Hillen's lecture, Oct. 31, 2024

$$\frac{\partial}{\partial t} u(x,t) + \underline{\underline{\operatorname{div} J(x,t)}} - f(u(x,t)) = 0 \quad \leftarrow$$

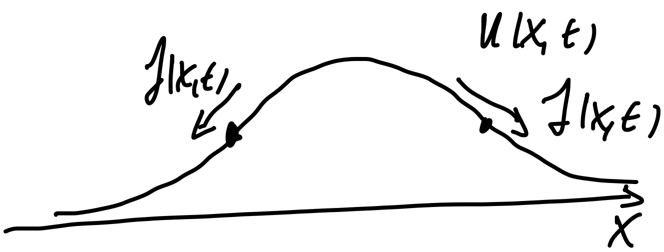
Anna Marciniac-Czochra, leukemia
Mark Chaplain, angiogenesis
Luigi Preziosi, tissue mechanics
Thomas Yankeelov, glioma
TH

$$\operatorname{div} J(x,t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} J_i(x,t) \quad \text{divergence}$$

What is $J(x,t)$?

Fickian law: for diffusion $\underline{J(x,t)} = -D\, \underline{\nabla u(x,t)}$

$$\nabla u = \left(\frac{\partial}{\partial x_1} u, \frac{\partial}{\partial x_2} u, \ldots \frac{\partial}{\partial x_n} u \right) \quad \text{gradient}$$



Fouriers law: $J(x,t) = -D \nabla u(x,t)$

Unbiased random walk $J(x,t) = -D \nabla u(x,t)$

(Cattaneo law: $\varepsilon J_t + D\nabla u = -J$)

Then $\frac{\partial u}{\partial t} = D \operatorname{div} \nabla u + f(u)$

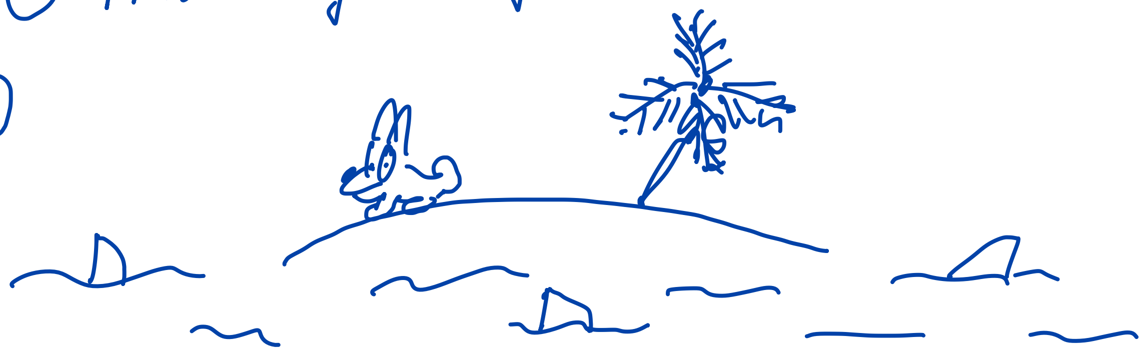
$$\operatorname{div} \nabla u = \Delta u \quad \text{Laplacian}$$

$$= \frac{\partial^2}{\partial x_1^2} u + \frac{\partial^2}{\partial x_2^2} u + \cdots + \frac{\partial^2}{\partial x_n^2} u$$

$$\boxed{\frac{\partial u}{\partial t} = D\Delta u + f(u)}$$ Reaction diffusion eq.

1. Critical domain size problem.
2. Travelling wave problem.

(1)



How large does the habitat have to be to support a population?

- ecology
- farming
- pest control
- cancer

We use the famous Fisher-KPP equation

$$\boxed{u_t = D u_{xx} + \mu u(1-u)}$$

on a finite domain $[0, l]$

Need boundary cond.

a) Homogeneous Dirichlet b.c. $u(0,t)=0$, $u(l,t)=0$

b) Homogeneous Neumann b.c (no flux b.c.)

$$u_x(0,t)=0 \qquad u_x(l,t)=0$$

b) Robin conditions, mixed. $\alpha u + \beta u_x = 0$ on $x=0$

Hillen's rule of thump:

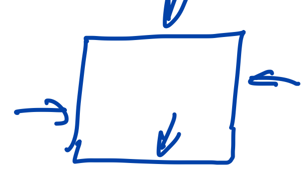
\# side conditions = \# highest order derivatives

RD-eq $\quad u_t = D u_{xx} + \mu u(1-u)$

time: 1 initial cond. $\quad u(x,0) = f(x)$

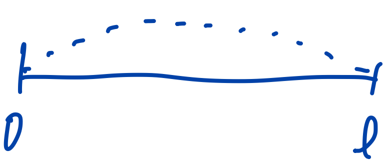
space: 2 boundary cond. as above

wave eq: $\quad u_{tt} = c^2 (u_{xx} + u_{yy})$ on a box



time: initial displacement + initial velocity

Critical domain size problem

$u(0,t) = 0$ 

$u(l,t) = 0$ Dirichlet.

(i) How large l to support a population?

(ii) How large l such that $u \equiv 0$ is unstable?

(iii) How large l such that a nontrivial steady state exists?

Lets do (iii): $u_t = 0 \qquad u_{xx} = -\frac{\mu}{D} u(1-u)$

introduce $v = u'$

$$\left.\begin{aligned} u' &= v \\ v' &= -\frac{\mu}{D} u(1-u) \end{aligned}\right\} (*)$$

Lets analyze (*):

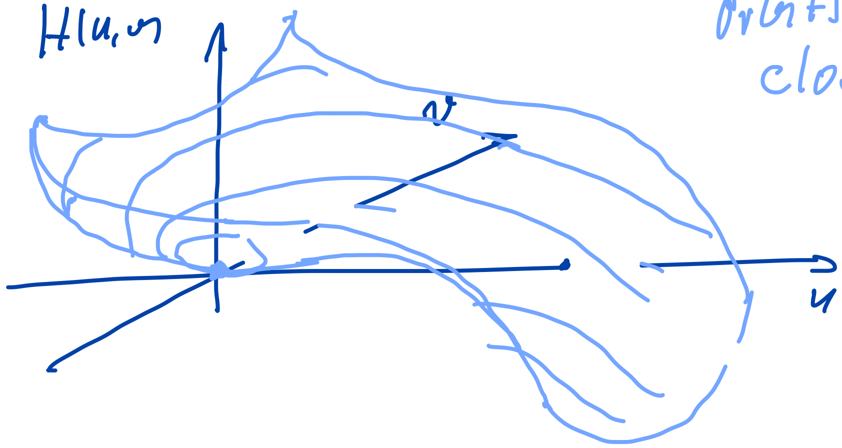
$P_1(0,0)$: linear center with eigenvalues

$$\lambda_{1/2} = \pm i \sqrt{\frac{\mu}{D}}.$$

$P_2(1,0)$: saddle point

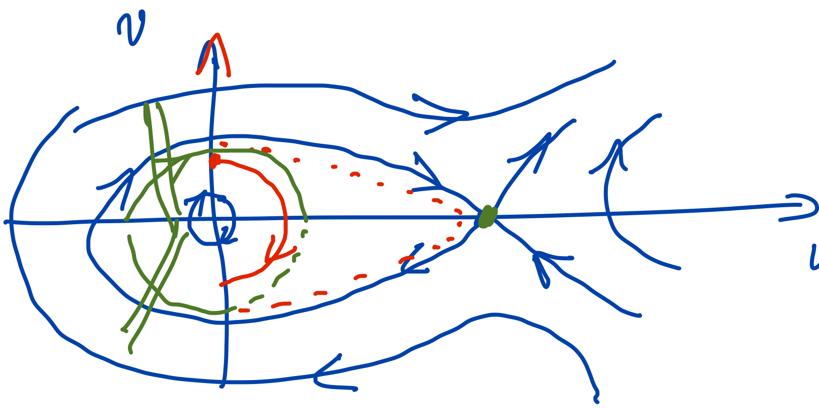
Hamiltonian fct: $H(u,v) = \frac{1}{2}v^2 + \frac{\mu}{D}\frac{u^2}{2} - \frac{\mu}{D}\frac{u^3}{3}$

then $\frac{\partial H}{\partial v} = u'$, $\frac{\partial H}{\partial u} = -v'$



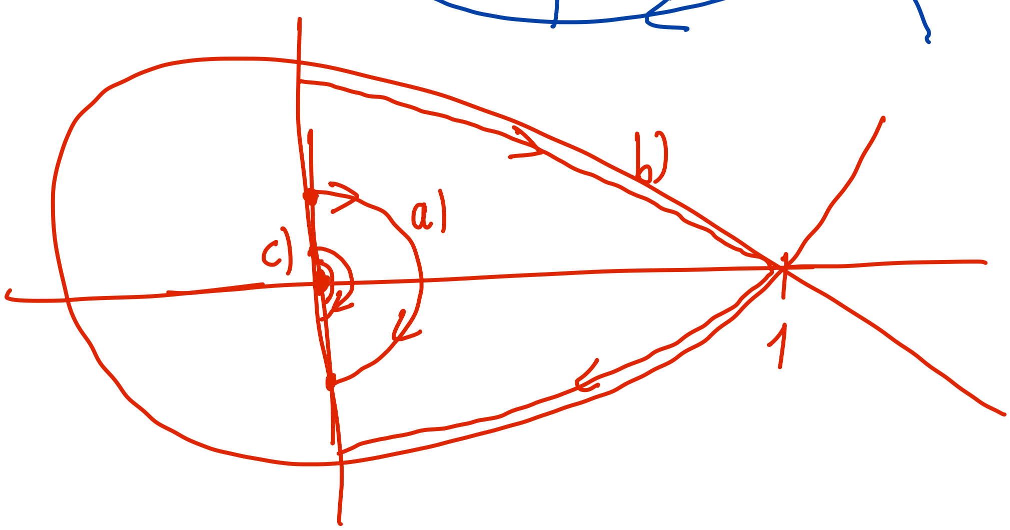
Orbits are closed curves

Phase plane:



Dirichlet: (red)
$u(0,t)=0$, $u(l,t)=0$

Neumann (green)
$u_v(0,t)=0$ $v(l,t)=0$
only $u\equiv 1$
$v \equiv 0$

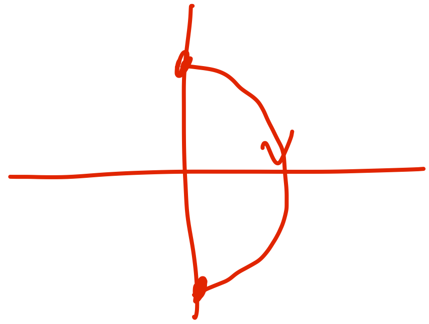


solution needs length l

~~then~~ length of b) $\to \infty$

length of c) given by linearization at $(0,0)$.

linearization at $(0,0)$ has eigenvalues $\lambda_{1/2} = \pm i\sqrt{\frac{\mu}{D}}$ we can write the solution as $\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = c_1 \begin{pmatrix} \cos(\sqrt{\frac{\mu}{D}}x + \phi) \\ \sin(\sqrt{\frac{\mu}{D}}x + \phi) \end{pmatrix}$



$\phi = \frac{\pi}{2}$

For length l: $\sqrt{\frac{\mu}{D}}\, l = \pi$

$$\boxed{l^* = \pi\sqrt{\frac{D}{\mu}}}$$

Minimum domain length!

② Travelling wave problem.

$$u_t = D u_{xx} + f(u)$$

assumptions: (i) $f(0) = f(1) = 0$

(ii) $f'(0) > 0$ $\quad f'(1) < 0$

(iii) $f(u) > 0$ $\quad 0 < u < 1$

Travelling wave coordinate $z = x - ct$

assume $\phi(z) = \phi(x - ct) = u(x,t)$ ←

ϕ: wave profile

boundary cond: $\phi(+\infty) = 0$, $\phi(-\infty) = 1$

$$-c\phi' = D\phi'' + f(\phi)$$

Introduce $\psi = \phi'$

$$\left.\begin{aligned} \phi' &= \psi \\ \psi' &= -\frac{c}{D}\psi - \frac{f(\phi)}{D} \end{aligned}\right\}$$

rescale $\tilde{c} = \frac{c}{D}$ $\tilde{f} = \frac{f}{D}$ remove $\sim$.

$$\left.\begin{aligned} \phi' &= \psi \\ \psi' &= -c\psi - f(\phi) \end{aligned}\right\} \quad (2)$$

equilibria: $(0,0)$, $(1,0)$

Jacobian $J = \begin{pmatrix} 0 & 1 \\ -f'(\phi) & -c \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 0 & 1 \\ -f'(0) & -c \end{pmatrix}$ $\left.\begin{aligned} \operatorname{tr} J(0,0) &= -c \\ \det J(0,0) &= f'(0) > 0 \end{aligned}\right\}$ stable node or spiral

eigenvalues: $\lambda_{1/2} = \frac{\operatorname{tr} J}{2} \pm \frac{1}{2}\sqrt{\operatorname{tr} J^2 - 4 \det J}$

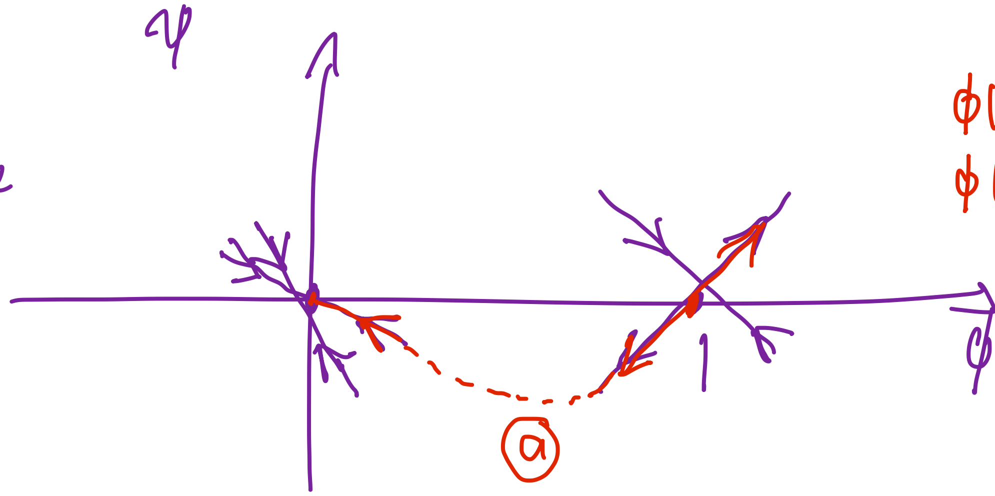
$$= -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2 - 4f'(0)}$$

b) spiral for $c^2 < 4f'(0)$ call $\boxed{c^* = 2\sqrt{f'(0)}}$

a) node for $c^2 > 4f'(0)$

$J(1,0) = \begin{pmatrix} 0 & 1 \\ -f'(1) & -c \end{pmatrix}$: saddle.

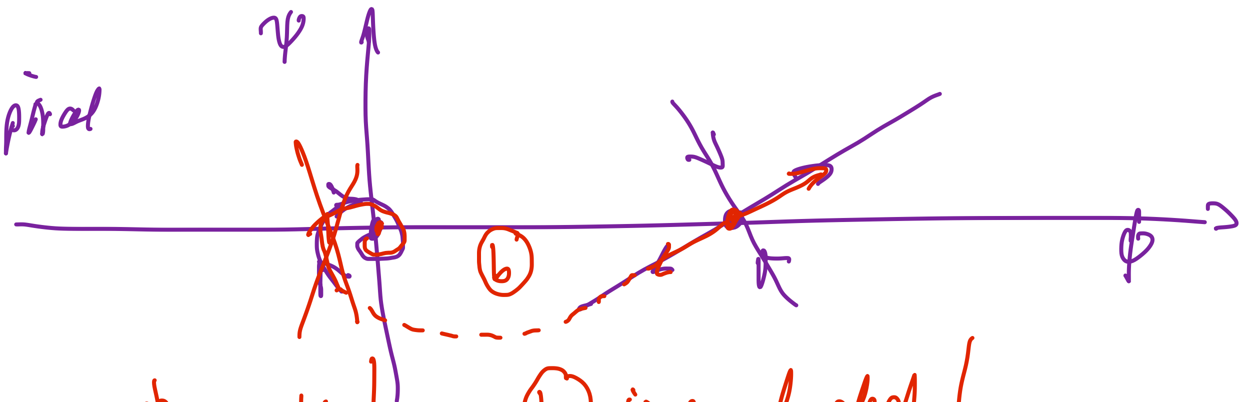
node



$\phi(+\infty) = 0$

$\phi(-\infty) = 1$

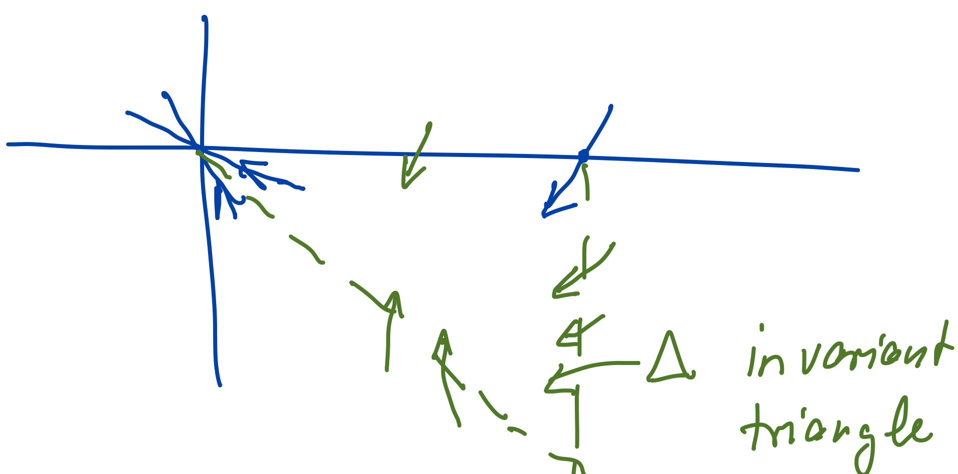
spiral



φ negative! (b) is excluded!

Then $c < c^*$ is not possible!

To do: (a) exists! (8 pages in my notes original proof by Källen)



and Poincaré-Bendixson

The minimal invasion speed is

$$c^* = 2\sqrt{f'(0)}$$

or if D is included,

$$\boxed{c^* = 2\sqrt{D f'(0)}}$$ Fisher speed.