal{X}$

Def: $\alpha \in W(\mathfrak{X})$ is nef (psef) if $\alpha_\pi \in H^{1,1}_{\mathbb{R}}(X_\pi)$ is — " — $\forall \pi$.

Lemma: α nef $\Rightarrow \alpha \le \alpha_\pi \quad \forall \pi$

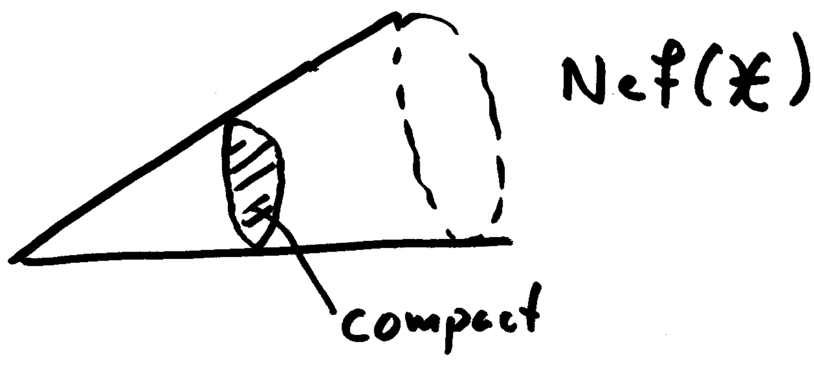
("Negativity lemma")

Lemma: $\mathrm{Nef}(\mathfrak{X}) \subset L^2(\mathfrak{X})$

"Pf:" α nef $\Rightarrow (\alpha \cdot \alpha) \geq 0 > -\infty$ $\square$

Key fact: $\mathrm{Nef}(\mathfrak{X})$, $\mathrm{Psef}(\mathfrak{X})$ are strict, convex closed cones in $W(\mathfrak{X})$ with compact basis: if $\omega \in \mathrm{Nef}(\mathfrak{X})$ and $(\omega \cdot \omega) > 0$ then

$$\{\alpha \in \mathrm{Nef}(\mathfrak{X}) \mid (\alpha \cdot \omega) = 1\} \text{ compact}$$



Functoriality

$f : X \dashrightarrow Y$ merom., dominant.

Fact 1 (Elimination of indet pts) $\Longrightarrow$

$\exists$ natural maps $\quad f_* : W(\mathcal{X}) \longrightarrow W(\mathcal{Y})$

$\qquad\qquad\qquad\qquad f^* : C(\mathcal{Y}) \rightarrow C(\mathcal{X})$

(can pretend f holomorphic!)

$$\begin{tikzcd} X_\pi \arrow[r, "\text{holo}"] \arrow[d] & Y_\varpi \arrow[d] \\ X \arrow[r, dashed, "f"] & Y \end{tikzcd}$$

Fact 2 (Elimination of contracted curves) $\Longrightarrow$

$$\begin{cases} f_* C(\mathcal{X}) \subset C(\mathcal{Y}) \\ f^* \text{ extends cont. to } f^* : W(\mathcal{Y}) \to W(\mathcal{X}) \end{cases}$$

$\beta \in C(\mathcal{Y}) \implies (f^*\beta \cdot f^*\beta) = \mathrm{topdeg}(f) \cdot (\beta \cdot \beta)$

$\Longrightarrow$

Prop: f_*, f^* extend to $L^2(\mathcal{X}), L^2(\mathcal{Y})$

Dynamics on $\mathfrak{X}$

$f : X \dashrightarrow X$ merom. dominant

Morally, get holo map $\mathfrak{X} \to \mathfrak{X}$
(but don't want to deal with $\mathfrak{X}$ itself)

$f_* : W(\mathfrak{X}) \circlearrowleft$
$\quad C(\mathfrak{X}) \circlearrowleft$ preserves nef, psef
$\quad L^2(\mathfrak{X}) \circlearrowleft$

f^* : same properties

$f^{n*} = f^{*n}$ etc

$f_* f^* = \lambda_2 \cdot \mathrm{id}$ $\qquad \lambda_2 =$ top deg

$(f_* \alpha \cdot \beta) = (\alpha \cdot f^* \beta)$

Def : $\lambda_1 := \lim_{n \to \infty} (f^{n*} \omega \cdot \omega)^{1/n}$
where $\omega \in \mathrm{Nef}(\mathfrak{X})$, $(\omega^2) > 0$
(Indep of ω)

Want to prove Perron-Frobenius type Thm in this setting:

Thm A : Existence of eigenclasses
Thm B : Spectral properties when $\lambda_1^2 > \lambda_2$

Eigenclasses on $\mathfrak{X}$

Thm A: $\exists\, \theta_*, \theta^* \in \mathrm{Nef}(\mathfrak{X})$ s.t

$$f_* \theta_* = \lambda_1 \theta_*$$
$$f^* \theta^* = \lambda_1 \theta^*$$

Thm B: If $\lambda_1^2 > \lambda_2$ then

$$\begin{cases} \lambda_1^{-n} f^{n*} \alpha = \text{const} \cdot \theta^* + O\left((\lambda_2/\lambda_1^2)^{n/2}\right) \\ \lambda_1^{-n} f_*^n \alpha = \text{const} \cdot \theta_* + O\left(\qquad\qquad\right) \end{cases} \quad n \to \infty$$

for every $\alpha \in L^2(\mathfrak{X})$

In fact, Thm B proved in exactly the same way as before!

What about Thm A?

Use finite-dimensional approximations!

S_n = induced map $H^{1,1}_{\mathbb{R}}(X_n) \circlearrowleft$ pushforward
T_n = ——— $''$ ——— pullback.

$\rho(S_n) = \rho(T_n) \searrow \lambda_1$

Eigenclasses for S_n, T_n converge to eigenclasses for f_*, f^*

$$\begin{tikzcd} \downarrow & \downarrow \\ X_2 \arrow[r, dashed] \arrow[dr] & X_2 \\ X_1 \arrow[u] \arrow[r, dashed] \arrow[dr] \arrow[ur, squiggly] & X_1 \arrow[u] \\ X \arrow[u] \arrow[r, dashed, "f"] \arrow[ur, squiggly] & X \arrow[u] \end{tikzcd}$$

Key ***fact***: $\mathrm{Nef}(\mathfrak{X})$ has compact basis!

<u>Cor</u> (of Thm B) If $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is merom + dominant and satisfies $\lambda_1^2 > \lambda_2$ then $\exists b > 0$ s.t

$$\deg f^n = b\lambda_1^n + O(\lambda_2^{n/2}) \quad \text{as } n \to \infty.$$

Concluding remarks/questions

1. Cantat used $L^2(\mathfrak{X})$ to study the Cremona group $Cr(2)$

2. Can we understand the eigenclasses $\theta_*, \theta^* \in \mathrm{Nef}(\mathfrak{X})$ when $\lambda_1^2 > \lambda_2$?

   a) Can understand θ_* when f is a polynomial map of $\mathbb{C}^2$ (Lecture 3)

   b) Thm [Diller-Dujardin-Guedj, Favre]
   θ_*, θ^* Cartier classes on $\mathfrak{X}$
   $\Updownarrow$
   $f: Y \to Y$ holo for some possibly singular model $Y \cong X$.

3. Can we use the eigenclasses to find models where f becomes AS ?