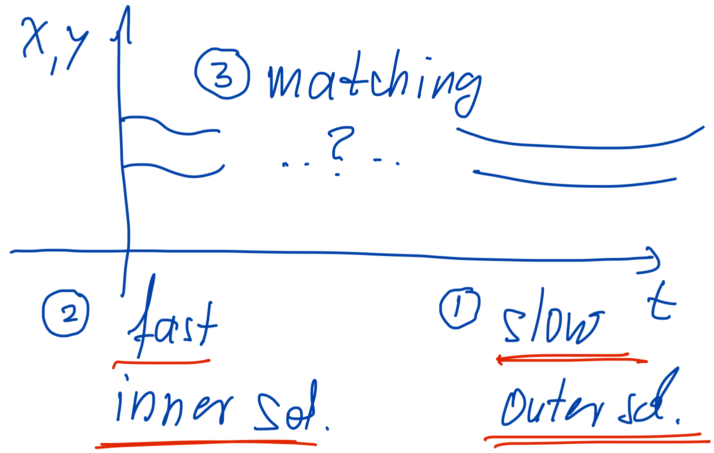
$$\left.\begin{aligned} \frac{d}{d\tau} x(\tau) &= -x + y(\alpha + x) \\ \varepsilon \frac{d}{d\tau} y(\tau) &= x - y(\varkappa + x) \end{aligned}\right\} \quad (1)$$

initial conditions $x(0) = \bar{x}$, $y(0) = \bar{y}$.

Singular perturbation:



① Outer solution.

Assume $S_0 \gg e_0$

$$\varepsilon = \frac{e_0}{S_0} \ll 1.$$

expansion in ε

$$x(\tau) = \sum_{n=0}^{\infty} \varepsilon^n x_n(\tau) \qquad y(\tau) = \sum_{n=0}^{\infty} \varepsilon^n y_n(\tau)$$

Substitute into (1):

$$\sum_{n=0}^{\infty} \varepsilon^n \dot{x}_n(\tau) = -\sum_n \varepsilon^n x_n + \sum_{n=0} \varepsilon^n y_n \left(\alpha + \sum_n \varepsilon^n x_n\right)$$

$$\sum_{n=0} \varepsilon^{n+1} \dot{y}_n = \sum_n \varepsilon^n x_n - \left(\sum_n \varepsilon^n y_n\right)\left(\varkappa + \sum_n \varepsilon^n x_n\right)$$

Compare orders of ε.

$$\varepsilon^0 : \quad \boxed{\left.\begin{aligned} \dot{x}_0 &= -x_0 + y_0(\alpha + x_0) \\ 0 &= x_0 - y_0(\varkappa + x_0) \end{aligned}\right\} \quad (2)} \qquad \text{slow}$$

the leading order approximation

$$\varepsilon^1: \quad \dot{x}_1 = -x_1 + \alpha y_1 + x_0 y_1 + x_1 y_0$$
$$\dot{y}_0 = x_1 - \underline{y_0 x_1 - y_1(\varkappa + x_0)}$$

Could continue to x_2, x_3, etc. ....

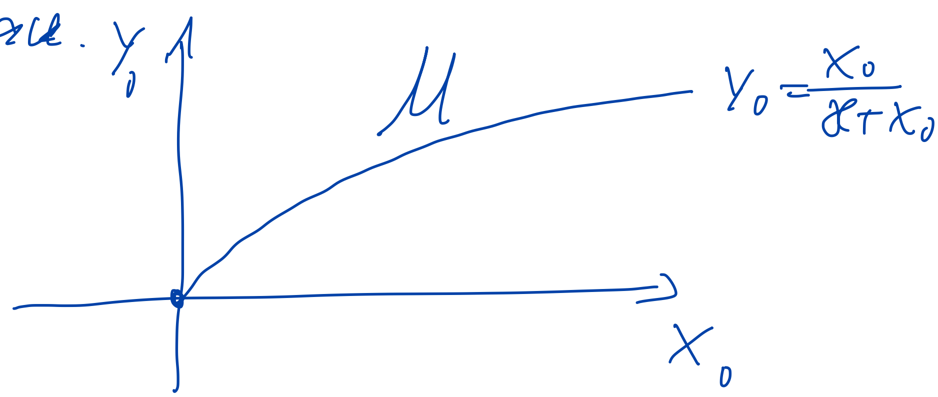
Lets solve

$$(\varepsilon^0 y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots)\cdot(\varkappa + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)$$

$$\varkappa y_0 + y_0 x_0 + \underline{\varepsilon y_0 x_1} + \varepsilon^2 y_0 x_2 + \dots$$
$$+ \underline{\varepsilon y_1 \varkappa} + \underline{\varepsilon y_1 x_0} + \varepsilon^2 y_1 x_1 + \varepsilon^3 y_1 x_2 + \dots$$

Solve 2nd eq of (2): $\quad y_0 = \dfrac{x_0}{\varkappa + x_0}$

This means $(x_0(\tau), y_0(\tau))$ lie on a manifold in phase space.



$$M = \left\{ (x, y) : y = \frac{x_0}{\varkappa + x_0} \right\} \quad \text{slow manifold.}$$

The first equation of (2) lives on $\mathcal{M}$.

$$\dot{x}_0 = -x_0 + \frac{x_0}{\varkappa + x_0}(\alpha + x_0)$$

$$= \frac{-x_0(\varkappa + x_0) + x_0(\alpha + x_0)}{\varkappa + x_0}$$

$$= \frac{(\alpha - \varkappa)x_0}{\varkappa + x_0} \qquad q = \alpha - \varkappa$$

$$= \frac{q\, x_0}{\varkappa + x_0}$$

Use separation of variables:

$$\dot{x}_0 \left(\frac{\varkappa + x_0}{x_0}\right) = q$$

$$\varkappa \frac{\dot{x}_0}{x_0} + \dot{x}_0 = q \qquad \text{integrate from } 0 \text{ to } \tau$$

$$\varkappa \ln x_0(\tau) - \varkappa \ln x_0(0) + x_0(\tau) - x_0(0) = q\tau$$

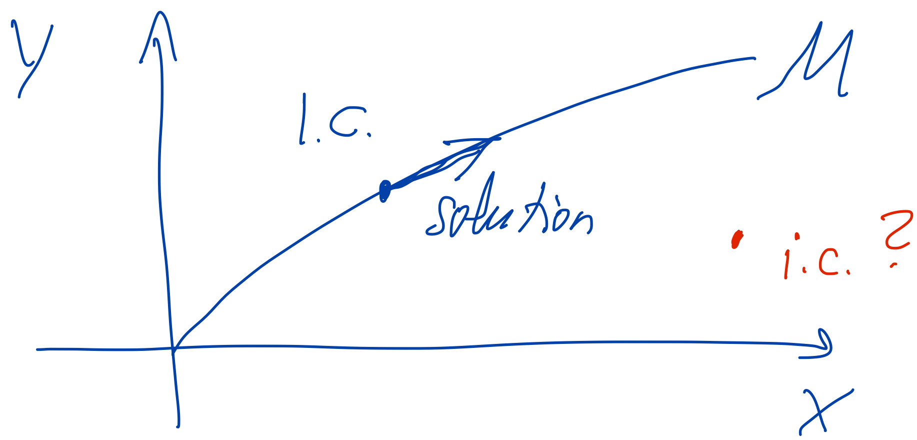
$$\boxed{x_0(\tau) + \varkappa \ln x_0(\tau) = x_0(0) + \varkappa \ln x_0(0) + q\tau}$$

$$\bar{x} \qquad\qquad \bar{x}$$

$$= \bar{x} + \varkappa \ln \bar{x} + q\tau$$

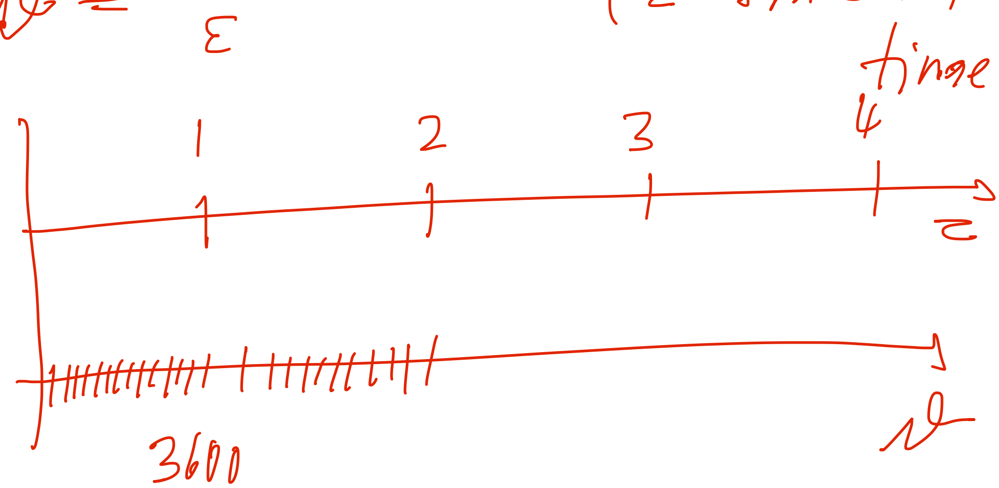
On $\mathcal{M}$: $y_0(0) = \bar{y} = \dfrac{\bar{x}}{\varkappa + \bar{x}}$

The choice of $\bar{y}, \bar{x}$ is restricted. That's why its called singular!



② Inner Solution

We rescale time to investigate a short time scale $\vartheta = \frac{\tau}{\varepsilon}$ (ε small)



$\varepsilon = 10^{-2}$

$$\frac{d}{d\vartheta} = \frac{d}{d\tau}\frac{d\tau}{d\vartheta} = \varepsilon \frac{d}{d\tau}$$

use this in (1).

$$\left.\begin{aligned}\frac{d}{d\vartheta}x &= \varepsilon\left(-x + y(\alpha+x)\right)\\ \frac{d}{d\vartheta}y &= x - y(\varkappa+x)\\ x(0) = \bar{x}, &\quad y(0) = \bar{y}\end{aligned}\right\}\quad (3)$$

Again expansion in ε

$$x = \sum_{j=0}^{\infty} \varepsilon^j x_j^{\vartheta} \qquad y = \sum_{j=0}^{\infty} \varepsilon^j y_j^{\vartheta}$$

$$\varepsilon^0): \quad \left.\begin{aligned}\dot{x}_0^{\vartheta} &= 0\\ \dot{y}_0^{\vartheta} &= x_0^{\vartheta} - y_0^{\vartheta}(\varkappa + x_0^{\vartheta})\end{aligned}\right\}\ \text{inner solution}$$

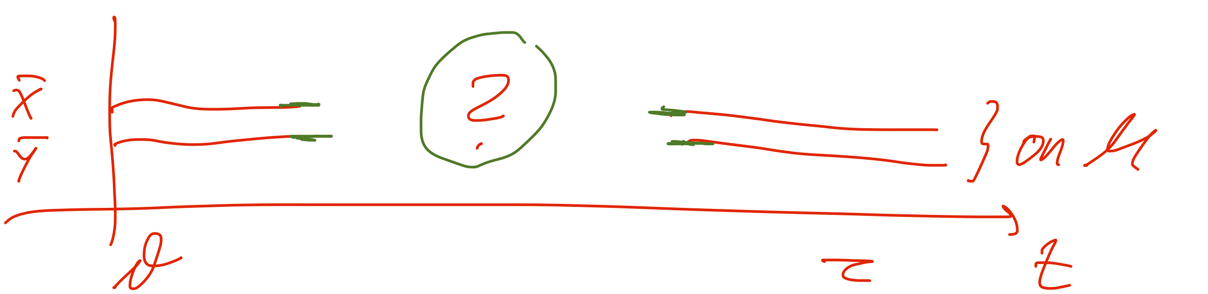
solve this $\quad x_0^{\vartheta}(\tau) = \text{const.} = \bar{x}$

solve with variation of parameters

$$y_0(\vartheta) = \frac{\bar{x}}{\varkappa+\bar{x}}\left(1 - e^{-(\varkappa+\bar{x})\vartheta}\right) + e^{-(\varkappa+\bar{x})\vartheta}\,\bar{y}$$

Then at $\vartheta = 0$: $\quad x_0^{\vartheta}(0) = \bar{x}$

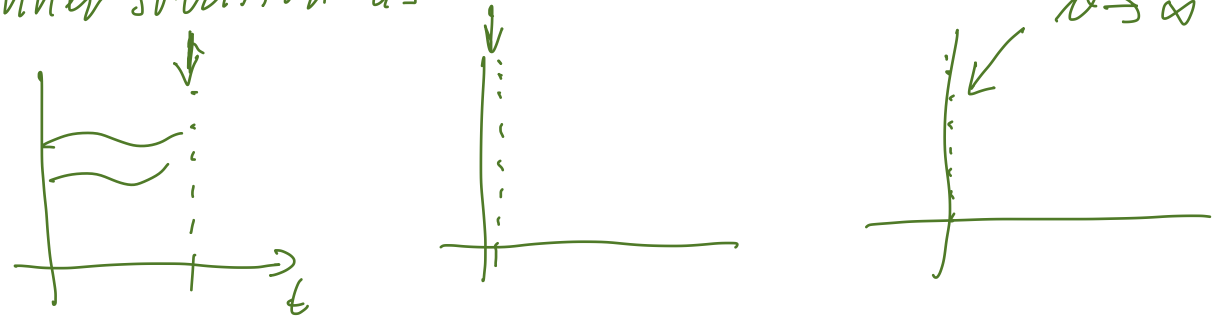
$\qquad\qquad\qquad\quad y_0^{\vartheta}(0) = \bar{y} \quad \checkmark$



③ Matching.

inner solution as $\varepsilon \to 0$.

$\lim_{\vartheta \to \infty} X_0(\vartheta)$

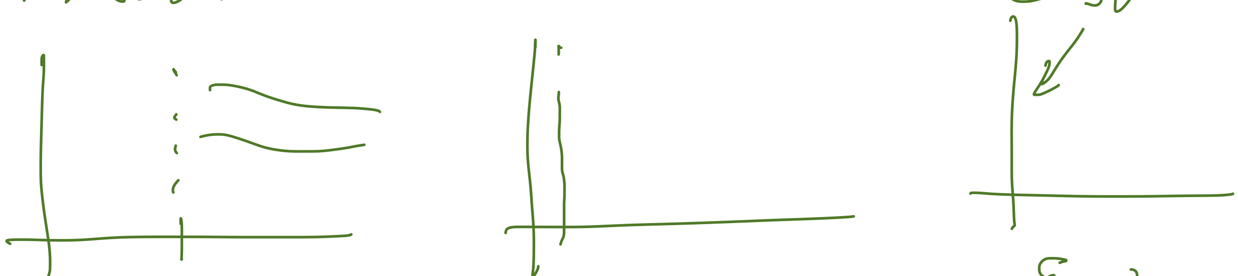


$\varepsilon > 0$ $\quad$ $\varepsilon \simeq 0$ $\quad$ $\varepsilon = 0$

$$\left.\begin{aligned} \lim_{\vartheta \to \infty} X_0(\vartheta) &= \bar{X} \\ \lim_{\vartheta \to \infty} Y^0(\vartheta) &= \frac{\bar{X}}{\mathcal{K} + \bar{X}} \end{aligned}\right\} (*)$$

outer solution

$\lim_{\tau \to 0} x_0(\tau)$



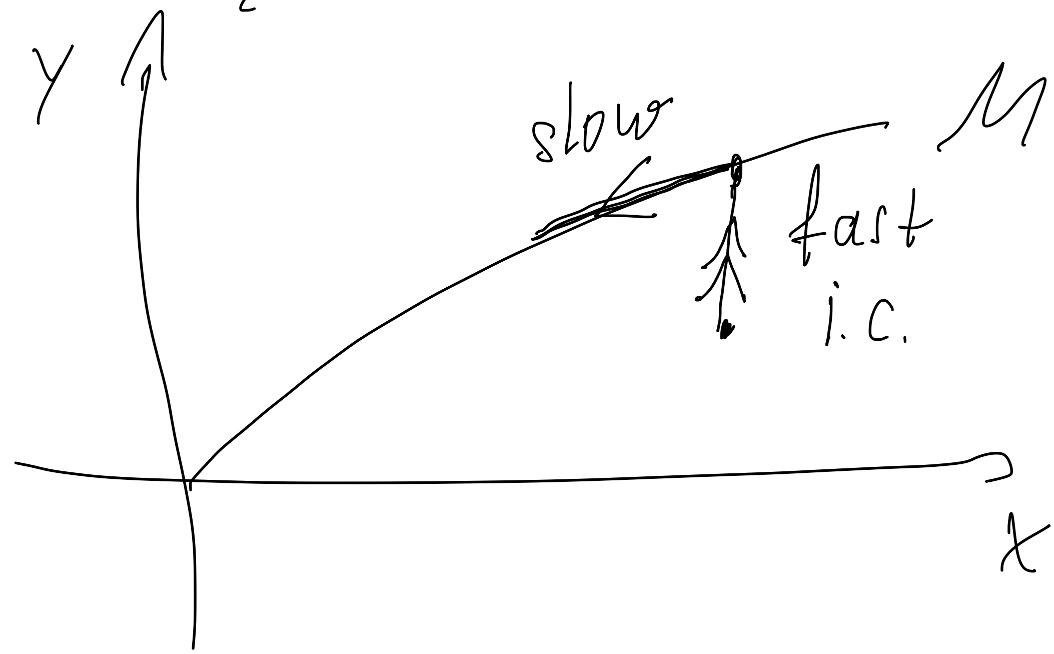
$\varepsilon > 0$ $\quad$ $\varepsilon \simeq 0$ $\quad$ $\varepsilon = 0$

$$\left.\begin{aligned} \lim_{\tau \to 0} x_0(\tau) &= \bar{X} \\ \lim_{\tau \to 0} y_0(\tau) &= \frac{\bar{X}}{\mathcal{K} + \bar{X}} \end{aligned}\right\} \text{is exactly } (*)$$

In general

$$\boxed{\lim_{\vartheta \to \infty} \text{"inner sol"} = \lim_{\tau \to 0} \text{"outer sol."}}$$

In phase space



A bit more theory. (Fenichel Theorems)

[→ Hek. J. Math. Biol., 2010, 60: 347–386]

ODE system: $\dot{u} = f(u, v, \varepsilon)$, $\dot{v} = \varepsilon g(u, v, \varepsilon)$ (1)

fast →

rescaled system $\tau = \varepsilon t$

slow → $\varepsilon u' = f(u, v, \varepsilon)$, $v' = g(u, v, \varepsilon)$ (2)

leading order fast

$$\left.\begin{aligned} \dot{u} &= f(u, v, 0) \\ \dot{v} &= 0 \end{aligned}\right\} \quad (3)$$

leading order slow

$$\left.\begin{aligned} 0 &= f(u, v, 0) \\ v' &= g(u, v, 0) \end{aligned}\right\} \quad (4)$$

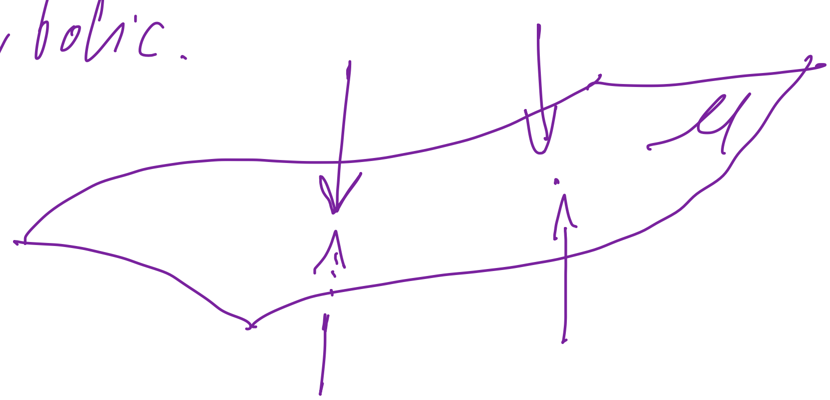
slow manifold

$$M = \{ (u, v) : f(u, v, 0) = 0 \}$$

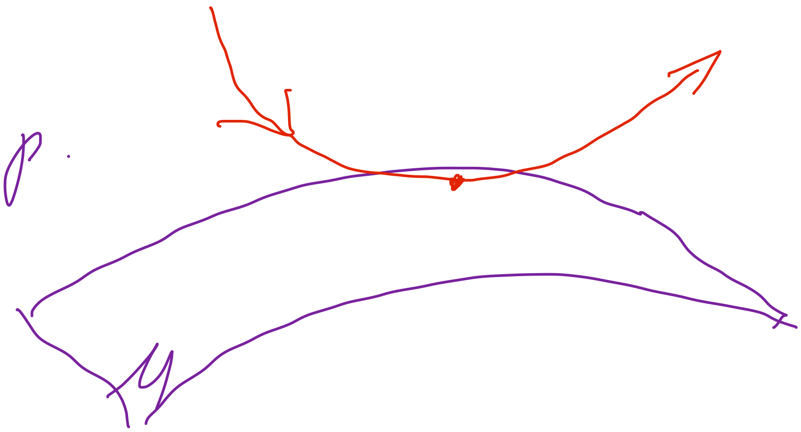
f, g are C^1 function

Def: The slow manifold is called normally hyperbolic if the eigenvalues of the Jacobian $\left.\frac{\partial f}{\partial u}(u, v, 0)\right|_M$ are uniformly hyperbolic.

norm. hyp.

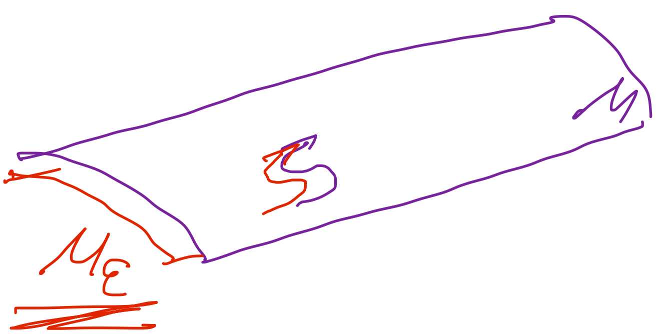


not norm. hyp.



Fenichel Theorem 1. Suppose M is norm. hyp. and compact. Then for ε small there exists a manifold M_ε that is

1) $O(\varepsilon)$-close to M
2) Diffeomorphic to M
3) locally invariant for the full system



Fenichel Theorem 2. In addition there exist stable and unstable manifolds $W^s(M_\varepsilon)$, $W^u(M_\varepsilon)$ that are $O(\varepsilon)$-close to $W^s(M)$, and $W^u(M)$, respectively.