Super Geometry and Supermoduli

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This is a very mild case of non commutative geometry.
Much of what can be done in commutative algebra and geometry carries over. But the main interest is in new phenomena that do not have straightforward 'bosonic' analogues.

Outline

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(1) Background:

- Background: particles
- Background: string perturbation theory
- Background: super string perturbation theory

(2) Supermanifolds

(3) Super symmetric manifolds

(4) Non splitness of supermoduli

(5) Atiyah classes vs obstructions

(6) Ramond boundary

(7) Further topics

Background: particles

bosons
fermions

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In its simplest form, "**super**" refers to: a $\mathbf{Z}$-graded ring A, which is graded-commutative:

$$b \cdot a = (-1)^{deg(a)deg(b)} a \cdot b,$$

or to its geometric spectrum. In fact, this needs only $\mathbf{Z}/2$-grading.

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Each: an integral over all metrics (and more).
Lots of symmetry =>
integrand depends only on the complex structure.
So: **amplitudes = integrals over** $M_{g,n}$.
($M_{g,n}$ = moduli space of complex structures.)

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More precisely, $\pi : \mathcal{C}_g := M_{g,n+1} \to M_{g,n}$ is the universal curve;
$\omega := \omega_{\mathcal{C}_g/M_{g,n}}$ is the bundle of holomorphic 1-forms on the moving curve;
$\omega^{\otimes i}$ is the bundle of holomorphic i-uple differentials;
$V_i := \pi_*(omega^{\otimes i})$ the vector bundle on $M_{g,n}$ of all global holomorphic i-uple differentials;
L_i is the detereminant of V_i, aka determinant of cohomology of $\omega^{\otimes i}$.

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d is the dimension of space-time.
Mumford's theorem: $L_2 = (L_1)^{13}$, independent of g.
$=>$ Bosonic string is consistent in $d = 26$ dimensions.
Also need to compactify M_g: Deligne-Mumford $\overline{M_g}$.

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There are actually two different kinds of punctures: Neveu-Schwarz and Ramond.

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It involves a ratio $(L_{\frac{3}{2}})/(L_{\frac{1}{2}})^{d/2}$.
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Also need to compactify $\mathcal{M}_g$: super Deligne-Mumford $\overline{\mathcal{M}_g}$.

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The actual calculation?

Elementary at "tree level" ($g = 0$) and elliptic ($g = 1$).

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- Push the integrand forward from $\mathcal{M}_g$ to M_g
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This led to very intensive work, trying to push this up to higher and higher genus. (up to 6? ∞?)
[DW]: this must fail at $g = 5$, maybe sooner.
Reason: there is no projection $\mathcal{M}_g \to M_g$.

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$M =$ body, $V =$ soul.
$dim(S) = (m|n)$ if $m = dim(M), n = rank(V)$.
Can define T_S, a (super) vector bundle on S of rank $(m|n)$.
Its restriction to M splits into even and odd parts:
$T_{S,+} = T_M, T_{S,-} = V$.

Supermanifolds

The **split** supermanifold $S = S(M, V)$ is $S = (M, \mathcal{O}_S)$ where

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A supermanifold $S = (M, \mathcal{O}_S)$ is **split** if $S = S(M, V)$ for some vector bundle V on M. It is **projected** if there is a projection $S \to M$.

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Is every supermanifold split and/or projected?
There is an obstruction class in cohomology:

$$\omega \in H^1(M, T_M \otimes \wedge^2 V^*)$$

Every C^∞ manifold is split. The obstruction class vanishes because the sheaf is fine. (There is a partition of unity.)
It can be non-0 in the holomorphic world. Which is where physics needs it.

Super symmetric manifolds

A supersymmetric manifold is a supermanifold $S = (M, \mathcal{O}_S)$ that has odd symmetries relating its even and odd tangent spaces. In particular,the even and odd tangent bundles are related: $V = T_{S,-}$ is a direct sum $V \cong \mathcal{S}^{\mathcal{N}}$ of $\mathcal{N}$ copies of a spinor bundle of T_M.
It is a much tighter structure than a supermanifold.
In particular, $dim(S) = (m|n)$ with $n = \mathcal{N}2^{m'}$, where $m' \cong [(m-1)/2]$.
First example: $m = \mathcal{N} = n = 1$ in the holomorphic world: a Super Riemann Surface.
Spinors = square root of T_M = spin structure, theta characteristic.

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$v : f(x) + \theta g(x) \mapsto g(x) + \theta f'(x)$
$v^2 : f(x) + \theta g(x) \mapsto f'(x) + \theta g'(x)$
$v^2 = \partial_x$.

Super Riemann Surfaces and supermoduli space

Key point: can do algebraic geometry with SRSs.

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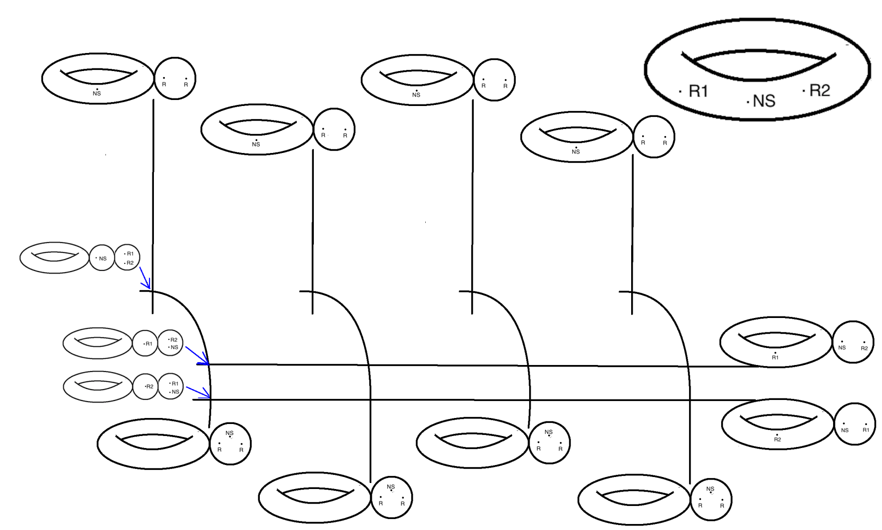
Underlying manifold: SM_g is the modui space of Riemann surfaces with a spin structure.

Super Riemann Surfaces and supermoduli space

Two types of puncture: N and R.
Neveu-Schwarz puncture lives at a point (a submanifold of dimension $(0|0)$), can be forgotten.
Ramond puncture lives on a divisor (a submanifold of dimension $(0|1)$), cannot be forgotten.
Local picture: $v := \partial_\theta + x\theta\partial_x$.
$v^2 = x\partial_x$: v is maximally non integrable except where $x = 0$.
DM compactification: two types of nodes
Gluing rules


NS
R
R1
R2
NS
R1
R2
R2
NS
R1
NS
R1
NS
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NS
R1

Non splitness of supermoduli

[DW1]: $\mathcal{M}_g$ (and others) are non split and non projected, for $g \geq 5$.
(Note: the analogous question for the DM compactification is easier.)
Idea: find compact curve C in M_g,
described as a family of branched covers.
Lift it to $\mathcal{M}_g$
Calculate: the obstruction, restricted to a neighborhood of C, is $\neq 0$.
Lift requires: all branching **odd**.

Atiyah classes vs obstructions

Atiyah class := obstruction to existence of a connection
On a manifold: in $H^1(X, \wedge^2 T^*X \otimes TX)$
On a vector bundle V : $H^1(X, T^*X \otimes V^* \otimes V)$
On a principal G-bundle P : $H^1(X, T^*X \otimes ad(P))$
(Case of a manifold: $V = T^*X$ but one component vanishes due to torsion freeness)
Bundles on supermanifolds have super Atiyah classes
[DW2]: The super Atiyah class of a supermanifold $S = (M, \mathcal{O}_S)$ has 3 components:

- the Atiyah class of M, i.e. of T_+S
- the Atiyah class of $V = T_-S$
- the obstruction class to splitting S.

Measure on moduli

The Mumford isomorphism;

$$L_2 \cong (L_1)^{13}$$

allows us to convert the natural metric on 1-forms on a Riemann surface C to a measure on the cotangent space $T^*M_g = H^1(C, K_C^2)$ to moduli.

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The construction generalizes to give a measure on moduli of punctured Riemann surfaces.

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Super Mumford gives a measure away from the bad locus.

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The case $r = 1$ remains open.

Morals:

- Origins in perturbative super string theory
- Supermanifolds vs supersymmetric manifolds
- Split vs non-split supermanifolds, e.g. M_g
- Obstruction theory, analogous to Atiyah classes
- Rich theory of supermoduli, DM compactifications
- R vs NS punctures and nodes
- The natural measure is defined via the Mumford isomorphism away from a bad locus, and almost always extends to the entire moduli space.

Thank you!!!