From symplectic deformation to isotopy, equivariantly

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Fields Workshop on Hamiltonian Geometry and Quantization

Joint with Pranav Chakravarthy, River Chiang, and Martin Pinsonnault.

Path-connectedness of the space of equivariant symplectic ball embeddings

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Here $B^4(r)$ is the closed ball in $\mathbb{R}^4$ of center 0 and radius r, with the standard symplectic form. The action of S^1 on the ball is through rotations of the coordinates, according to the isotropy weights at Σ.

Translation to uniqueness of equivariant blowup

An embedding of the closed ball $B(r)$ is the restriction of an embedding of an open ball $B^o(r+\delta)$. The standard symplectic blowup of size $\frac{r^2}{2}$ in $B^o(r+\delta)$ transports to (M,ω) through i. The G-action descends to the blowup.

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Symplectic forms ω_0 and ω_1 on M are **isotopic** if they are connected by a **deformation**: a family ω_t, $0 \leqslant t \leqslant 1$, of symplectic forms, and, moreover, the forms ω_t are all cohomologous:

$$[\omega_t] = [\omega_0] \text{ for all } t.$$

Moser showed that, if M is compact, an isotopy between ω_0 and ω_1 is **strong**, i.e., it has the form

$$\omega_t = \varphi_t^* \omega_0$$

where

$$\varphi_t \colon M \to M$$

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The definitions have analogs in the equivariant setting; Moser's result also holds in the equivariant setting.

McDuff's result in the non-equivariant setting

Theorem (McDuff, 1996)

Let (M, ω) be a ruled symplectic four-manifold or the projective plane with a multiple of the Fubini-Study form. For any $k > 0$ there is at most one way of symplectically blowing up k points to specific sizes, up to isotopy.

Recall that a **ruled symplectic four-manifold** is an S^2-bundle over a closed Riemann surface with a symplectic form that is non-degenerate on each fiber.

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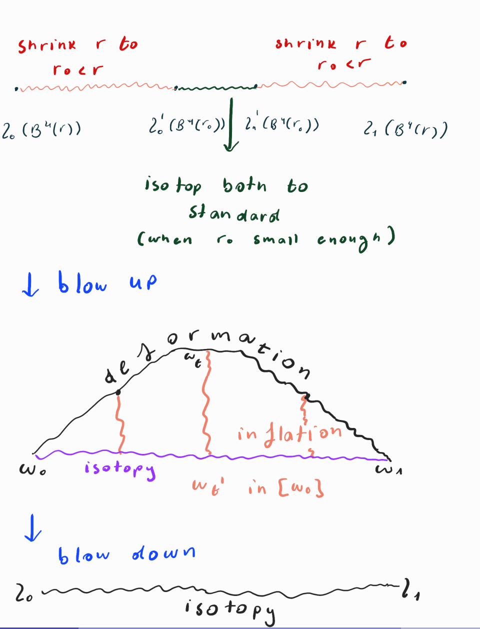
Equivalently:

Theorem (McDuff, 1996)

The space of symplectic ball embeddings of given sizes into the projective plane or a ruled surface is path-connected.

McDuff's proof



Deformation to isotopy through inflation

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If J is ω-tame, an embedded J-holomorphic curve is a symplectic submanifold.

Lemma (Lalonde 1994, Lalonde and McDuff 1996, McDuff 2001, Buse 2011, Chakravarthy-Payette-Pinsonnault 2024)

Let Z be an embedded 2-submanifold of (M, ω) that is J-holomorphic w.r.t. an ω-tame almost complex structure J. Then there exists a family ω_t, $0 \leqslant t \leqslant 1$ starting at ω of symplectic forms taming J in class $[\omega] + t\lambda PD[Z]$, for $\lambda \in [0, T)$, where

$$T = \begin{cases} \infty & \text{if } Z \cdot Z \geqslant 0 \\ \frac{\omega(Z)}{-Z \cdot Z} & \text{if } Z \cdot Z < 0. \end{cases}$$

Moreover, for a family (ω^s, J^s, Z^s), we get a family ω_t^s of symplectic forms in $[\omega^s] + t\lambda^s PD[Z^s]$ taming J^s, with $\omega_0^s = \omega^s$.

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McDuff applied Taubes' "Seiberg-Witten equals Gromov" Theorem to get J-holomorphic curves in the required classes.

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An equivariant version of McDuff's result, for a Hamiltonian S^1-action.

Remark

We cannot get equivariant isotopy by averaging the forms in the (non-invariant) symplectic isotopy constructed by McDuff. The averaged forms need not be non-degenerate. The forms in the symplectic isotopy might not be tamed by any ***invariant*** *almost complex structure.*

Comparing with Karshon's result

Karshon (1999) classified compact, connected symplectic four-manifolds with Hamiltonian S^1-actions by the decorated graphs associated to them.

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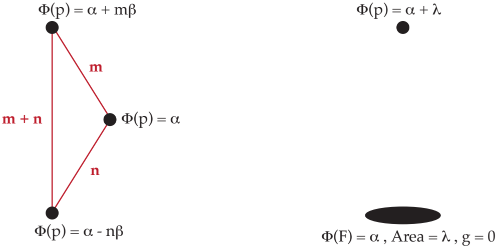

Figure: The decorated graphs for two Hamiltonian $S^1 \circlearrowright \mathbb{C}P^2$: with only isolated fixed points on the left, and with a fixed surface on the right. Fixed points and surfaces are labelled with the *momentum map* Φ *label*. A fixed surface is also labelled with the *area label*, and the *genus* g. An edge of label k represents an embedded 2-sphere on which the circle acts by rotations of speed k.

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If $S^1 \circlearrowright (M_0, \omega_0)$ and $S^1 \circlearrowright (M_1, \omega_1)$ are obtained from a Hamiltonian $S^1 \circlearrowright (M^4, \omega)$ by an equivariant symplectic blowup of the same size at the same fixed component then they have the same decorated graph and hence differ by an equivariant symplectomorphism.

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However, this equivariant symplectomorphism might not be connected to the identity by a family of equivariant diffeomorphisms, so we cannot conclude that ω_0 and ω_1 are equivariantly isotopic.

Remark

Karshon also showed that the underlying symplectic manifold of a compact, connected Hamiltonian S^1-manifold of dimension four is a k-blowup of either the projective plane or a ruled symplectic four-manifold. So we can restrict to such manifolds.

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This is not true for Hamiltonian S^1-actions. In that case, the preimage of a generic point is of dimension 3 and is a union of orbits.

For example, for the standard S^1-action on $\mathbb{C}^2$, with momentum map

$$\phi(z_1, z_2) = \frac{z_1^2 + z_2^2}{2},$$

the preimage of a point $\neq 0$ is a 3-sphere, which is a union of Hopf circles.

Our algorithm in the equivariant setting

Following McDuff's argument, we first need an equivariant version of the construction of the deformation between the blowup forms, and of the Inflation lemma.

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Then, we need to bypass the application of Taubes' "SW=Gr" since it does not guarantee the existence of J-holomorphic curves for some **invariant** tame almost complex structure; for that we use invariant curves that we read from the decorated graph.

Inflation using curves that we read from the decorated graph

Example

Let $\mu \geqslant 1 > c > \gamma > 0$. Consider a Hamiltonian S^1-action on $(S^2 \times S^2, \mu\tau \oplus \tau)$ and a fixed component Σ. Let ω and ω' be S^1-equivariant symplectic blowups of sizes c and γ, performed at Σ. Then there is a family of invariant symplectic forms connecting ω' and an invariant symplectic form in $[\omega]$.

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The S^1-action on each blowup extends to a toric action whose momentum polytope is one of the following two (the same one for both blowups):

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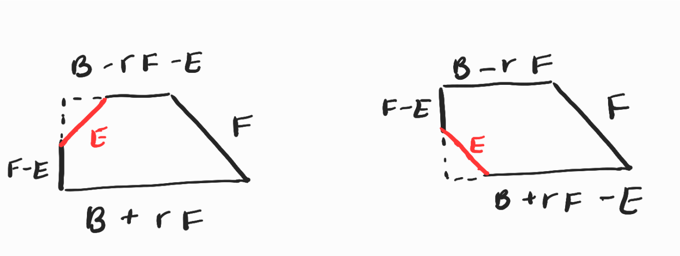

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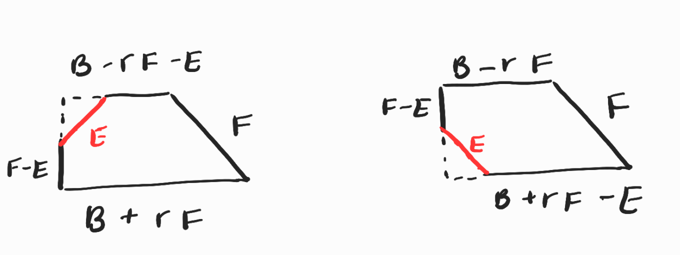

The preimages of the edges are J-holomorphic spheres for an invariant complex structure J that is compatible with both ω and ω'.

In particular, in the left case there are J-holomorphic spheres in

$$B + rF,\ F - E,\ E,\ \text{and } B - rF - E \text{ for some } r \geqslant 0;$$

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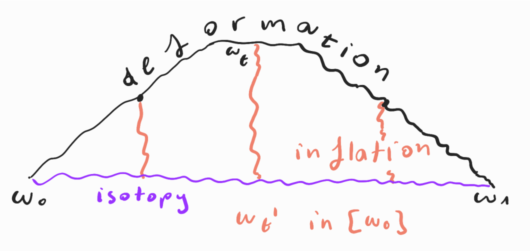
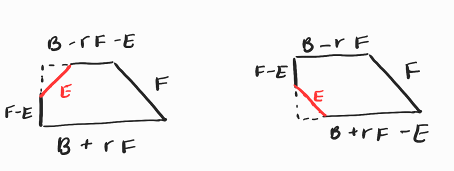
where B and F are the images in $H_2(S^2 \times S^2 \# \overline{\mathbb{CP}^2}; \mathbb{Z})$ of

$$[S^2 \times \mathrm{pt}],\ [\mathrm{pt} \times S^2] \text{ in } H_2(S^2 \times S^2; \mathbb{Z}),$$

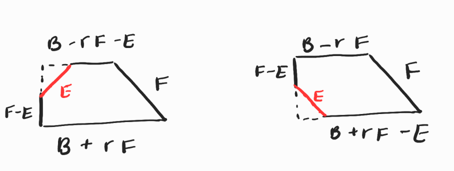
and E is the class of the exceptional divisor.

To increase the size of E from γ to c we inflate along a class $A - E$; this will affect the size of classes other than E; to re-adjust, we inflate along other classes and normalize.





In the left case, we equivariantly inflate ω' along $B + rF$ and then along $F - E$; in the right case, the equivariant inflation is along $B + rF - E$ and along F by appropriate sizes, depending on μ, c, γ and the configuration.



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The classes $B + rF$ and F are of non-negative self-intersection and so is $B + rF - E$ if $r > 0$, so we can inflate along each by any positive size, while the class $F - E$ is of self-intersection -1 so we can inflate along it only by a positive size that is smaller than its area.

In general, the S^1-action might not extend to a toric action.

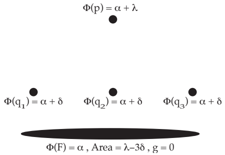

Figure: 3 vertices at the same height imply that the circle action does not extend.

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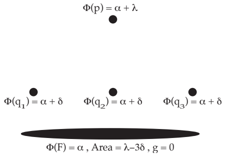

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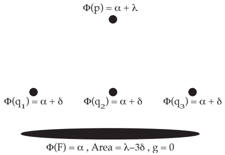

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Still, the fat vertices and the edges of the decorated graph correspond to embedded J-holomorphic curves for some invariant compatible J.

Moreover, we will need to inflate not just along J-holomorphic curves but also along cusp curves.

Invariant cusp curves

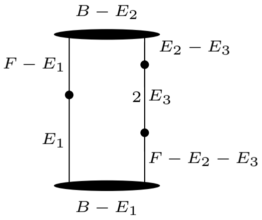

Figure: An S^1-action obtained by a sequence of 3 equivariant blowups from an S^1-ruled symplectic four-manifold.

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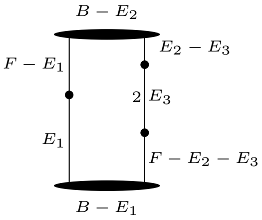

Figure: An S^1-action obtained by a sequence of 3 equivariant blowups from an S^1-ruled symplectic four-manifold.

$$E_2 = (E_2 - E_3) + E_3;$$

$$F = (F - E_2 - E_3) + (E_2 - E_3) + 2E_3;$$

$$F - E_3 = (F - E_2 - E_3) + (E_2 - E_3) + E_3;$$

$$B = (B - E_1) + E_1.$$

Inflation along a cusp curve

Theorem (Buse-Li, 2022)

Assume that $E \in H_2(M;\mathbb{Z})$ of self-intersection -1 is represented by a J-cusp curve with exactly two rational embedded components in $\{D_1, D_2\}$, where $D_1 \cdot D_1 = -1$, $D_2 \cdot D_2 = -2$, $D_1 \cdot E = 0$, $D_2 \cdot E = -1$, and $D_1 \cdot D_2 = 1$. Here J is ω-compatible.
Then for λ in $[0, \omega(D_2))$ there is a J-tamed symplectic form in

$$[\omega] + \lambda \, \mathrm{PD}(E).$$

Inflation along a cusp curve

Theorem (Buse-Li, 2022)

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Then for λ *in* $[0, \omega(D_2))$ *there is a* J*-tamed symplectic form in*

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The proof is by alternate inflations. We generalize Buse-Li's algorithm beyond this special case. We use the equivariant Inflation lemma to do so in the equivariant setting as well. Moreover, we get a path between ω and the end-form.

Further questions

- Application to extending homologically trivial cyclic actions to Hamiltonian circle actions. For that, we also need connectedness for $\mathbb{Z}_n$-equivariant symplectic ball embeddings.

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- Application to extending homologically trivial cyclic actions to Hamiltonian circle actions. For that, we also need connectedness for $\mathbb{Z}_n$-equivariant symplectic ball embeddings.
- Calculation of the homotopy type of the space of equivariant symplectic ball embeddings of a given size centered at a given connected component of the fixed point set, in special cases.

Happy Birthday to **Lisa**, **Yael**, and **Jonathan**

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and to my son David, who will be 10 tomorrow