On the moment-angle manifolds over Bier spheres (based on j.w.w. Rade Živaljević and Matvey Sergeev)

Ivan Limonchenko

Mathematical Institute SASA, Belgrade, Serbia
ilimonchenko@gmail.com

Workshop on Polyhedral Products
Fields Institute, Toronto, Canada
August 1, 2024

Simplicial complexes

Let us fix our terminology and notations related to simplicial complexes.

Abstract simplicial complex

By an (abstract) simplicial complex K on $[m] := \{1, 2, \ldots, m\}$ we mean a subset of $2^{[m]}$ s.t.

$$\sigma \in K, \tau \subseteq \sigma \Rightarrow \tau \in K.$$

If $\{i\} \in K$, then i is a vertex of K, otherwise i is a ghost vertex of K; elements of K are called its faces (or, simplices). The dimension of K is one less than the maximal number of elements in a face of K.

Minimal non-faces and maximal faces

The set of maximal faces (w.r.t. inclusion) of K will be denoted by $\mathrm{M}(K)$. A subset $I \subseteq [m]$ s.t. $K_I := K \cap 2^I = \partial\Delta_I$ is called a minimal non-face of K and we write: $I \in \mathrm{MF}(K)$.

Flag complexes and flag polytopes

Note that 1-dimensional simplicial complexes are simple graphs.
The following class of simplicial complexes, clique complexes of simple graphs, will be of particular importance for us.

Flag complex

A simplicial complex is called flag if one of the following two equivalent conditions holds:

- each minimal non-face has no more than 2 vertices;
- any subset of vertices pairwise joint by edges forms a face.

Recall that a simple polytope P is a flagtope (or, flag polytope) if its nerve complex $K_P := \partial P^*$ is a flag complex.

The flag polytopes we'll see today all belong to the class of 2-truncated cubes introduced by Buchstaber and Volodin in 2011: that is, they are combinatorially equivalent to simple polytopes that can be obtained from the n-cube I^n after cutting off certain faces of codimension 2, one by one.

Bier spheres: construction (Thomas Bier'92)

Here is a simple and powerful construction producing a big class of spheres. We'll discuss different classes of spheres later on.
Let $[m] := \{1, 2, \ldots, m\}$ and $[m'] := \{1', 2', \ldots, m'\}$ be two ordered sets of the same cardinality with the map $\varphi\colon i \mapsto i', 1 \leq i \leq m$ being an order preserving bijection between them. We denote $I' := \varphi(I)$ for any $I \subset [m]$.

Suppose K is a simplicial complex on $[m]$ and $K \neq \Delta_{[m]}$. Then

- its Alexander dual is a simplicial complex $K^\vee$ on $[m']$ such that
$$I' \in \mathrm{M}(K^\vee) \Longleftrightarrow I^c \in \mathrm{MF}(K).$$
- its Bier sphere is a simplicial complex $\mathrm{Bier}(K)$ on $[m] \sqcup [m']$ s.t.
$$\mathrm{Bier}(K) := \{I \sqcup J' \mid I \in K, J' \in K^\vee, I \cap J = \varnothing\};$$
that is, the deleted join of K and $K^\vee$.

Bier spheres: first examples

We have:

$$\varnothing_{[m]}^{\vee} = \partial\Delta_{[m']} = \mathrm{Bier}(\varnothing_{[m]}),$$

an $(m-2)$-dimensional sphere on $[m] \sqcup [m']$ with the set of vertices $[m']$ and the set of ghost vertices $[m]$.

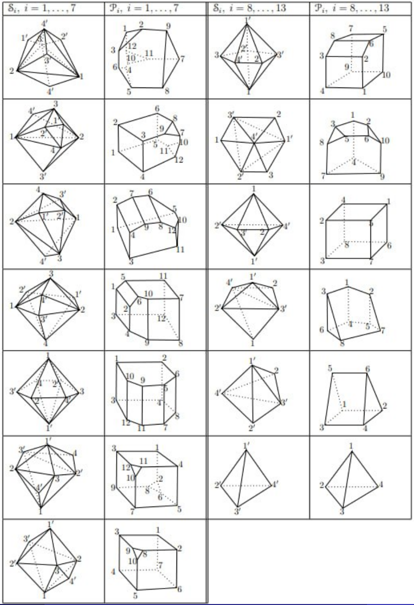
Since $K^{\vee\vee} = K$, one has: $\mathrm{Bier}(K) = \mathrm{Bier}(K^{\vee})$ for any K on $[m]$.

Classification in dimension 1

It turns out that a 1-dimensional Bier sphere can only be combinatorially equivalent to the boundary of

- a triangle, if $\mathrm{M}(K) = \{12, 13, 23\}$;
- a square, if $\mathrm{M}(K) = \{12, 23\}$ and $\mathrm{M}(K) = \{12\}$;
- a pentagon, if $\mathrm{M}(K) = \{1, 23\}$; or
- a hexagon, if $\mathrm{M}(K) = \{1, 2, 3\}$.

Classification in dimension 2: L., Sergeev'24



Bier spheres: open problems

The previous examples motivate the following problems which remain open:

1. Find a K s.t. $\mathrm{Bier}(K)$ is **not** isomorphic to the boundary of a simplicial polytope;
2. Find necessary and sufficient conditions on K for $\mathrm{Bier}(K)$ to be isomorphic to a nested set complex; that is, a nerve complex of a nestohedron.

In fact, see Matoušek (2003), for a fixed m large enough one has:

- the number of different combinatorial types of Bier spheres of dimension $m-2$ is more than

$$2^{(2^m/m)-2m^2};$$

- the number of different combinatorial types of polytopal spheres of dimension $m-2$ is not larger than

$$2^{4m^3}.$$

Simplicial complexes and polyhedra

In order to discuss general properties of Bier spheres we need to introduce a few classes of combinatorial spheres.

Geometric simplicial complex

By a geometric simplicial complex (or, polyhedron) we mean a set P of simplices of arbitrary dimensions in $\mathbb{R}^n$ s.t.

- each face of a simplex in P is itself in P;
- intersection of any two simplices from P is a face of both.

The notions of a face, vertex and dimension are defined similarly to the ones above.

We say that a polyhedron P is a geometric realization of a simplicial complex K and write: $P = |K|$ if there is a bijection between their vertex sets, which maps simplices of K to vertex sets of simplices in P.

By the Nöbeling-Pontryagin Theorem, any simplicial complex K of dimension n has a geometric realization in $\mathbb{R}^{2n+1}$.

Maps of polyhedra

A map $\varphi\colon P_1 \to P_2$ between two polyhedra is called:

- simplicial map, if it maps vertices to vertices so that faces go to faces, and is linear on each face;
- simplicial isomorphism, if it is simplicial and the inverse simplicial map exists;
- PL-map, if it is a simplicial map of a certain subdivision of P_1 to a certain subdivision of P_2;
- PL-homeomorphism, if it is PL and the inverse PL-map exists.

We shall also need the following crucial notion.
For any $\sigma \in K$ we define its link to be the subcomplex:

$$\mathrm{link}_K(\sigma) := \{\tau \in K \,|\, \sigma \cap \tau = \varnothing, \sigma \cup \tau \in K\}.$$

In what follows, we always assume that $\varnothing \in K$ and $\mathrm{link}_K(\varnothing) = K$.

The 5 classes of spheres

We call a simplicial complex K of dimension $n-1$:

- polytopal sphere, if it is isomorphic to the boundary of a simplicial n-polytope;
- starshaped sphere, if it is isomorphic to the underlying complex of a complete simplicial fan $\Leftrightarrow \exists$ a geometric realization and a point in $\mathbb{R}^n$ s.t. each ray emanating from this point meets this realization in exactly one point;
- PL-sphere, if it is PL-homeomorphic to $\partial\Delta^n$;
- simplicial sphere, if it is homeomorphic to S^{n-1};
- (rational) homology sphere, if for each $\sigma \in K$ one has:

$$\tilde{H}_i(\mathrm{link}_K(\sigma);\mathbb{Q}) = \begin{cases} 0, & \text{if } i < n-1-|\sigma|; \\ \mathbb{Q}, & \text{if } i = n-1-|\sigma|. \end{cases}$$

Bier spheres: general properties I

Here is the main property of the Bier's construction.

Theorem (Bier'92; de Longueville'04)

For any $K \neq \Delta_{[m]}$ on $[m]$ its Bier sphere $\mathrm{Bier}(K)$ is an $(m-2)$-dimensional PL-sphere with the number of vertices varying between m and $2m$.
More precisely, if one replaces each simplex

$$\sigma \sqcup \tau \in \mathrm{Bier}(K)$$

by a join $(s = (s_1 \subset \ldots \subset s_k) \in \mathrm{bs}(2^\sigma)$ and $t = (t_1 \subset \ldots \subset t_\ell) \in \mathrm{bs}(2^\tau))$

$$\mathrm{bs}(2^\sigma) * \mathrm{bs}(2^\tau) \ni s \sqcup t \colon (s_1 \subset \ldots \subset s_k \subset t_\ell^c \subset \ldots \subset t_1^c) \in \mathrm{bs}(\partial\Delta_{[m]})$$

of the first barycentric subdivisions, one gets the whole of $\mathrm{bs}(\partial\Delta_{[m]})$; that is, the nerve complex of the m-dimensional permutohedron Pe^m.

One of our key results concerns face vectors of Bier spheres.

Convex polytopes: face vectors

Let S be an n-dimensional simplicial polytope. Its f-vector is $(f_0, \dots, f_{n-1})$, where f_i is the number of i-dimensional faces in S. We also set $f_{-1} = 1$. By definition, for the dual simple polytope S^* we set $f(S^*) = f(S)$.

- h-vector $h(S) = h(S^*) = (h_0, h_1, \dots, h_n)$:

$$h_0 t^n + \ldots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \ldots + f_{n-1};$$

- g-vector $g(S) = g(S^*) = (g_0, g_1, \dots, g_{[n/2]})$:

$$g_0 = 1, g_i = h_i - h_{i-1} \text{ for } i > 0;$$

- γ-vector $\gamma(S) = \gamma(S^*) = (\gamma_0, \gamma_1, \dots, \gamma_{[n/2]})$:

$$h_0 + h_1 t + \ldots + h_n t^n = \sum_{i=0}^{[n/2]} \gamma_i t^i (1+t)^{n-2i}.$$

The last representation exists, because, the h-vector of any simple/simplicial polytope is **symmetric** by Dehn-Sommerville equations.

Flag homology spheres: conjectures

We can define the 4 face vectors above for any homology sphere in a similar way, since it was shown by Adiprasito in 2018 that the famous g-theorem (necessity part) holds for all (rational) homology spheres; in particular, the Dehn-Sommerville equations hold for them.

CONJECTURE (Gal'05): if K is a **flag** homology sphere, then its face vector $\gamma(K)$ is componentwise nonnegative.

In other words, $\gamma(K^{n-1}) \geq \gamma(I^n) = (1, 0, \ldots, 0)$.
The previous conjecture was soon generalized as follows.

CONJECTURE (Nevo-Petersen'10): if K is a **flag** homology sphere, then its face vector $\gamma(K)$ is the f-vector of a flag simplicial complex.

- Nevo, Petersen'10: for K with $\gamma_1(K) \leq 3$;
- Aisbett, Volodin'20: for $K = K_P$, where P is a 2-truncated cube.

Flag homology spheres: conjectures

The next two conjectures were stated by Chudnovsky and Nevo (2020). As a vertex link in a flag homology sphere is again a flag homology sphere, the following conjecture implies Gal's conjecture.

LINK CONJECTURE: if K is a **flag** homology sphere, then $\gamma(K) \geq \gamma(\mathrm{link}_K(v))$, coefficientwise, for any vertex v of K.

An equator in a flag homology sphere is a full subcomplex being a flag homology sphere of codimension 1. Therefore, the following conjecture is a generalization of the previous one; however they are in fact equivalent as was shown by Chudnovsky and Nevo.

EQUATOR CONJECTURE: if K is a **flag** homology sphere and E is its equator, then $\gamma(K) \geq \gamma(E)$, coefficientwise.

- Chudnovsky, Nevo'20: for $K = K_P$, where P is a 2-truncated 3-cube.

Flag Bier spheres: main result

Theorem (L., Živaljević'24)

Both the Nevo-Petersen and Equator Conjectures are true in the class of flag Bier spheres.

The key step is the classification of all flag Bier spheres: any flag Bier sphere is a polytopal sphere, which is combinatorially equivalent to the nerve complex of one of the following simple polytopes:

1. I^n for $n \geq 1$;
2. $I^n \times Q_5$ for $n \geq 0$, where Q_5 is a pentagon;
3. $I^n \times Q_6$ for $n \geq 0$, where Q_6 is a hexagon;
4. $I^n \times \mathcal{P}_4$ for $n \geq 0$, where $\mathcal{P}_4$ is a 3-dimensional Bier polytope that can be obtained from I^3 by performing two adjacent edge cuts.

Polyhedral products over Bier spheres

Now, we turn to the topological properties of moment-angle-complexes associated with Bier spheres.
Since K is a full subcomplex in $\mathrm{Bier}(K)$ on the vertex set $[m]$, we immediately obtain:

- A moment-angle-complex over a Bier sphere might have an arbitrary torsion in integer homology;
- Cohomology algebra of a moment-angle-complex over a Bier sphere might contain a higher order (in)decomposable nontrivial Massey product (dates back to the work of Denham-Suciu'07).

Let us recall the following results on the manifold structure for polyhedral products of the type $\mathcal{R}_K := (\mathbb{D}^1, \mathbb{S}^0)^K$ and $\mathcal{Z}_K := (\mathbb{D}^2, \mathbb{S}^1)^K$.

Combinatorial spheres and moment-angle-complexes

Let K be a simplicial complex on $[m]$ with $\dim K = n-1, n \geq 3$.

Theorem (Cai Li'17)

- $\mathcal{R}_K$ is a topological n-manifold $\Leftrightarrow$ K is a s.c. homology sphere;
- $\mathcal{Z}_K$ is a topological $(n+m)$-manifold $\Leftrightarrow$ K is a homology sphere.

Theorem (Panov, Ustinovsky'12; Tambour'12)

If K is a starshaped sphere, then $\mathcal{R}_K$ and $\mathcal{Z}_K$ are both homeomorphic to smooth manifolds.

Bier spheres: general properties II

It turns out that all Bier spheres satisfy the conditions of the last theorem.

Theorem (Jevtić, Timotijević, Živaljević'19)

Let $K \subset 2^{[m]}$ be a simplicial complex. Then $\mathrm{Bier}(K)$ has a geometric realization as a starshaped sphere in the hyperplane $H_0 := \{x \in \mathbb{R}^m \,|\, \langle u, x \rangle = 0\}$, where u is the sum of the standard basis vectors $e_i, 1 \leq i \leq m$ in $\mathbb{R}^m$.

Thus, (real) moment-angle-complexes over Bier spheres acquire equivariant smooth structures.

Bier spheres: general properties III

Furthermore, it turns out that all Bier spheres acquire characteristic maps.

Theorem (L., Sergeev'24)

Let $K \subset 2^{[m]}$ be a simplicial complex with $m \geq 2$. Then one has:

$$s_{\mathbb{R}}(\mathrm{Bier}(K)) = s(\mathrm{Bier}(K)) = m+1,$$

that is, the real and complex Buchstaber numbers of $\mathrm{Bier}(K)$ are both maximal possible.

In the above notations applied to $\mathbb{R}^{m-1}$ the characteristic map sends

$$i, i' \mapsto e_i, \text{ for all } 1 \leq i \leq m-1$$

and

$$m, m' \mapsto u.$$

Toric manifolds over Bier spheres

Using the two previous constructions we get their generalization.

Theorem (L., Živaljević'24)

Let $K \subset 2^{[m]}$ be a simplicial complex with $m \geq 2$.
Then $\mathrm{Bier}(K)$ is isomorphic to the underlying complex of a complete regular fan $\Sigma(K)$ in $\mathbb{R}^{m-1}$, whose cones have the form:

$$C(\sigma, \tau) := \mathbb{R}_{\geq 0}\langle -e_p, e_q \mid p \in \sigma, q' \in \tau\rangle \text{ for } \sigma \in K, \tau \in K^{\vee}, \sigma \sqcup \tau \in \mathrm{Bier}(K),$$

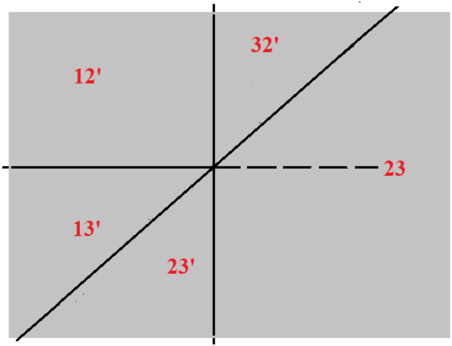
where we set $e_m := -u$.

This result shows that starting with an **arbitrary** simplicial complex K on $[m]$ one gets:

- a canonical smooth $(3m-1)$-dim **moment-angle manifold** $\mathcal{Z}_{\mathrm{Bier}(K)}$ and
- a canonical smooth $2(m-1)$-dim **toric manifold** $X_{\Sigma(K)}$

associated to K.

Example



Let $m = 3$ and $\mathrm{M}(K) = \{1, 23\}$. Then

$$\mathrm{M}(\mathrm{Bier}(K)) = \{12', 13', 23, 23', 32'\}$$

and we get the complete regular fan $\Sigma(K)$ drawn above.

Immediate applications

As immediate corollaries we get the next two statements.

Corollary 1

Dehn-Sommerville equations hold for all Bier spheres.

This was first proved by Björner et al. (2005) using purely combinatorial technics.

Corollary 2

There are infinitely many toric manifolds whose orbit spaces are not homeomorphic to any simple polytope as manifolds with corners.

This was first proved by Suyama (2015) using non-polytopality of the Barnette sphere.

References

- T. Bier, *A remark on Alexander duality and the disjunct join*, preprint (1992), 7pp.
- A. Björner, A. Paffenholz, J. Sjöstrand, G. M. Ziegler, *Bier spheres and posets*, Disc. Comp. Geom. **34** (2005), 71-86.
- F. D. Jevtić, M. Timotijević, R. T. Živaljević, *Polytopal Bier spheres and Kantorovich-Rubinstein polytopes of weighted cycles*, Disc. Comp. Geom. **65** (2019), 1275-1286.
- I. Limonchenko and M. Sergeev, *Bier spheres and toric topology*; preprint (2024); arXiv:2406.03597.
- M. de Longueville, *Bier spheres and barycentric subdivision*, J. Comb. Th., Series A **105** (2004), 355-357.
- J. Matoušek. *Using the Borsuk–Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry.* Universitext, Springer-Verlag, Berlin, 2003, 214pp.

THANK YOU FOR YOUR ATTENTION!