Motives and Automorphic Representations

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Contents

References: 1. J. Arthur, A note on the Automorphic Langlands Group, Bull. Canad. Math Soc. 45 (2002), 466–482.

2. ———, The work of Robert Langlands, arXiv, to appear in H. Holden and R. Piene, The Abel Laureates 2018–2022, Springer, 2024

1. Motives (pure); Grothendieck (c 1965)

2 simultaneous roles:

(i) Fundamental (but hidden) building blocks of smooth, projective, alg. varieties

(ii) Universal cohomology theory, through which all other cohom. theories in algebraic geom. factor. (Betti, de Rham, l-adic, crystalline, Hodge theory, etc.)

- F, Q : fields of char 0, $F \subset \mathbb{C}$.

(Usually F is number field + $Q = \mathbb{Q}$)

- Expect : a semisimple, Q-linear category $Mot_{F,Q}$, pure motives over F with coeff^s in Q.

- Would come with 2 functors

$$(SProj)_F \xrightarrow{M_F} Mot_{F,Q}$$

(smooth proj. varieties over F)

and

$$Mot_{F,Q} \xrightarrow{R_B} Vect_Q$$

(vector spaces over Q)

whose composition

$$H_B = R_B \circ M_F,$$

$$(SProj)_F \longrightarrow Mot_{F,Q} \longrightarrow (Vect)_Q$$

building blocks

universal cohomology

is just Betti (singular) cohom. of complex manifolds with Q-coefficients.

(4)

- $M_{F,Q}$ would be a Tannakian category with fibre functor H_B ; this means that $Mot_{F,Q}$ is (anti) isomorphic to the category $Rep_Q(\mathcal{G}_{F,Q})$ of finite dim. representations of a reductive, proalgebraic gp $\mathcal{G}_{F,Q}$ defined over Q
- Assume now F is a number field + $Q = \mathbb{Q}$.

Then $\mathcal{G}_F = \mathcal{G}_{F,Q}$ would be a (reductive) group over $\mathbb{Q}$, with a canonical mapping

$$\mathcal{G}_F \longrightarrow \Gamma_F, \qquad \Gamma_F = Gal(\bar{F}/F)$$

What is it !

2. Automorphic Representations (cuspidal), Langlands (1967) – letter to Weil.

G reductive algebraic gp over F.

Recall: $G(F)$ (discrete) $\subset G(\mathbb{A})$ (locally compact)

- Informal def: An automorphic rep. $\pi \in \Pi_{aut}(G)$ is an irreducible representation that "occurs in" the decomp. of $L^2(G(F) \backslash G(\mathbb{A}))$
- $\pi = \widetilde{\bigotimes_v} \pi_v$ – (restricted) direct product of irred rep^s $\pi_v \in \Pi(G_v)$ of the local completions $G_v = G(F_v)$, almost all of which,

  $\{\pi_v : v \notin S\}$, ← finite set of valuations including archimedean places S_∞

  are unramified.

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They should be governed / classified by certain loc. c^{pt} groups attached to F & its local completions:

$$
\begin{tikzcd}
L_{F_v} \arrow[r, two heads] \arrow[d, hook] & W_{F_v} \arrow[r] \arrow[d, hook] & \Gamma_{F_v} \arrow[d, hook] \\
L_F \arrow[r, two heads] & W_F \arrow[r] & \Gamma_F ,
\end{tikzcd}
$$

embeddings defined up to conjugacy

"Langlands groups"   Weil groups   Galois groups

for the local Langlands groups $L_{F_v} = \begin{cases} W_{F_v}, & v \in S_\infty, \\ W_{F_v} \times SU(2), & v \notin S_\infty, \end{cases}$

–well understood–, and the hypothetical global Langlands gp L_F (automorphic Galois group) –not well understood–, a precursor to the motivic Galois gp.

These locally compact gp^s are ingredients for a conjectured classification of irred. rep^s of $G_v = G(F_v)$, and aut. rep^s

$$\pi = \widetilde{\bigotimes_v}\, \pi_v, \qquad \pi_v \in \Pi_{irred}(G_v),$$

of $G(\mathbb{A})$ – very deep.

3. Conjectural candidate for aut. Galois gp L_F

2 ingredients

(i) An indexing set $\mathcal{C}_F = \{(G, c)\}$

where G/F is a quasisplit, simple, simply connected gp over F, and $c = \{c_v : v \notin S\}$ S_π is a concrete datum attached to a primitive aut. rep π of $G(\mathbb{A})$

(the set of conj. classes $\{c_v(\pi) = c(\pi_v) : v \notin S\}$ in the local Langlands "L-group" ${}^L G_v = \hat{G} \rtimes \Gamma_{F_v}$ that classify unramified rep^s $\{\pi_v : v \notin S\}$) "complex dual group"

(ii) For each $c \sim (G, c)$ in $\mathcal{C}_F$, an ext^n

$$1 \longrightarrow K_c \longrightarrow L_c \longrightarrow W_F \longrightarrow 1$$

of W_F (the global Weil gp) by a compact, simply connected gp K_c (a compact real form of the s.c. cover $\hat{G}_{sc}$ of the complex dual gp $\hat{G}$).

Note: Both (i) + (ii) depend on Langlands' conjectural Principle of Functoriality

(for the values of global L-functions $L^S(s, c, r)$ near $s = 1$, in order to define primitive aut rep π)

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Given (i) + (ii), we define Automorphic Galois group

$$L_F = \prod_{c \in \mathcal{C}_F} (L_c \to W_F), \quad \text{— fibre product over } W_F \text{ —}$$

a loc. c^{pc} gp, with local embeddings

$$\begin{tikzcd} L_{F_v} \arrow[r, hook, two heads] \arrow[d, hook] & W_{F_v} \arrow[d, hook] \\ L_F \arrow[r, hook, two heads] & W_F \end{tikzcd}$$

—

Conjecture 1: There is a bijection

$\{$ irred. rep^s $r: L_F \to GL(N, \mathbb{C})\} \to \{$cuspidal aut rep^s of $GL(N, \mathbb{A})\}$,

which is compatible with the localizations F_v.

Conjecture 2: A more general mapping

$$\{\phi : L_F \to {}^LG\} \xrightarrow{\sim} \{\Pi_\phi \subset \Pi_{aut, temp}(G)\}$$

for any G/F quasisplit, from bounded, global, Langlands parameters, to disjoint, global L-packets, whose union is $\Pi_{aut, temp}(G)$ — the set of tempered, aut. rep^s of $G(\mathbb{A})$.

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4. Conjectural candidate for motivic Galois gp $\mathcal{G}_F$

The const^{n} of L_F leads to a complex, ~~pro-alg.~~ (reductive) pro-alg.

gp $\mathcal{G}_F$, depending on the embedding $F \subset \mathbb{C}$, with maps

$$\begin{tikzcd}
L_F \arrow[r] \arrow[d] & W_F \arrow[r] \arrow[d] & \Gamma_F \arrow[d, equal] \\
\mathcal{G}_F \arrow[r] & \mathcal{T}_F \arrow[r] & \Gamma_F
\end{tikzcd}$$

— (loc. compact gp^s)

— complex, proalg. gp^s, whose reps. $r: \mathcal{G}_F \longrightarrow GL(n, \mathbb{C})$ would parametrize motives

fibre product of complexifications of those L_c of Hodge type – i.e. $\forall\, r: L_c \longrightarrow GL(n,\mathbb{C})$, & $\forall\, v \in S_\infty$, the restr^{n} of r to the sub gp $\mathbb{C}^* \subset W_{F_v} = L_{F_v} \hookrightarrow L_c$ is a Hodge structure

Langlands Taniyama group (Corvallis conf. proceedings, 1979) – the "algebraic hull of the motivic part of W_F"

This would be a general form of the Shimura-Taniyama-Weil conjecture, whose proof we can hope is tied up in the (future) proof of Langlands' Princ. of Functoriality

(rephrase) More work needed on the conjectural gp $\mathcal{G}_F$

(i) The motivic Gal. gp $\mathcal{G}_F$ is supposed to be defined over $\mathbb{Q}$, while our const^{n} here is for its gp of complex points. Determine its structure over $\mathbb{Q}$ explicitly from the families (G, c) in $\mathcal{G}_F$ that parametrize factors of L_F — or rather the subset that parametrize factors of $\mathcal{G}_F$.

(ii) Construct the cohomological realizations of $\mathcal{G}_F$, again explicitly in terms of the families (G, c) attached to $\mathcal{G}_F$.

(iii) Describe periods

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5. Mixed motives

We should also have the mixed motivic Galois group $\mathcal{G}_F \ltimes \mathcal{U}_F = \mathcal{G}_F^+$, with

$$\mathcal{G}_F \ltimes \mathcal{U}_F = \mathcal{G}_F^+ \longrightarrow \mathcal{G}_F \longrightarrow \mathcal{T}_F \longrightarrow \Gamma_F,$$

which plays role of $\mathcal{G}_F$ for singular / open varieties.

Problem: • Find explicit (conjectural) constⁿ for its unip. radical $\mathcal{U}_F$, as a prounipotent alg. gp with an action of $\mathcal{G}_F$.

– closely related to Beilinson conjectures & BSD-conj

• Could we also think of a purely automorphic version $L_F^+ = L_F \ltimes U_F$, constructed in terms of Eisenstein series?

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6. Exponential motives

An important generalization of motives, which

(i) gives periods (such as e, γ) not attached to ord. motives,

(ii) is needed for fund. physics (Kontsevich-Soibelman),

(iii) plays role of diff. eq^{tns} with irregular sing. points in Riemann-Hilbert corresp. (Deligne, Katz, Bloch, Esnault).

They give (conjectural) exponential motivic Galois gp $\tilde{\mathcal{G}}_F$ + its mixed ext^n $\tilde{\mathcal{G}}_F^+$, (ext^n of $\mathcal{J}_F$ defined by Greg Anderson)

with

$$\begin{tikzcd}
\tilde{\mathcal{G}}_F \ltimes \tilde{\mathcal{R}}_F = \hat{\mathcal{G}}_F^+ \arrow[r] \arrow[d] & \tilde{\mathcal{G}}_F \arrow[r] \arrow[d] & \hat{\mathcal{J}}_F \arrow[r] \arrow[d] & \Gamma_F \arrow[d, equal] \\
\mathcal{G}_F \ltimes \mathcal{R}_F = \mathcal{G}_F^+ \arrow[r] & \mathcal{G}_F \arrow[r] & \mathcal{J}_F \arrow[r] & \Gamma_F
\end{tikzcd}$$

Problem: Find explicit construction (conjectural) for $\tilde{\mathcal{G}}_F$ like that of $\mathcal{G}_F$. For this, would need further generalization of S-T-W conjecture – i.e. an exponential automorphic Galois group

$$\tilde{L}_F \longrightarrow E_F$$

What could it possibly be ??