Hillen lecture Oct 17, 2024

likelihood $\mathcal{L}(\text{data} \mid p)$

(3.5) AIC Akaike Information Criterion

Which model is the best?

$p = (A, B, C, D)$

If two models have the same number of param.

$$\mathcal{LL}(\hat{p}_{model 1}) > \mathcal{LL}(\hat{p}_{model 2})$$

then model 1 is better.

If model 2 has more param. it is punished!

n_p = number of param.

$$AIC = 2\mathcal{LL}(\hat{p}) - 2n_p$$

The larger AIC, the better the model

If the number of data points is small

$N \leq 40$, then we use a corrected AIC

$$\boxed{AIC_c = 2\mathcal{LL}(\hat{p}) - 2n_p \frac{N}{N - n_p - 1}}$$

Example. Paramecium growth data from Gausse.

de Vries et. al. p 308.

3 Models			# param.

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logistic	$\dot{x} = rx\left(1 - \frac{x}{K}\right)$	2
Gompertz	$\dot{x} = -rx \ln\left(\frac{x}{K}\right)$	2
Bernoulli	$\dot{x} = rx\left(1 - \left(\frac{x}{K}\right)^{\theta}\right)$	3

$\theta \leq 1$

least squares fit, compute $\mathcal{LL}$

	$\mathcal{LL}$	AIC_c
logistic	-63.7	-131.9
Gompertz	-74.4	-153.4
Bernoulli	-60.3	-127.8 ⟸

Now AIC_c for 24 measurements

(typo in book: AIC_c logistic -313.9)

(3.6) Likelihood ratio test

Test for nested models.

Nested: one model is a special case of another. Logistic model is nested inside the Bernoulli model for $\theta = 1$.

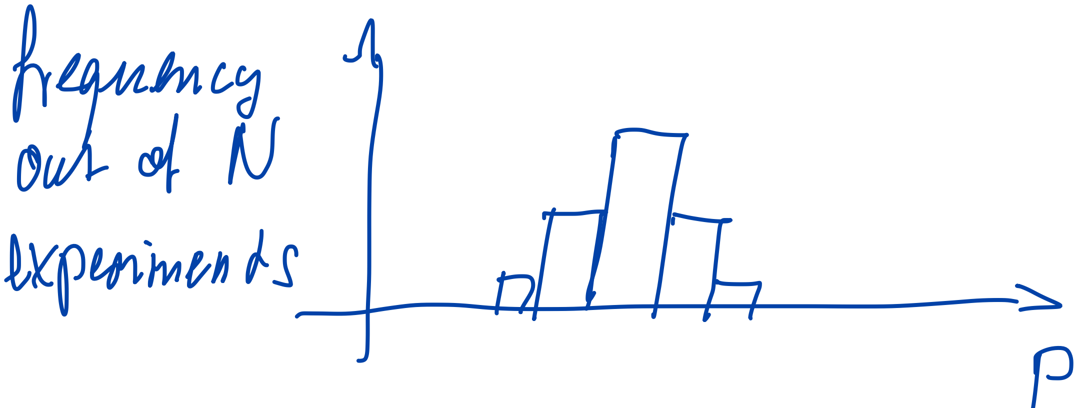
$M(0)$ the special model

$M(p)$ is the more general model

$$\lambda = -2\left(\mathcal{LL}(p) - \mathcal{LL}(0)\right)$$

Assumptions:

- sample size large enough, $N \geq 25$
- the fitted param. p are normally distributed.



Now: λ is $\chi^2_{n,\gamma}$ distributed, where n is the difference in the numbers of param.

Statistical test

$(\log\text{-}\theta = 1)$

null hypothesis: H_0 $\quad p = 0$

alternative hypothesis: H_p $\quad p \neq 0$ $\quad$ (Bernoulli $\theta \neq 1$)

Take a tolerance level α (0.05)

$\gamma = 1 - \alpha$ find $\chi^2_{n,\gamma}$ value and compute the confidence interval.

If 0 is not in the conf. interval then we reject H_0.

The P-value is the α value for which

$\lambda = \chi^2_{n,\gamma}$

$P > 0.05$	n.s.
$P \leq 0.05$	*
$P \leq 0.01$	**
$P \leq 0.001$	***

P is the prob. to falsely reject the null hypothesis.

For the paramecium data

$$\lambda = -2\left(\mathcal{LL}(\text{Bernoulli}) - \mathcal{LL}(\text{logistic})\right)$$

$$= 6.76$$

and we $\chi^2_{1,\gamma}$ distribution

Find confidence interval

$$\mathcal{LL}(p_{low}) - \mathcal{LL}(p_{up}) = \mathcal{LL}(p) - \chi^2_{1,\gamma}$$

For α values 0.1, 0.05, 0.025, $\underline{0.01}$

we have $\underline{\chi^2_{1,\gamma} < \lambda = 6.76}$

This means for these α values $0 \notin [p_{low}, p_{up}]$

Hence the probability to erroneously reject the logistic model is $P = 0.01$.

(3.6) Sensitivity Analysis

Given $M(p)$ a model that depends on parameters $p \in \mathbb{R}^n$ and gives output ϕ in $\mathbb{R}^m$. Let π be the log likelihood estimator (base parameter)

Def. The sensitivity of an outcome $\phi \in \mathbb{R}$ on a parameter $p \in \mathbb{R}$ is defined as

$$S(\phi) = \frac{\text{relative change of the outcome}}{\text{relative change of parameter}}$$

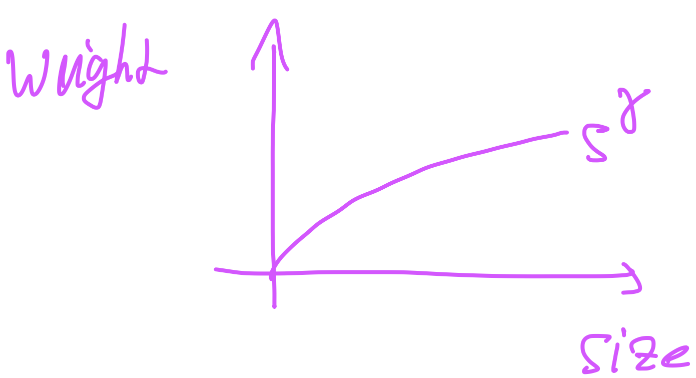
$$= \frac{\Delta\phi / \phi}{\Delta p / p} \qquad \longleftarrow \text{ to remember}$$

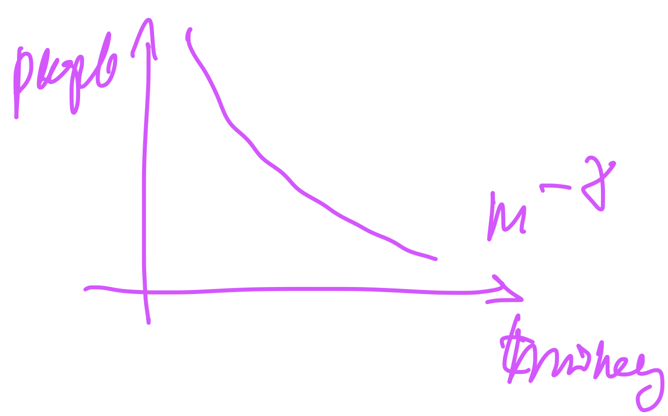
$$= \frac{\Delta\phi}{\Delta p} \cdot \frac{p}{\phi} \qquad \longleftarrow \text{ todo}$$

$$= \frac{\Delta\phi / \Delta p}{\phi / p} = \frac{\partial\phi / \partial p}{\phi / p} \qquad \text{differential form}$$

$$= \frac{\partial \ln \phi}{\partial \ln p} \; ?$$

Powerlaws!





power law: $\phi(p) = c p^{\alpha}$

$$S(\phi) = \frac{\frac{\partial \phi}{\partial p}}{\frac{\phi}{p}} = \frac{c \alpha p^{\alpha-1}}{c p^{\alpha} / p} = \alpha$$

"powerlaw": $\phi(p) \approx p^{\text{"sensitivity"}}$!

4) Spatial Movement

(4.1) Measurements

A Population level: X_t random variable for the location at time t. We assume

A1 mean location $\bar{X}_t = E(X_t)$

A2 mean square distance

$$(\bar{X}_t - X_t)^2 = V(X_t)$$

B Individual level follow individual path

B1 mean speed γ

B2 turning rate μ

(B3) distribution of new directions

$T(v, v')$. v, v' velocities

Q1 How to model measurements of type (A) or type (B).

Q2 How are type (A) and (B) models related?