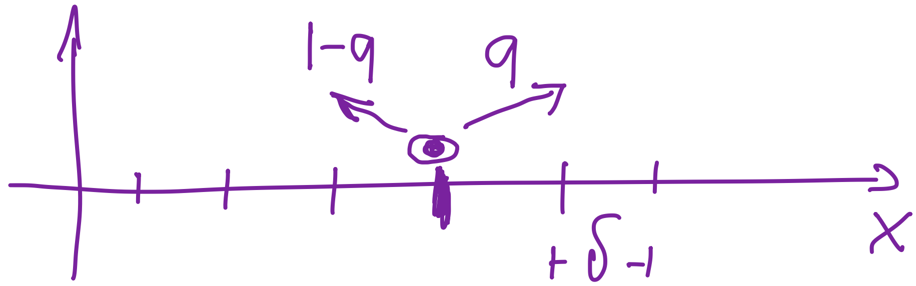
Hillen's lecture Oct 29, 2025

(4.1) Random walk on a grid



$P(X_1$ is here$) = 0$

q: prob. to move right

$1-q$: prob to move left

[→ Zauderer, PDE's of Applied Math 1983

→ Okubo, Levin, Diffusion and Ecological Problems, 2001]

X_n: random variable of position after n steps

$$E(X_1) = \sum_y x\, P(x=y) = q\delta + (1-q)(-\delta)$$

$$= \delta(2q-1)$$

$$E(X_n) = \delta(2q-1) + E(X_{n-1}) = \dots$$

$$= n\cdot\delta(2q-1)$$

$q = \frac{1}{2}$: $E(X_n) = 0$

$q > \frac{1}{2}$: $E(X_n) > 0$ movement to the right

$q < \frac{1}{2}$: $E(X_n) < 0$ —— " —— left

Variance: $V(X_1) = E(X_1^2) - E(X_1)^2$

$E(X_1^2) = q\delta^2 + (1-q)(-\delta)^2 = \delta^2$

$V(X_1) = \delta^2 - \delta^2(2q-1)^2 = \delta^2(1 - 4q^2 + 4q - 1)$

$= 4q\delta^2(1-q)$

independent random walkers

$V(X_n) = n \cdot V(X_1) = 4nq\delta^2(1-q)$

Compare to measurements

$\underline{\overline{X}_n} = \underline{n \cdot \delta(2q-1)}$ $\quad \underline{V(X_n)} = 4n\delta^2 q(1-q)$

time step τ: n time steps have time

$t = n\tau$ or $n = \frac{t}{\tau}$

mean velocity $c = \frac{\overline{X}_t}{t} = \underline{\frac{\delta}{\tau}(2q-1)}$

Diffusion const. $D := \frac{1}{2}\frac{V(X_t)}{t} = \frac{2\delta^2}{\tau} q(1-q)$

(4.2) Master equation approach.

$p(x,t)$: prob. to find a random walker who started at 0 at location x at time t.

Master equation (balance of probabilities)

$$\rightarrow \quad p(x, t+\tau) = q\, p(x-\delta, t) + (1-q)\, p(x+\delta, t)$$

Use Taylor expansion for τ, δ small

$$p(x, t+\tau) = p(x,t) + \frac{\partial}{\partial t} p(x,t)\,\tau + \frac{\partial^2}{\partial t^2} p(x,t)\frac{\tau^2}{2} + \dots$$

$$= q\left(p(x,t) - \frac{\partial}{\partial x} p(x,t)\,\delta + \frac{\partial^2}{\partial x^2} p(x,t)\frac{\delta^2}{2} - \dots\right)$$

$$(1-q)\left(p(x,t) + \frac{\partial}{\partial x} p(x,t)\,\delta + \frac{\partial^2}{\partial x^2} p(x,t)\frac{\delta^2}{2} + \dots\right)$$

change notation $\frac{\partial}{\partial t} p = p_t \qquad \frac{\partial^2}{\partial x^2} p = p_{xx}$

$$\cancel{p} + \tau p_t + \underbrace{\frac{\tau^2}{2} p_{tt}}_{\text{small} \simeq 0} = \cancel{p} + \delta(1-2q)\, p_x + \frac{\delta^2}{2} p_{xx} + \underbrace{\dots}_{\simeq 0}$$

Then $p_t = \underbrace{\frac{\delta}{\tau}(1-2q)}_{=-c} p_x + \frac{\delta^2}{2\tau} p_{xx}$

$$\frac{\delta^2}{2\tau} = \frac{D}{4q(1-q)}$$

Now we study different scalings:

$\delta \to 0$, $\tau \to 0$, $q \to \frac{1}{2}$

a) $\frac{\delta}{\tau} \to \alpha = \text{const.}$ then $\frac{\delta^2}{2\tau} = \frac{\delta}{2}\frac{\delta}{\tau} \to 0.$

Then $\boxed{p_t + c p_x = 0}$ $\quad c = \alpha(1-2q)$

a pure drift model, advection model

b) $\frac{\delta^2}{\tau} \to \beta$ const.

(b.1) if $q \neq \frac{1}{2}$ then $\frac{\delta}{\tau} = \frac{1}{\delta}\frac{\delta^2}{\tau} \to \infty$

doesn't work

(b.2) if $q \to \frac{1}{2}$ then $\underset{\downarrow \atop \infty}{\frac{\delta}{\tau}}\underset{\downarrow \atop 0}{(1-2q)} \to -c$ bdd.

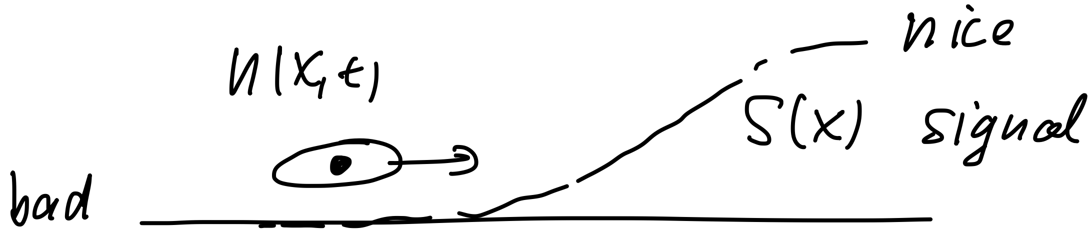
$$\frac{D}{4q(1-q)} \to D$$

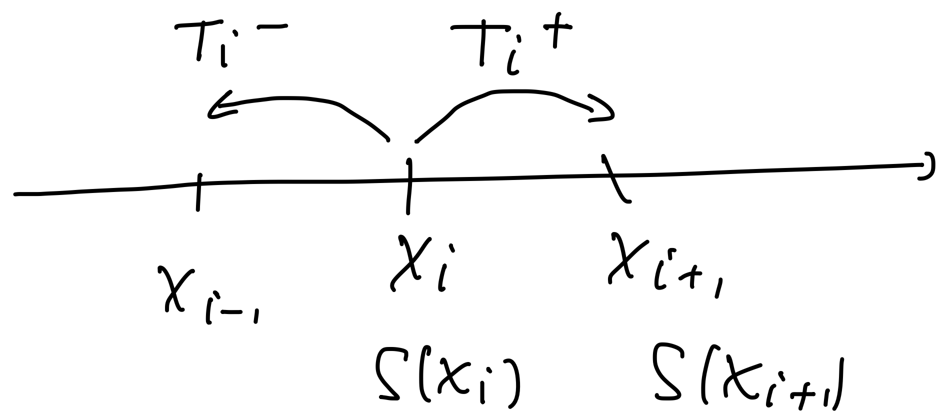
and we get $\boxed{p_t = -c p_x + D p_{xx}}$ ☺

an advection diffusion equation.

where $c = \frac{\overline{X_t}}{t}$ $\qquad D = \frac{1}{2}\frac{V(X_t)}{t}$

(4.3) Application to chemotaxis





Master equation:

$$\frac{du_i}{dt} = T_{i-1}^+ u_{i-1} + T_{i+1}^- u_{i+1} - (T_i^+ + T_i^-)u_i$$

How to model the transition probabilities?

$$T_i^+ = \alpha + \beta\left(S(x_{i+1}) - S(x_i)\right)$$

$$T_i^- = \alpha + \beta\left(S(x_{i-1}) - S(x_i)\right)$$

abbreviate $S_i = S(x_i)$

$$\frac{du_i}{dt} = u_{i-1}\left(\alpha + \beta(S_i - S_{i-1})\right) + u_{i+1}\left(\alpha + \beta(S_i - S_{i+1})\right)$$

$$- u_i\left(2\alpha + \beta(S_{i+1} - S_i + S_{i-1} - S_i)\right)$$

$$= \alpha(u_{i-1} - 2u_i + u_{i+1})$$

$$- \beta\left((u_{i+1} + u_i)(S_{i+1} - S_i) - (u_{i-1} + u_i)(S_i - S_{i-1})\right)$$

$$= \alpha \Delta x^2 \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2}$$ 2nd derivative

$$- 2\beta\Delta x^2 \frac{\frac{u_{i+1}+u_i}{2}\frac{S_{i+1}-S_i}{\Delta x} - \frac{u_i + u_{i-1}}{2}\frac{S_i - S_{i-1}}{\Delta x}}{\Delta x}$$

derivative

$$\xrightarrow{\Delta x \to 0} D u_{xx} - \chi(u S_x)_x$$

$\lim \alpha \Delta x^2 = D$ $\qquad$ $\lim 2\beta \Delta x^2 = \chi$

The full chemotaxis model has two equations:

$$\boxed{\begin{aligned} u_t &= D u_{xx} - \chi (u S_x)_x \\ S_t &= D_S S_{xx} + \alpha u - \beta S \end{aligned}}$$

Keller-Segel eq for chemotaxis.

see [→ Hillen, Painter, The User's guide of chemotaxis, 2009, JoMB].

(4.5) Reaction-Diffusion Equations

→ J.D. Murray Math Biol.

→ N. Britton. React. Diff. Eq's

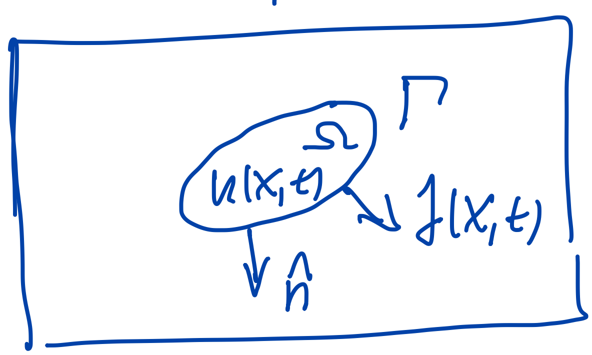
→ Okubo, Levin

$$u_t = D u_{xx} + f(u)$$

Derivation

test domain Ω

with boundary Γ



balance all possible changes in Ω.

$$\begin{pmatrix}\text{change of } u\\ \text{in } \Omega\end{pmatrix} = \begin{pmatrix}\text{flux through}\\ \Gamma\end{pmatrix} + \begin{pmatrix}\text{change due to}\\ \text{birth, death}\\ \text{interactions}\end{pmatrix}$$

$$\frac{d}{dt}\int_{\underline{\underline{\Omega}}} u(x,t)\,dx = -\int_{\underline{\underline{\Gamma}}} J(x,t)\cdot\hat{n}\,dS + \int_{\underline{\underline{\Omega}}} f(u(x,t))\,dx$$

Use divergence theorem

$$\int_{\Gamma} J \cdot \hat{n} \, ds = \int_{\Omega} \operatorname{div} J(x,t) \, dx$$

Then

$$\int_{\Omega} \frac{\partial}{\partial t} u(x,t) + \operatorname{div} J(x,t) - f(u(x,t)) \, dx = 0$$

for all Ω

$$\Rightarrow \quad \frac{\partial}{\partial t} u(x,t) + \operatorname{div} J(x,t) - f(u(x,t)) = 0 \quad a.e.$$

say it is true for all x !

Q1 How to choose J ?   next lecture !