The Moore–Tachikawa conjecture via shifted symplectic geometry

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Joint work with Peter Crooks

Overview

Theorem (Kostant 1963). Let G be a complex semisimple group, $\mathfrak{g} := \mathrm{Lie}(G)$.

(1) $\exists$ *global slice* $\mathcal{S} \subset \mathfrak{g}^*_{\mathrm{reg}}$ for the coadjoint action

(2) The stabilizers G_ξ are *abelian* for all $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$

(3) $\mathfrak{g}^*_{\mathrm{reg}}$ is *Hartogs* : holomorphic functions on $\mathfrak{g}^*_{\mathrm{reg}}$ extend to $\mathfrak{g}^*$

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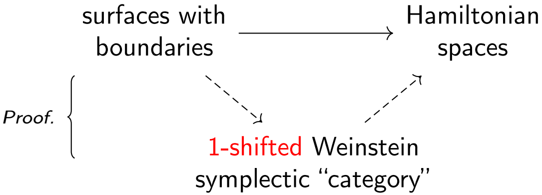
Upshot of the talk.
Any not necessarily semisimple Lie algebra satisfying (1)–(3), or, more generally, Poisson affine variety satisfying analogues of (1)–(3), defines a Topological Quantum Field Theory valued in Hamiltonian spaces.



$$\begin{tikzcd}
\text{surfaces with boundaries} \arrow[rr] \arrow[dr, dashed] & & \text{Hamiltonian spaces} \\
& \text{1-shifted Weinstein symplectic ``category''} \arrow[ur, dashed] &
\end{tikzcd}$$

Proof.

The case $\mathfrak{g}$ complex semisimple is part of the Moore–Tachikawa conjecture.

The Moore–Tachikawa conjecture

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Three Hamiltonian spaces:

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$$G \times G \circlearrowright T^*G$$

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$$\begin{aligned} G \times G \circlearrowright\ & T^*G = G \times \mathfrak{g}^* \\ G \circlearrowright\ & G \times \mathcal{S} \end{aligned}$$

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$$\begin{aligned} G \times G \circlearrowright\ & T^*G = G \times \mathfrak{g}^* \\ G \circlearrowright\ & G \times \mathcal{S} \\ & \{(g, \xi) \in G \times \mathcal{S} : \mathrm{Ad}^*_g \xi = \xi\} \end{aligned}$$

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Three Hamiltonian spaces:

$\longmapsto \quad G \times G \circlearrowright T^*G \quad =: M_{\square}$

$\longmapsto \quad G \circlearrowright G \times \mathcal{S} \quad =: M_{\Subset}$

$\longmapsto \quad \{(g, \xi) \in G \times \mathcal{S} : \mathrm{Ad}^*_g \xi = \xi\} \quad =: M_{\ominus}$

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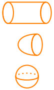

$$\begin{array}{lrl} \longmapsto & G \times G \circlearrowright T^*G & =: M_{\text{▭}} \\ \longmapsto & G \circlearrowright G \times \mathcal{S} & =: M_{\text{◖}} \\ \longmapsto & \{(g,\xi) \in G \times \mathcal{S} : \mathrm{Ad}^*_g \xi = \xi\} & =: M_{\ominus} \end{array}$$

$$(M_{\text{◖}} \times M_{\text{◖}}) /\!\!/ G \cong M_{\ominus}$$

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$$\ominus \longmapsto \{(g,\xi) \in G \times \mathcal{S} : \mathrm{Ad}^*_g \xi = \xi\} =: M_{\ominus}$$

$$\text{◖◗} \cong \ominus \qquad\qquad \text{◖▭} \cong \text{◖}$$
$$(M_{\text{◖}} \times M_{\text{◖}}) /\!\!/ G \cong M_{\ominus} \qquad (M_{\text{◖}} \times M_{\text{▭}}) /\!\!/ G \cong M_{\text{◖}}$$

Conjecture (Moore–Tachikawa 2011). This extends to a functor (TQFT)

η_G : **2-dim cobordisms** $\longrightarrow$ **Hamiltonian spaces**

Objects: unions of circles

Morphisms: surfaces 

Composition: gluing

Objects: complex semisimple groups

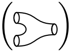
Morphisms: $G \xrightarrow{M} H$

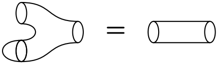
M Hamiltionian $G \times H$-space

Composition: $G \xrightarrow{M} H \xrightarrow{N} I$

$N \circ M := (M \times N) /\!\!/ H$

The Moore–Tachikawa conjecture

It suffices to construct η_G () and verify a finite number of relations such as



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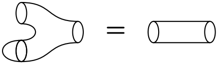

Examples.

$\eta_{\mathrm{SL}(2,\mathbb{C})} \Big($  $\Big) = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

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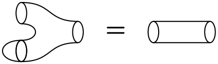

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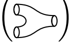
$\eta_{\mathrm{SL}(3,\mathbb{C})}$ () $= \overline{\mathcal{O}_{\min}(E_6)}$

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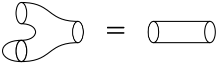

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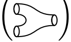
$\eta_{\mathrm{SL}(3,\mathbb{C})}$ () $= \overline{\mathcal{O}_{\min}(E_6)}$

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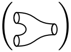
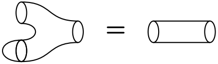
It suffices to construct η_G  and verify a finite number of relations such as



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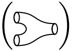
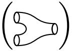
$\eta_{\mathrm{SL}(2,\mathbb{C})}$  $= \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$\eta_{\mathrm{SL}(3,\mathbb{C})}$  $= \overline{\mathcal{O}_{\min}(E_6)}$

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Partial solutions.

- Braverman–Finkelberg–Nakajima: $G = \mathrm{SL}(n, \mathbb{C})$ (Coulomb branches)
- Ginzburg–Kazhdan: scheme version (*ad hoc* construction)

2-dim cobordisms $\longrightarrow$ **Hamiltonian schemes**

- Bielawski : regular version ($M \xrightarrow{\mu} \mathfrak{g}^*_{\mathrm{reg}}$)

Differentiable stacks (Behrend–Xu 2011)

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$$(G_1 \times M_1 \to M_1) \sim (G_2 \times M_2 \to M_2)$$

then $M_1/G_1 \cong M_2/G_2, \quad H^*_{G_1}(M_1) \cong H^*_{G_2}(M_2), \quad$ etc?

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Example. $\mathcal{G} = G \times M \rightrightarrows M$, $\quad x \xrightarrow{(g,x)} g \cdot x$

Differentiable stacks

Two Lie groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are **Morita equivalent**, denoted $\mathcal{G}_1 \sim \mathcal{G}_2$, if

$$\exists \quad \begin{tikzcd} & \mathcal{H} \arrow[dl] \arrow[dr] & \\ \mathcal{G}_1 & & \mathcal{G}_2 \end{tikzcd} \quad \text{such that } \mathcal{H} \cong f_1^*\mathcal{G}_1 \text{ and } \mathcal{H} \cong f_2^*\mathcal{G}_2.$$

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Summary. *We replace singular quotients by equivalence classes of manifolds with Lie group actions, or more generally, Lie groupoids.*

Shifted symplectic geometry (Pantev–Toën–Vaquié–Vezzosi 2012)

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$\begin{tikzcd} \mathrm{Lie}(\mathcal{G}) \arrow[r] & TM \arrow[d] & \\ & T^*M \arrow[r] & \mathrm{Lie}(\mathcal{G})^* \end{tikzcd}$	$\begin{tikzcd} \mathrm{Lie}(\mathcal{G}) \arrow[r] \arrow[d] & TM \arrow[d] \\ T^*M \arrow[r] & \mathrm{Lie}(\mathcal{G})^* \end{tikzcd}$	$\begin{tikzcd} & \mathrm{Lie}(\mathcal{G}) \arrow[r] \arrow[d] & TM \\ T^*M \arrow[r] & \mathrm{Lie}(\mathcal{G})^* & \end{tikzcd}$
0-shifted symplectic symplectic geometry	**1-shifted symplectic**	**2-shifted symplectic**

Shifted symplectic geometry

1-shifted symplectic stack.

$$\begin{tikzcd}
\mathcal{G} \arrow[d, shift left] \arrow[d, shift right] & \omega \in \Omega^2_{\mathcal{G}} \\
M & \phi \in \Omega^3_M
\end{tikzcd}$$

satisfying a compatibility condition ($d\omega = \mathsf{s}^*\phi - \mathsf{t}^*\phi$) and a non-degeneracy condition

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quasi-isomorphism

This is exactly the notion of ***quasi-symplectic groupoids*** of Bursztyn–Craininc–Weinstein–Zhu and Xu (2004).

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Example. G Lie group, $\mathfrak{g} := \mathrm{Lie}(G)$.

$$\begin{tikzcd} T^*G \arrow[d, shift left] \arrow[d, shift right] & \omega = \text{canonical} \\ \mathfrak{g}^* & \phi = 0 \end{tikzcd}$$

Shifted symplectic geometry

1-shifted Lagrangians.

$$\begin{tikzcd}
\mathcal{L} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\
(N, \gamma) \arrow[r] & M
\end{tikzcd}$$

1-Lagrangian

1-shifted
symplectic

where $\gamma \in \Omega^2_N$ satisfies some compatibility and non-degeneracy conditions.

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Theorem (Pantev–Toën–Vaquié–Vezzosi 2012).

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1-shifted Lagrangian correspondences. $\mathcal{L} \to \mathcal{G}_1 \times \mathcal{G}_2^-$

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Moore–Tachikawa conjecture. Every complex semisimple group G induces a TQFT

$\eta_G :$ **2-dim cobordisms** $\longrightarrow$ **Hamiltonian spaces**

Objects: complex semisimple groups

Morphisms: $G \overset{M}{\to} H$: M Hamil. $G \times H$-space

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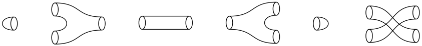
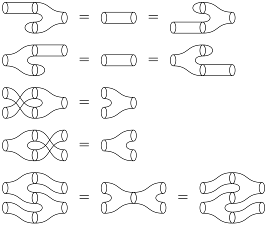
and on **morphisms** by  with relations



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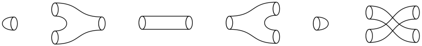

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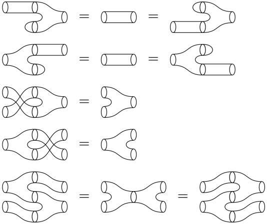

(⋆)

It suffices to specify an object $X \in \mathbf{C}$ ($\bigcirc \mapsto X$, $\bigcirc\bigcirc \mapsto X \otimes X$, $\ldots$) and morphisms

	$\longmapsto$	$(I \to X)$	*“unit”*
	$\longmapsto$	$(X \otimes X \to X)$	*“product”*
	$\longmapsto$	$(X \to X \otimes X)$	*“co-product”*
	$\longmapsto$	$(X \to I)$	*“co-unit”*

satisfying analogues of (⋆), i.e. X is a ***commutative Frobenius object*** in $(\mathbf{C}, \otimes, I)$.

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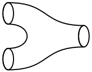

$$\longmapsto$$

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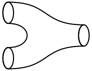

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Corollary. Every quasi-symplectic groupoid $\mathcal{G}$ ***Morita equivalent*** to an abelian Lie groupoid induces a TQFT

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(2) The stabilizers G_ξ are *abelian* for all $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$

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solves the scheme version of the Moore–Tachikawa conjecture.

Summary and generalizations

- Let $M \subset \mathbb{C}^n$ be a smooth complex affine variety with a Poisson structure.

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Example. $M = \mathfrak{g}^*$, $\mathfrak{g}$ complex semisimple $\implies$ Moore–Tachikawa conjecture

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