Homotopy types of moment-angle complexes associated to almost linear resolutions

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(joint with Ben Briggs)

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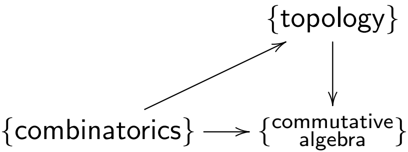
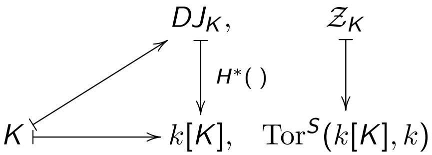
Definitions/notation





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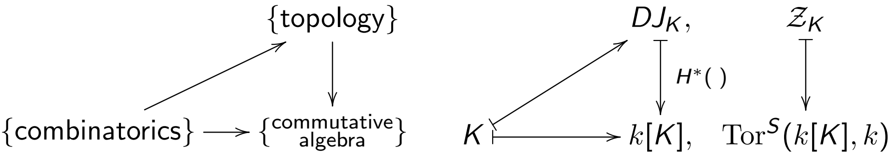

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- the *Stanley–Reisner ring* $k[K] = S/I_K$,

$$S = k[v_1, \ldots, v_m],\ |v_i| = 2, \quad I_K = (v_{i_1} \cdots v_{i_r} \mid \{i_1, \ldots, i_r\} \notin K)$$

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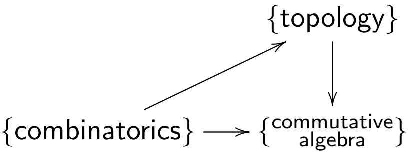

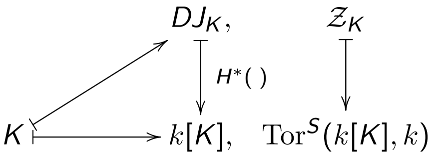

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- the *moment-angle complex*, the *Davis–Januszkiewicz space*

$$\mathcal{Z}_K = (D^2, S^1)^K \qquad DJ_K = (\mathbb{C}P^\infty, *)^K$$

- the homotopy fibration

$$\mathcal{Z}_K \xrightarrow{\omega} DJ_K \hookrightarrow BT^m.$$

- **Hard problem:** Determine the homotopy type of $\mathcal{Z}_K$ from the combinatorics of K.
- **Equivalent problem:** Determine the homotopy type of $\mathcal{Z}_K$ from the Stanley–Reisner ring $k[K]$.
- **Easier(?) problem:** Determine the homotopy type of $\mathcal{Z}_K$ from the minimal free resolution of $k[K]$.

$$F_p \xrightarrow{\partial_p} \cdots \to F_1 \xrightarrow{\partial_1} F_0 \to k[K] \to 0$$

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Cohomology of moment-angle complexes

Theorem (Baskakov–Buchstaber–Panov, Franz)

There are natural (in K) isomorphisms of graded algebras

$$\begin{aligned} H^*(\mathcal{Z}_K) &\cong \mathrm{Tor}_*^S(k[K], k) \\ &\cong \bigoplus_{J \subseteq [m]} \widetilde{H}^*(K_J). \end{aligned}$$

←*Hochster's formula*

- This gives a topological interpretation of the modules appearing in the minimal free resolution $F_\bullet$ of $k[K]$.

  Avramov–Golod (1971): $\mathrm{Tor}_*^S(k[K], k)$ is a Poincaré duality alg. $\Leftrightarrow K$ is Gorenstein.

  Cai (2017): $\mathcal{Z}_K$ is a manifold $\Leftrightarrow K$ is Gorenstein.

- What about the differentials?

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Cohomology operations

Let $\Lambda = \Lambda(\iota_1, \ldots, \iota_m)$, $|\iota_j| = -1$.

- $T^m \curvearrowright \mathcal{Z}_K$ induces derivations $\iota_j \colon H^*(\mathcal{Z}_K) \to H^{*-1}(\mathcal{Z}_K)$

$$\begin{aligned} S^1_j \times \mathcal{Z}_K &\longrightarrow \mathcal{Z}_K \\ H^*(\mathcal{Z}_K) &\longrightarrow \Lambda(u) \otimes H^*(\mathcal{Z}_K) \\ \alpha &\longmapsto 1 \otimes \alpha + u \otimes \iota_j(\alpha) \end{aligned}$$

making $H^*(\mathcal{Z}_K)$ a graded Λ-module.

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- Define a dg Λ-module structure on the *reduced Koszul complex*

$$R^*(K) := \big(k[K] \otimes \Lambda(u_1, \dots, u_m)\big)/(v_i^2, v_i u_i), \quad du_i = v_i$$

  by letting each ι_j act by the graded derivation $\frac{\partial}{\partial u_j}$:

$$\iota_j(v_i) = 0, \qquad \iota_j(u_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Cohomology operations

Proposition (A.–Briggs)

- There is a natural (in K) isomorphism of dg Λ-modules

$$C^*_{cw}(\mathcal{Z}_K) \cong R^*(K).$$

  Therefore $H^*(\mathcal{Z}_K) \cong \mathrm{Tor}^S_*(k[K], k)$ as graded Λ-modules.

- There are commutative diagrams

$$\begin{array}{ccc} H^n(\mathcal{Z}_K) & \xrightarrow{\cong} & \bigoplus_{J\subseteq[m]} \widetilde{H}^{n-|J|-1}(K_J) \\ \downarrow \iota_j & & \downarrow \\ H^{n-1}(\mathcal{Z}_K) & \xrightarrow{\cong} & \bigoplus_{J\subseteq[m]} \widetilde{H}^{n-|J|-2}(K_J) \end{array}$$

  where the right vertical map is induced by inclusions $K_{J\setminus\{j\}} \hookrightarrow K_J$.

Cohomology operations $\Lambda(\iota_1, \ldots, \iota_m) \otimes H^*(\mathcal{Z}_K) \to H^*(\mathcal{Z}_K)$

- $\Lambda(\iota_1, \ldots, \iota_m)$ embeds into an algebra of *higher cohomology operations* induced by the T^m-action (in the sense of GKM):

$$\delta_s \colon H^*(X) \to H^{*-2\deg s+1}(X)$$

for each monomial $s \in S = k[v_1, \ldots, v_m]$ (where $\delta_{v_j} = \iota_j$).

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Theorem (A.–Briggs)

The cohomology operations δ_s assemble into a differential

$$\delta = \sum_{\substack{\text{sq-free} \\ \text{monomials}}} s \otimes \delta_s = \sum_{U \subseteq [m]} v_U \otimes \delta_U$$

making $\big(S \otimes H^*(\mathcal{Z}_K), \delta\big)$ the minimal free resolution of $k[K]$.

$$H^*(\mathcal{Z}_K) \text{ as } \Lambda(\iota_1, \ldots, \iota_m)\text{-module} \leftrightsquigarrow \textit{linear part} \text{ of the resolution}$$

Homotopy type of $\mathcal{Z}_K$

There is a hierarchy of families of simplicial complexes:

shifted $\subset$ vertex-decomposable $\subset$ shellable $\subset$ sequentially Cohen–Macaulay

Theorem (Grbić–Theriault, Welker–Grujić, Iriye–Kishimoto)

If the dual of K belongs to one of the classes above, then $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres.

A classical result in commutative algebra characterizes these simplicial complexes in terms of the minimal free resolution of $k[K]$:

Theorem (Eagon–Reiner, 1998)

K is dual Cohen–Macaulay if and only if $k[K]$ has a linear resolution.

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"sweep action"

$$T^m \curvearrowright \mathcal{Z}_K \quad \rightsquigarrow \quad \Lambda(\lambda_1, \ldots, \lambda_m) \otimes H_*(\mathcal{Z}_K) \to H_*(\mathcal{Z}_K)$$

$$\begin{array}{ccccc} S^n & & S^2 & & \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mu_i} & & \\ \mathcal{Z}_K & \xrightarrow{\omega} & DJ_K & \longrightarrow & BT^m \end{array}$$

Theorem (A.–Briggs)

If $f \in \pi_n(\mathcal{Z}_K)$ has Hurewicz image $h(f) \in H_n(\mathcal{Z}_K)$, then

$$\lambda_i h(f) = h\big([\overline{\mu_i, \omega \circ f}]\big) \in H_{n+1}(\mathcal{Z}_K)$$

where $[\overline{\mu_i, \omega \circ f}]$ is the lift of the Whitehead prod. $[\mu_i, \omega \circ f] \in \pi_{n+1}(DJ_K)$. In particular, the Hurewicz image of $\mathcal{Z}_K$ is closed under the sweep action.

Homotopy type of $\mathcal{Z}_K$

Corollary

For any simplicial complex K, the image of the Hurewicz map

$$h\colon \pi_*(\mathcal{Z}_K) \longrightarrow H_*(\mathcal{Z}_K)$$

contains the linear strand of $k[K]$, i.e., the $\Lambda(\lambda_1,\dots,\lambda_m)$-submodule of $H_*(\mathcal{Z}_K) \cong \mathrm{Ext}_S(k[K],k)$ generated by $\mathrm{Ext}^1_S(k[K],k)$.

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Corollary

If $k[K]$ has a linear resolution over $k = \mathbb{Z}$ (resp. $\mathbb{Z}/p$), then $\mathcal{Z}_K$ has the homotopy type (resp. p-local homotopy type) of a wedge of spheres.

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If $k[K]$ is dual seq. Cohen–Macaulay over $k = \mathbb{Z}$ (resp. $\mathbb{Z}/p$), then $\mathcal{Z}_K$ has the homotopy type (resp. p-local homotopy type) of a wedge of spheres.

Proof.

Use Corollary above plus
Herzog–Hibi (1999): K is seq. dual Cohen–Macaulay iff $k[K]$ has componentwise linear resolution. □

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Corollary

If K is flag with chordal 1-skeleton, then $\mathcal{Z}_K \simeq$ wedge of spheres.

Proof.

Use Corollary above plus
Fröberg (1990): K flag with chordal 1-skeleton iff $k[K]$ has a 2-linear resolution. □

Homotopy types of moment-angle manifolds

If $K \neq \partial\Delta^{m-1}$ is a sphere triangulation (or Gorenstein complex), then $k[K]$ cannot have a linear or componentwise linear resolution.

Definition

$k[K]$ has an *almost linear resolution* if it has a d-linear resolution for $p-1$ steps, for some d, where $p = \operatorname{proj\,dim} k[K]$.

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Examples:

- Boundary complexes of polygons
- Boundary complexes of cyclic polytopes $C(2n, m)$
- Odd-dimensional neighbourly sphere triangulations
  (An $(n-1)$-dimensional K is *neighbourly* if every set of $k \leqslant \lfloor \frac{n}{2} \rfloor$ vertices is a face of K.)

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Conjecture (Kalai): $\lim\limits_{m\to\infty} \frac{\#\,\text{nbrly } (n-1)\text{-spheres on } m \text{ vertices}}{\#\,(n-1)\text{-spheres on } m \text{ vertices}} = 1$
for $n \geqslant 4$.

Homotopy types of moment-angle manifolds

Definition

$k[K]$ being *componentwise almost linear (CAL)* is defined similarly and is equivalent to the minimal free resolution $F_\bullet$ satisfying

$$H_i(\mathrm{lin}(F_\bullet)) = 0 \text{ for } 1 < i < \operatorname{proj\,dim} k[K].$$

Examples:

- All Gorenstein complexes with almost linear resolution
- Boundary complexes of stacked polytopes
- Connected sums of above examples

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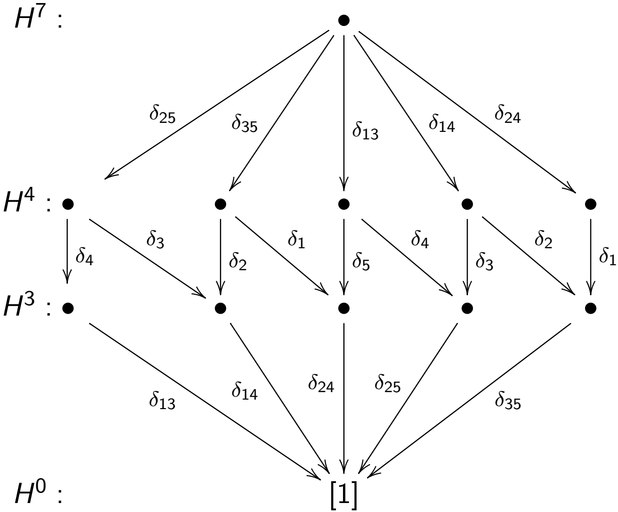
Theorem (A.–Briggs)

If K is a Gorenstein complex which is CAL, then

$$\mathcal{Z}_K \simeq_{\mathbb{Q}} \textit{ connected sum of sphere products}$$
$$\Omega\mathcal{Z}_K \simeq \textit{ product of loops of spheres.}$$

$H^*(\mathcal{Z}_K)$ for $K = \pentagon$



THANK YOU!