Intrinsic volumes of symmetric cones

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(joint work with Peter Bürgisser)

From Dynamics to Complexity
A conference celebrating the work of Mike Shub

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Motivating question

What is the probability that
the solution of a random semidefinite program
has rank r?

Overview

We will work over

- the real numbers $\mathbb{R}$,
- the complex numbers $\mathbb{C}$,
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$\beta \in \{1, 2, 4\}$ indicates the ground (skew-)field

$$\mathbb{F}_\beta := \begin{cases} \mathbb{R} & \text{if } \beta = 1, \\ \mathbb{C} & \text{if } \beta = 2, \\ \mathbb{H} & \text{if } \beta = 4. \end{cases}$$

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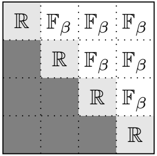

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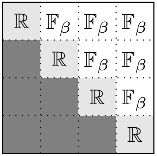

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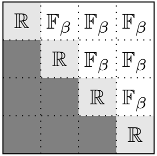

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- *Normal distribution* on $\mathrm{Her}_{\beta,n}$:

  Gaussian Orthogonal/Unitary/Symplectic Ensemble

  (GOE/GUE/GSE), short: GβE.

LP

Input: $z, a_i \in \mathbb{R}^d,\ b_i \in \mathbb{R}$,
$(i = 1, \ldots, m)$

Problem: $$\begin{bmatrix} \max & z \cdot x \\ \text{s.t.} & a_i \cdot x = b_i \\ & x \geq 0 \end{bmatrix}$$

LP	SDP
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Definition

$$\text{“random LP”} := \big(a_i, z \in \mathcal{N}(0, I_d)\,,\ b_i \in \mathcal{N}(0,1)\,,\ \text{ all independent}\big)$$

$$\text{“random SDP”} := \big(A_i, Z \in \mathrm{G}\beta\mathrm{E}\,,\ b_i \in \mathcal{N}(0,1)\,,\ \text{ all independent}\big)$$

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Remark

The unique solution of a random

$$\begin{cases} \text{LP has exactly } m \text{ nonzero components.} \\ \text{SDP satisfies } d_{\beta,r} \leq m \leq d_{\beta,r} + \beta r(n-r), \\ \qquad \text{where } r = \text{(rank of the solution).} \end{cases}$$

LP

Linear programming is connected with the *binomial distribution*:

$$\text{Prob}[\text{LP is infeasible}] = \sum_{j=0}^{m-1} \frac{\binom{n}{j}}{2^n} ,$$

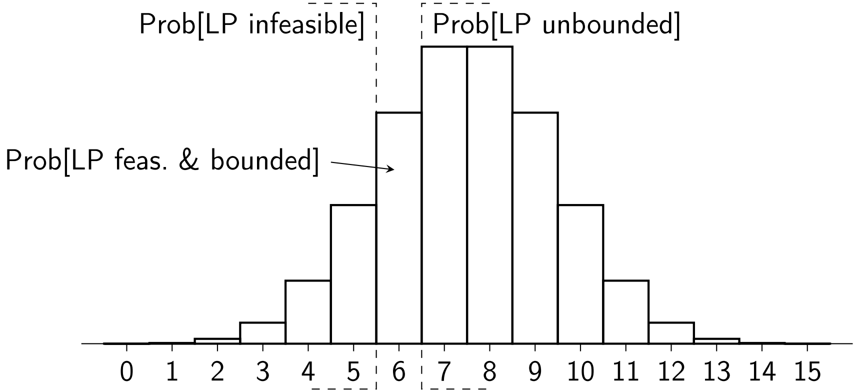
$$\text{Prob}[\text{LP is unbounded}] = \sum_{j=m+1}^{n} \frac{\binom{n}{j}}{2^n} ,$$

$$\text{Prob}[\text{LP is feas. \& bounded}] = \frac{\binom{n}{m}}{2^n} .$$

($\rightarrow$ Sporyshev, Todd, Cheung/Cucker)

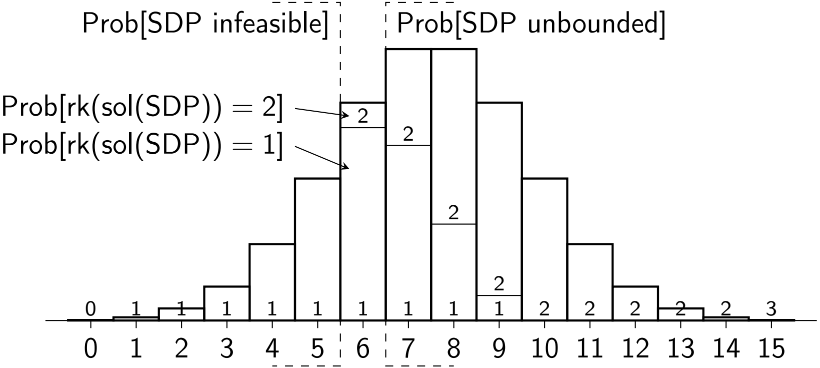
LP

Linear programming is connected with the *binomial distribution*:
$(d = 15, m = 6)$



SDP

Semidefinite programming is connected with a different distribution: $(\beta = 4, n = 3, m = 6)$



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Then

$$V_j(C) = \sum_{F \in \mathcal{F}_j} \operatorname*{Prob}_{x \in \mathcal{N}(0,I_d)} \left[\Pi_C(x) \in F\right]\,.$$

Curvature measures

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Note that $\hat{\mathscr{B}}(\mathbb{R}^d) = \hat{\mathscr{B}}_0(\mathbb{R}^d) \mathbin{\dot{\cup}} \hat{\mathscr{B}}_\emptyset(\mathbb{R}^d)$, where

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Note also

$$\hat{\mathscr{B}}_0(\mathbb{R}^d) \cong \hat{\mathscr{B}}_\emptyset(\mathbb{R}^d) \cong \mathscr{B}(S^{d-1}) .$$

Definition

The *jth curvature measure* of a polyhedral cone $C \subseteq \mathbb{R}^d$ is

$$\Phi_j(C,.)\colon \hat{\mathscr{B}}(\mathbb{R}^d) \to \mathbb{R}_+$$

$$\Phi_j(C, M) = \sum_{F \in \mathcal{F}_j} \operatorname*{Prob}_{x \in \mathcal{N}(0, I_d)} \left[\Pi_C(x) \in F \cap M\right] .$$

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- $\Phi_0(C,M) = \begin{cases} V_0(C) & \text{if } 0 \in M \\ 0 & \text{if } 0 \notin M. \end{cases}$

Remark

1. Φ_j can be extended to nonpolyhedral cones,
2. the curvature measures appear in a tube formula similar to the one defining the intrinsic volumes.

Recall SDP

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Let $C = \{X \in \mathrm{Her}_{\beta,n} \mid X \succeq 0\}$, $M_r = \{X \in C \mid \mathrm{rk}(X) = r\}$.
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main ingredient of the proof: kinematic formula

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Then for the *random projection* $\Pi_W(C)$

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and for the *random intersection* $C \cap W$

$$\mathbb{E}\big[\Phi_j(C \cap W, M \cap W)\big] = \Phi_{m+j}(C, M)\ , \quad \text{for } j = 1, 2, \ldots, d-m\ ,$$

$$\mathbb{E}\big[V_0(C \cap W)\big] = V_0(C) + V_1(C) + \ldots + V_m(C)\ .$$

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Mehta's integral

$$\frac{1}{(2\pi)^{n/2}} \cdot \int_{z \in \mathbb{R}^n} e^{-\frac{\|z\|^2}{2}} \cdot |\Delta(z)|^\beta \, dz = \prod_{j=1}^{n} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)}$$

$$=: M_{n,\beta}$$

Fix $0 \leq r \leq n$, write $z = (\underbrace{z_1, \ldots, z_r}_{=:x}, \underbrace{z_{r+1}, \ldots, z_n}_{=:y})$.

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$$= \prod_{1\leq i<j\leq r} (z_i - z_j)^\beta \cdot \prod_{r+1\leq i<j\leq n} (z_i - z_j)^\beta \cdot \prod_{i=1}^{r} \prod_{j=1}^{n-r} (z_i - z_j)^\beta$$

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&= \Delta(x)^\beta \cdot \Delta(y)^\beta \cdot \prod_{i=1}^{r} \prod_{j=1}^{n-r} (x_i - y_j)^\beta \\
&= \Delta(x)^\beta \cdot \Delta(y)^\beta \cdot \sum_{k=0}^{\beta r(n-r)} f_{\beta,k}(x; -y),
\end{aligned}$$

where $f_{\beta,k}(x;y) = \begin{pmatrix} x\text{-homog. part of } \prod_{i=1}^r \prod_{j=1}^{n-r} (x_i + y_j)^\beta \\ \text{of degree } k \end{pmatrix}$.

Recall

$$M_{n,\beta} = \frac{1}{(2\pi)^{n/2}} \cdot \int\limits_{z\in\mathbb{R}^n} e^{-\frac{\|z\|^2}{2}} \cdot |\Delta(z)|^{\beta} \, dz \, ,$$

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Definition

For $0 \le k \le \beta r(n-r)$

$$\boxed{J_\beta(n,r,k) := \frac{1}{(2\pi)^{n/2}} \cdot \int_{z\in\mathbb{R}^n_+} e^{-\frac{\|z\|^2}{2}} \cdot |\Delta(x)|^\beta \cdot |\Delta(y)|^\beta \cdot f_{\beta,k}(x;y) \, dz \,,}$$

where $z = (x, y)$, $x \in \mathbb{R}^r$, $y \in \mathbb{R}^{n-r}$.

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where $z = (x, y)$, $x \in \mathbb{R}^r$, $y \in \mathbb{R}^{n-r}$.

$\left[J_\beta(n,r,k) := 0, \text{ if } k < 0 \text{ or } k > \beta r(n-r). \right]$

Theorem (A, Bürgisser)
Let $C = \{X \in \mathrm{Her}_{\beta,n} \mid X \succeq 0\}$, $M_r = \{X \in C \mid \mathrm{rk}(X) = r\}$.
Then

$$\Phi_m(C, M_r) = \binom{n}{r} \cdot \frac{J_\beta(n, r, m - d_{\beta,r})}{M_{n,\beta}} \ ,$$

where $d_{\beta,r} = \dim \mathrm{Her}_{\beta,r} = r + \beta\binom{r}{2}$.

Writing the integral as an expectation: Let

$$1_+(A) := \begin{cases} 1 & \text{if } A \succeq 0, \\ 0 & \text{else.} \end{cases}$$

Then

$$\Phi_m(C, M_r) = c_{\beta,n,r} \cdot \underset{\substack{A \in \mathrm{G}\beta\mathrm{E}(r) \\ B \in \mathrm{G}\beta\mathrm{E}(n-r)}}{\mathbb{E}} \Big[1_+(A) \cdot 1_+(B) \cdot f_{\beta,k}(A; B) \Big],$$

with $k = m - d_{\beta,r}$ and $c_{\beta,n,r} := \binom{n}{r} \cdot \frac{M_{r,\beta} \cdot M_{n-r,\beta}}{M_{n,\beta}}$.

Remark

We have

$$\Phi_m(C, M_r) > 0 \iff d_{\beta,r} \leq m \leq d_{\beta,r} + \beta r(n-r)\,. \qquad (*)$$

Remark

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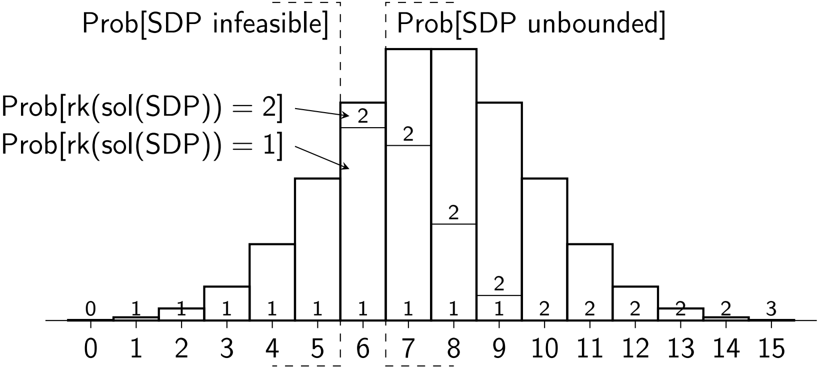
These are known as *Pataki's inequalities*:

> If r denotes the rank of the solution of a random SDP, then almost surely the inequalities $(*)$ are satisfied.

SDP (revisit)

Semidefinite programming is connected with a different distribution: $(\beta = 4, n = 3, m = 6)$



SDP (revisit)

Semidefinite programming is connected with a different distribution:

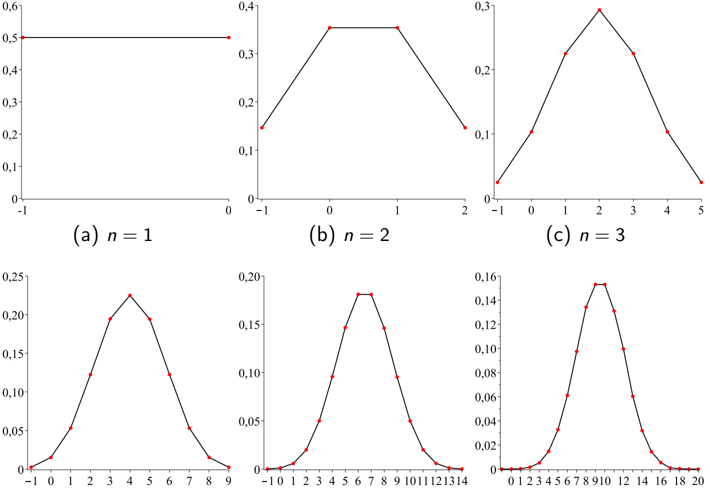
$$\text{Prob}[\text{SDP is infeasible}] = \sum_{j=0}^{m-1} V_j(\mathcal{C}_{\beta,n}) \ ,$$

$$\text{Prob}[\text{SDP is unbounded}] = \sum_{j=m+1}^{n} V_j(\mathcal{C}_{\beta,n}) \ ,$$

$$\text{Prob}[\text{rk}(\text{sol}(\text{SDP})) = r] = \Phi_m(C, M_r) \ .$$

	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
$\mathcal{C}_{\beta,1}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	0
$\mathcal{C}_{1,2}$	$\frac{1}{2} - \frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{4}$	$\frac{1}{2} - \frac{\sqrt{2}}{4}$	0	0	0	0	0
$\mathcal{C}_{2,2}$	$\frac{1}{4} - \frac{1}{2\pi}$	$\frac{1}{4}$	$\frac{1}{\pi}$	$\frac{1}{4}$	$\frac{1}{4} - \frac{1}{2\pi}$	0	0	0	0
$\mathcal{C}_{4,2}$	$\frac{1}{4} - \frac{2}{3\pi}$	$\frac{1}{8}$	$\frac{2}{3\pi}$	$\frac{1}{4}$	$\frac{2}{3\pi}$	$\frac{1}{8}$	$\frac{1}{4} - \frac{2}{3\pi}$	0	0
$\mathcal{C}_{1,3}$	$\frac{1}{4} - \frac{\sqrt{2}}{2\pi}$	$\frac{\sqrt{2}}{4} - \frac{1}{4}$	$\frac{\sqrt{2}}{2\pi}$	$1 - \frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2\pi}$	$\frac{\sqrt{2}}{4} - \frac{1}{4}$	$\frac{1}{4} - \frac{\sqrt{2}}{2\pi}$	0	0
$\mathcal{C}_{2,3}$	$\frac{1}{8} - \frac{3}{8\pi}$	$\frac{3}{16} - \frac{1}{2\pi}$	$\frac{1}{4\pi}$	$\frac{1}{2\pi}$	$\frac{3}{16} + \frac{1}{8\pi}$	$\frac{3}{16} + \frac{1}{8\pi}$	$\frac{1}{2\pi}$	$\frac{1}{4\pi}$	$\ldots$
$\mathcal{C}_{4,3}$	$\frac{1}{8} - \frac{47}{120\pi}$	$\frac{11}{64} - \frac{8}{15\pi}$	$\frac{1}{40\pi}$	$\frac{4}{15\pi} - \frac{1}{16}$	$\frac{19}{120\pi}$	$\frac{3}{32}$	$\frac{13}{120\pi} + \frac{7}{64}$	$\frac{11}{30\pi} + \frac{1}{16}$	$\ldots$

$\beta = 1$, intrinsic volumes



(a) $n = 1$

(b) $n = 2$

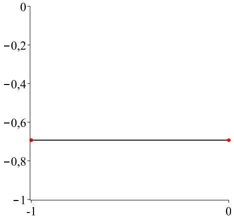
(c) $n = 3$

(d) $n = 4$

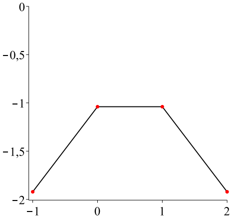
(e) $n = 5$

(f) $n = 6$

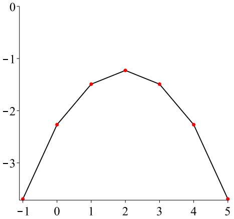
$\beta = 1$, log(intrinsic volumes)



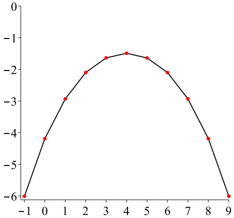
(g) $n = 1$



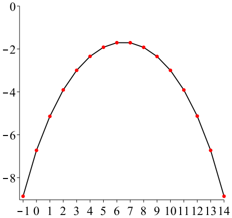
(h) $n = 2$



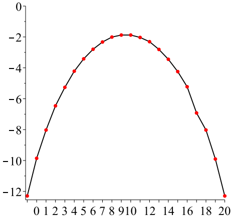
(i) $n = 3$



(j) $n = 4$



(k) $n = 5$



(l) $n = 6$

Final remarks

(Very) Soon: Preprint paper on the arXiv.

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3. Applications in physics?
4. Connections with the "algebraic degree" of SDP?

Thank you!

&

Happy Birthday Mike!