Supersolvable Posets & Fiber-Type Arrangements

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Configuration Spaces

Inputs $\begin{cases} X = \text{manifold of dimension} \geq 2 \\ n \in \mathbb{Z}_{>0} \end{cases}$

$$\mathrm{Conf}_n(X) = \{(x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\}$$

Applications to physics, robotics, ...

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Theorem [Fadell-Neuwirth '62] There is a fiber bundle

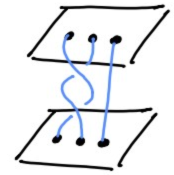
$$\begin{tikzcd}
X - \{n \text{ points}\} \arrow[r] & \mathrm{Conf}_{n+1}(X) \arrow[d] & (x_1, \dots, x_n, x_{n+1}) \arrow[d, mapsto] \\
& \mathrm{Conf}_n(X) & (x_1, \dots, x_n)
\end{tikzcd}$$

"forget the last point"

* This is a tool to study topological invariants of configuration spaces (eg. homotopy type, (co)homology, stability)

Motivating Example / $X = \mathbb{C}$

* $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$ is a pure braid group



[Fox-Neuwirth '62]

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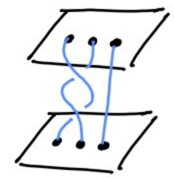

[Fox-Neuwirth '62]

- $\mathrm{Conf}_n(\mathbb{C})$ is $K(\pi, 1)$

LES $\cdots \to \pi_i(\mathbb{C} - \{n \text{ pts}\}) \to \pi_i(\mathrm{Conf}_{n+1}(\mathbb{C})) \to \pi_i(\mathrm{Conf}_n(\mathbb{C})) \to \cdots$

- $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$ is an iterated semidirect product of free groups

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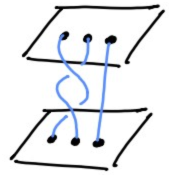
* $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$ is a pure braid group  [Fox-Neuwirth '62]
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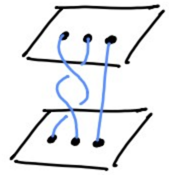
* $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$ is a pure braid group  [Fox-Neuwirth '62]
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* $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$ is an iterated semidirect product of free groups
* $H^*(\mathrm{Conf}_{n+1}(\mathbb{C})) \cong H^*(\mathrm{Conf}_n(\mathbb{C})) \otimes H^*(\mathbb{C} - \{n \text{ pts}\})$

  as $H^*(\mathrm{Conf}_n(\mathbb{C}))$-modules

  Serre spectral sequence $E_2^{pq} \cong H^p(\mathrm{Conf}_n(\mathbb{C}); \mathcal{H}^q(\mathbb{C} - \{n \text{ pts}\})) \Longrightarrow H^{p+q}(\mathrm{Conf}_{n+1}(\mathbb{C}))$

  or Leray-Hirsch Theorem

* The Poincaré polynomial has rational roots: [Arnol'd '69]

$$\sum_{i \geq 0} \dim H^i(\mathrm{Conf}_n(\mathbb{C}))\, t^i = \prod_{j=1}^{n-1} (1 + jt)$$

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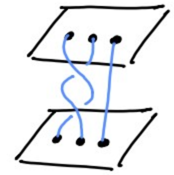

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* For each i, the sequence of S_n-representations

  $\{H^i(\mathrm{Conf}_n(\mathbb{C}))\}_n$ stabilizes [Church-Farb '13]

* and more...

Combinatorics

* view $\boxed{\mathrm{Conf}_n(X) = X^n - \bigcup_{1 \le i < j \le n} \Delta_{ij}}$

the complement of an arrangement of submanifolds

$\mathcal{A}_n = \{ \Delta_{ij} : 1 \le i < j \le n \}$ where $\Delta_{ij} = \{ (x_1, \ldots, x_n) \in X^n : x_i = x_j \}$

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Set of intersections $\bigcap_{\Delta \in S} \Delta$, $S \subseteq \mathcal{A}_n$ partially ordered by reverse inclusion	$\xleftrightarrow[\text{isomorphism}]{\text{poset}}$	set of partitions of $\{1, 2, \dots, n\}$ partially ordered by refinement

$$\begin{tikzcd}[column sep=small]
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\Delta_{12} \arrow[dr, dash] & \Delta_{13} \arrow[d, dash] & \Delta_{23} \arrow[dl, dash] \\
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\end{tikzcd}
\quad \cong \quad
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Question For what arrangements do we obtain a similar fibration?

$$
\begin{tikzcd}
X^{n+1} \arrow[d] & \supseteq X^{n+1} - \bigcup_{H \in \mathcal{A}} H \\
X^n &
\end{tikzcd}
$$

Goal Find a combinatorial characterization

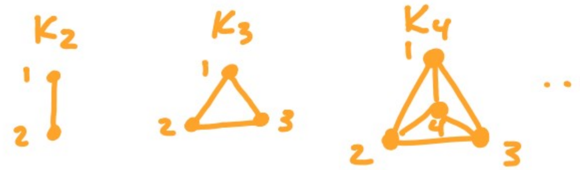
Generalized Configuration Spaces

"Partial configuration space"

Γ = simple graph on vertices $1, 2, \ldots, n$.

$$\boxed{\mathrm{Conf}_\Gamma(X)} = \{(x_1, \ldots, x_n) \in X^n \ : \ x_i \neq x_j \text{ when } \{i,j\} \text{ is an edge of } \Gamma\}$$

Note $\quad \mathrm{Conf}_{K_n}(X) = \mathrm{Conf}_n(X)$



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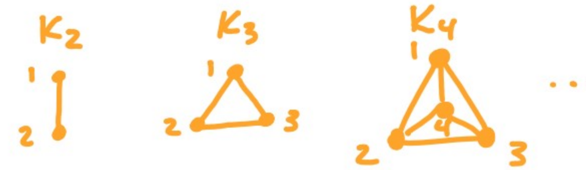

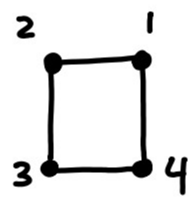

$$\mathrm{Conf}_{\square}(X)$$

$$\downarrow$$

$$\mathrm{Conf}_{\Gamma}(X) \ni (x_1, x_2, x_3)$$

if $x_1 = x_3$ fiber is $X - \{x_1\}$

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fibers not homeomorphic

$\Rightarrow$ not a fiber bundle

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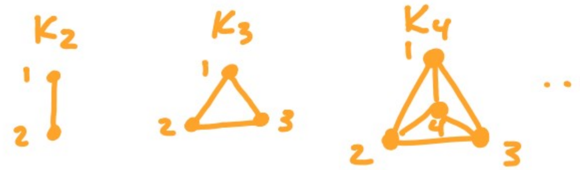
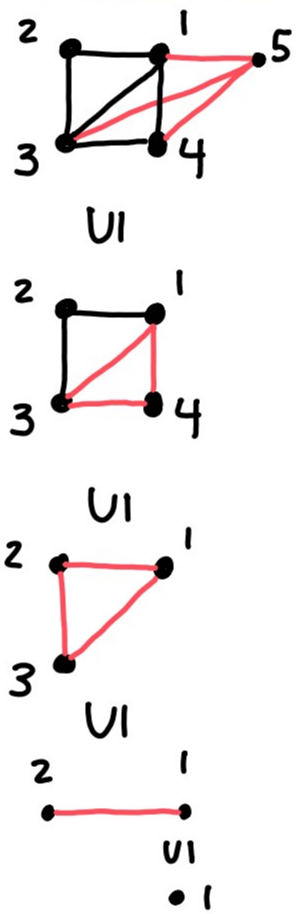


Γ chordal



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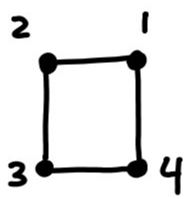

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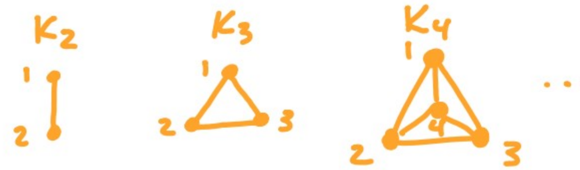

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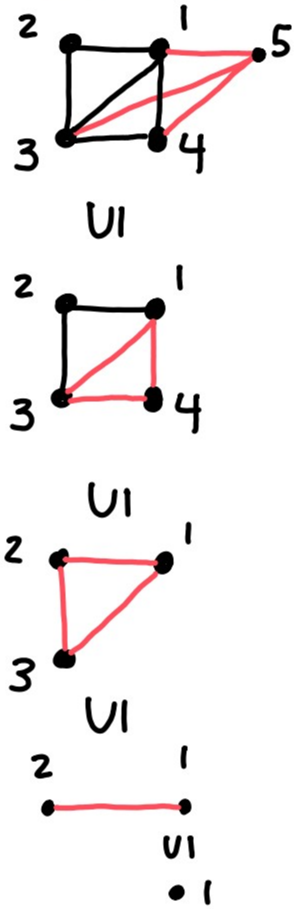

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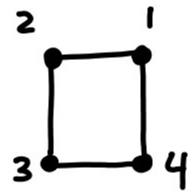

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"orbit configuration space"

$G \circlearrowright X$ free group action

$$\mathrm{Conf}_n^G(X) = \{(x_1, \dots, x_n) \in X^n : i \neq j \Rightarrow Gx_i \cap Gx_j = \emptyset\}$$

$\mathrm{Conf}_{n+1}^G(X)$ is a fiber bundle

$$\downarrow$$

$\mathrm{Conf}_n^G(X)$

[Xicotencatl '97]

Abelian Arrangements

X = connected abelian Lie group

$\cong (S^1)^d \times \mathbb{R}^v$ where $d+v \geq 2$

my favorite examples: $\mathbb{C}$, $\mathbb{C}^\times$, $S^1 \times S^1$

$$(a_1, \dots, a_n) \in \mathbb{Z}^n \longleftrightarrow \alpha : X^n \to X \rightsquigarrow H_\alpha = \ker(\alpha) \leq X^n$$

$$(x_1, \dots, x_n) \longmapsto \sum_{i=1}^n a_i x_i$$

An abelian arrangement is a finite set $\mathcal{A}$ of H_α in X^n.

The topological space:

arrangement complement

$$M(\mathcal{A}) = X^n - \bigcup_{H \in \mathcal{A}} H$$

* we understand $H^*(M(\mathcal{A}))$ better when X is not compact

(even $H^*(\mathrm{Conf}_n(S^1 \times S^1))$ not fully understood)

The combinatorial structure:

poset of layers $P(\mathcal{A})$

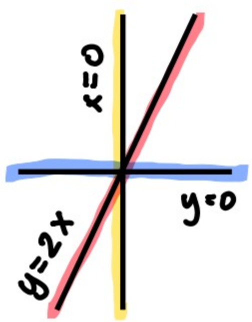
connected components of intersections $\bigcap_{H \in S} H$ $(S \subseteq \mathcal{A})$

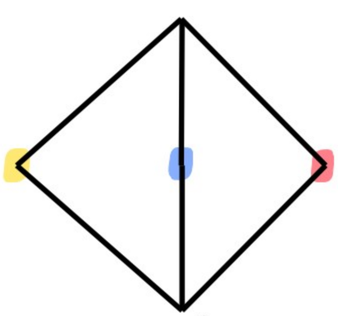
partially ordered by reverse inclusion

Examples/ $n=2$, $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$, $\alpha_3 = (2,-1)$

$x = 0$ $\quad$ $y = 0$ $\quad$ $2x - y = 0$

$X = \mathbb{C}$
(drawn $\mathbb{R}$)





X^2

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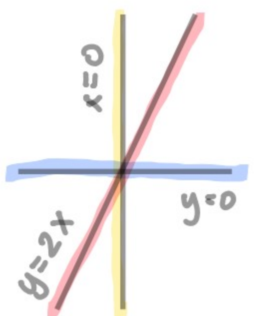
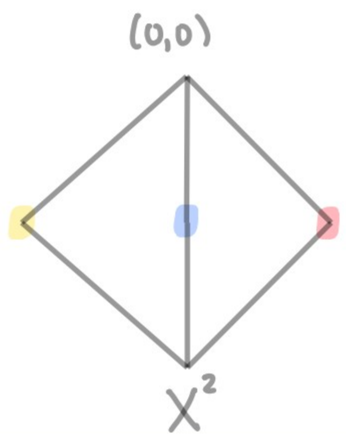
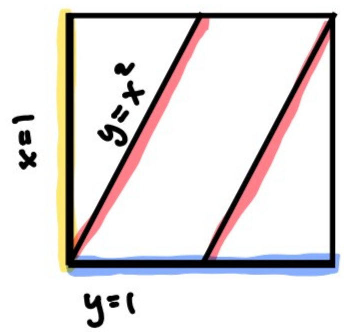
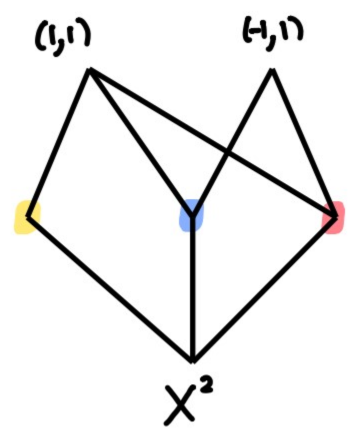




$X = \mathbb{C}^\times$

(drawn S^1)





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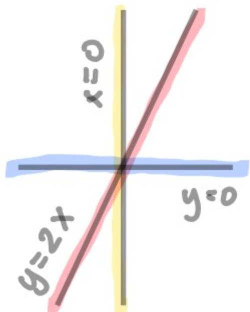

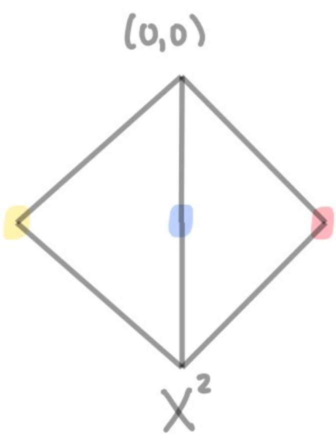

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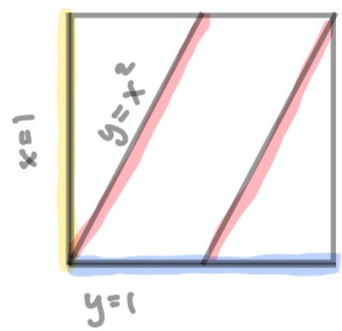

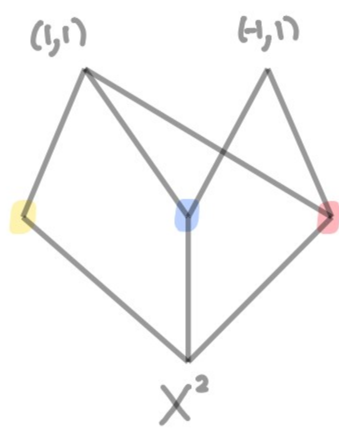

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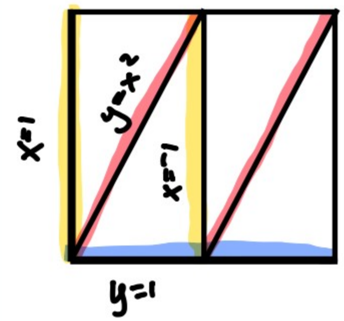

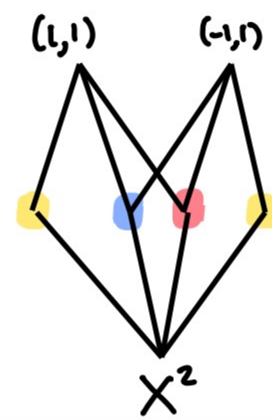

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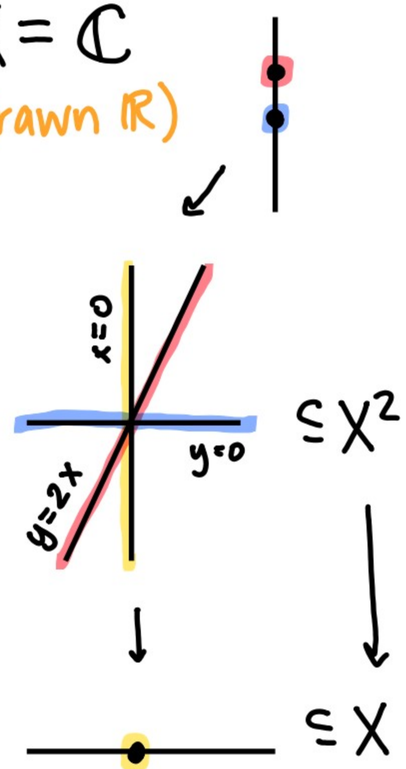

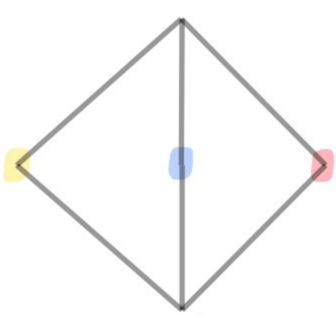

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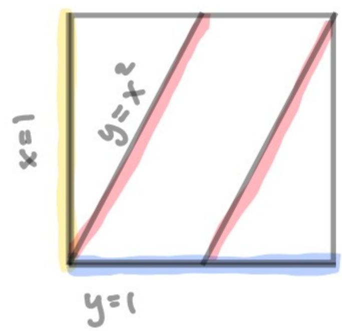

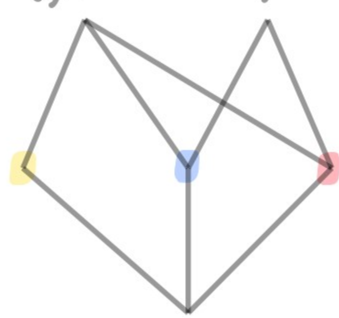

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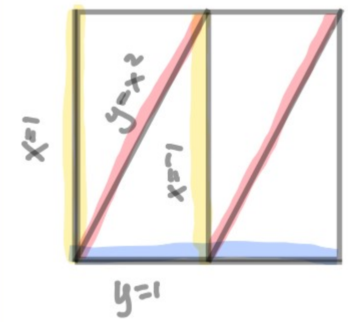

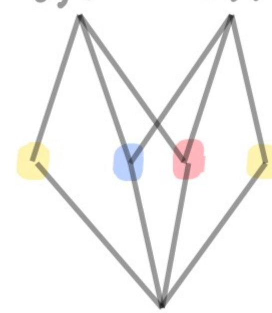

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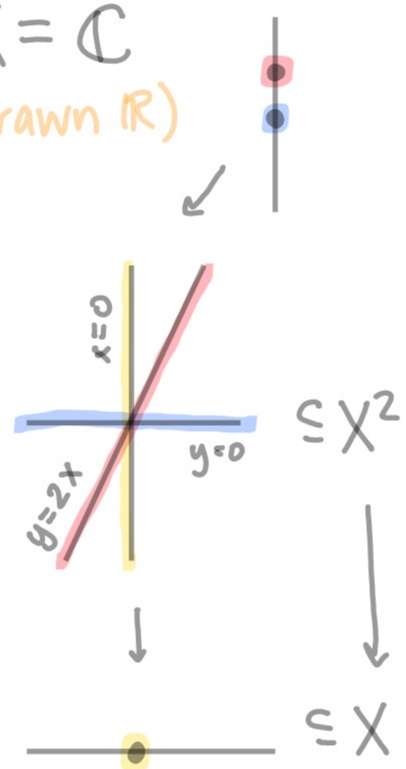

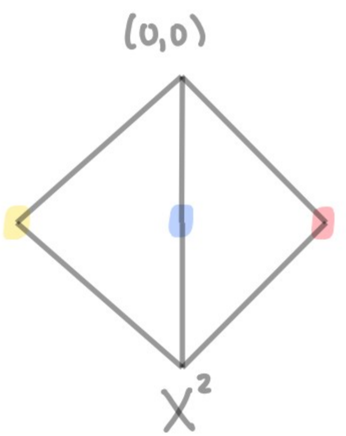

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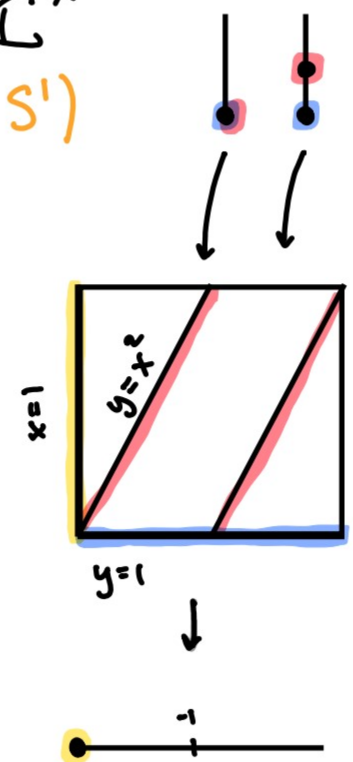

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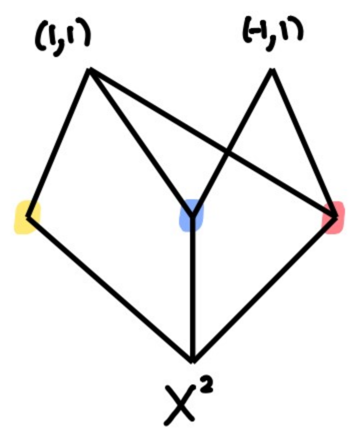
not a

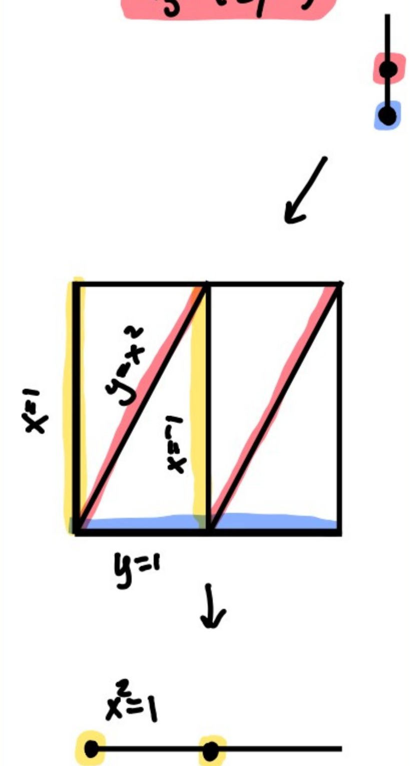
fiber bundle



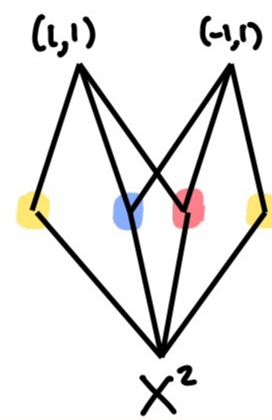
$\alpha_1=(2,0)$

$\alpha_2=(0,1)$

$\alpha_3=(2,-1)$

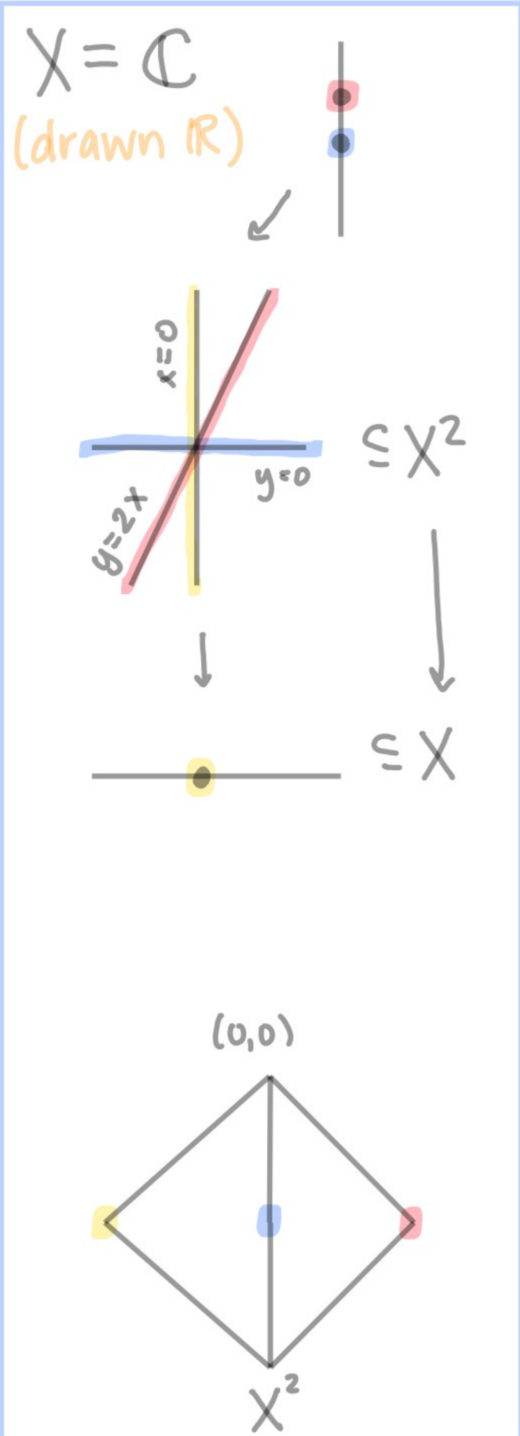


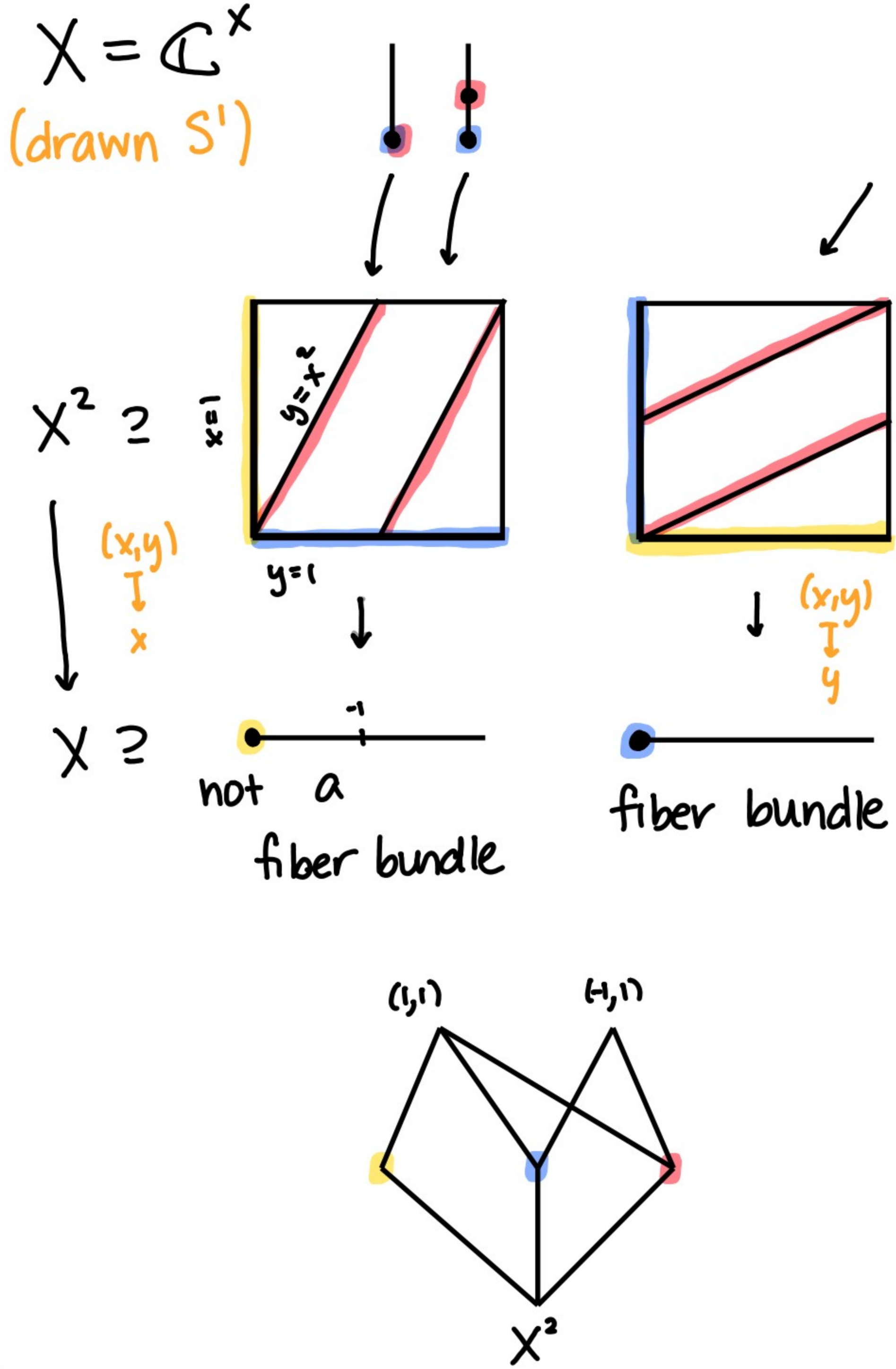
fiber bundle

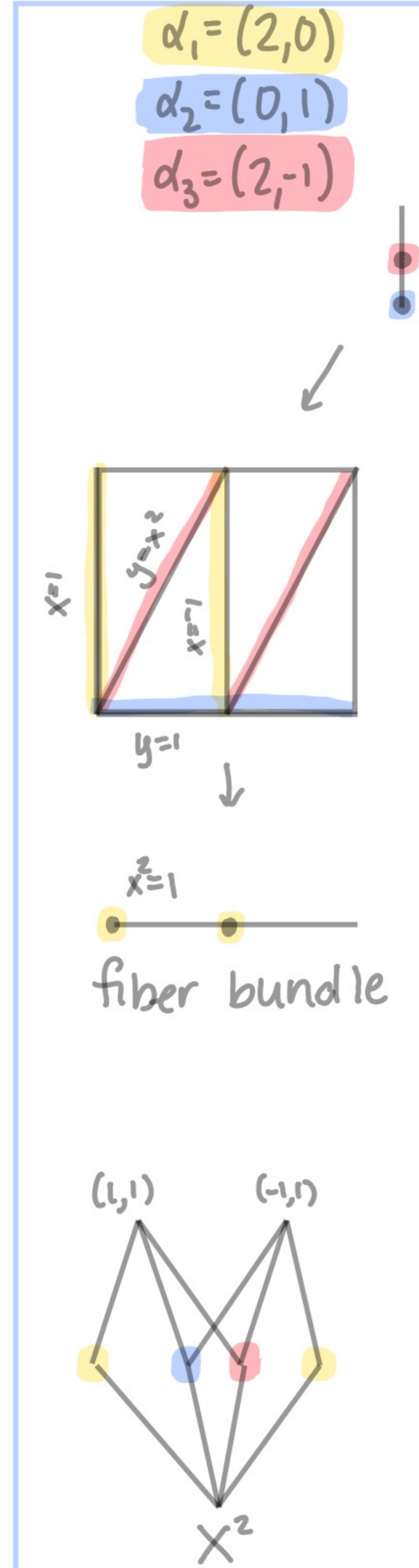


Examples/

$n=2$, $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$, $\alpha_3 = (2,-1)$







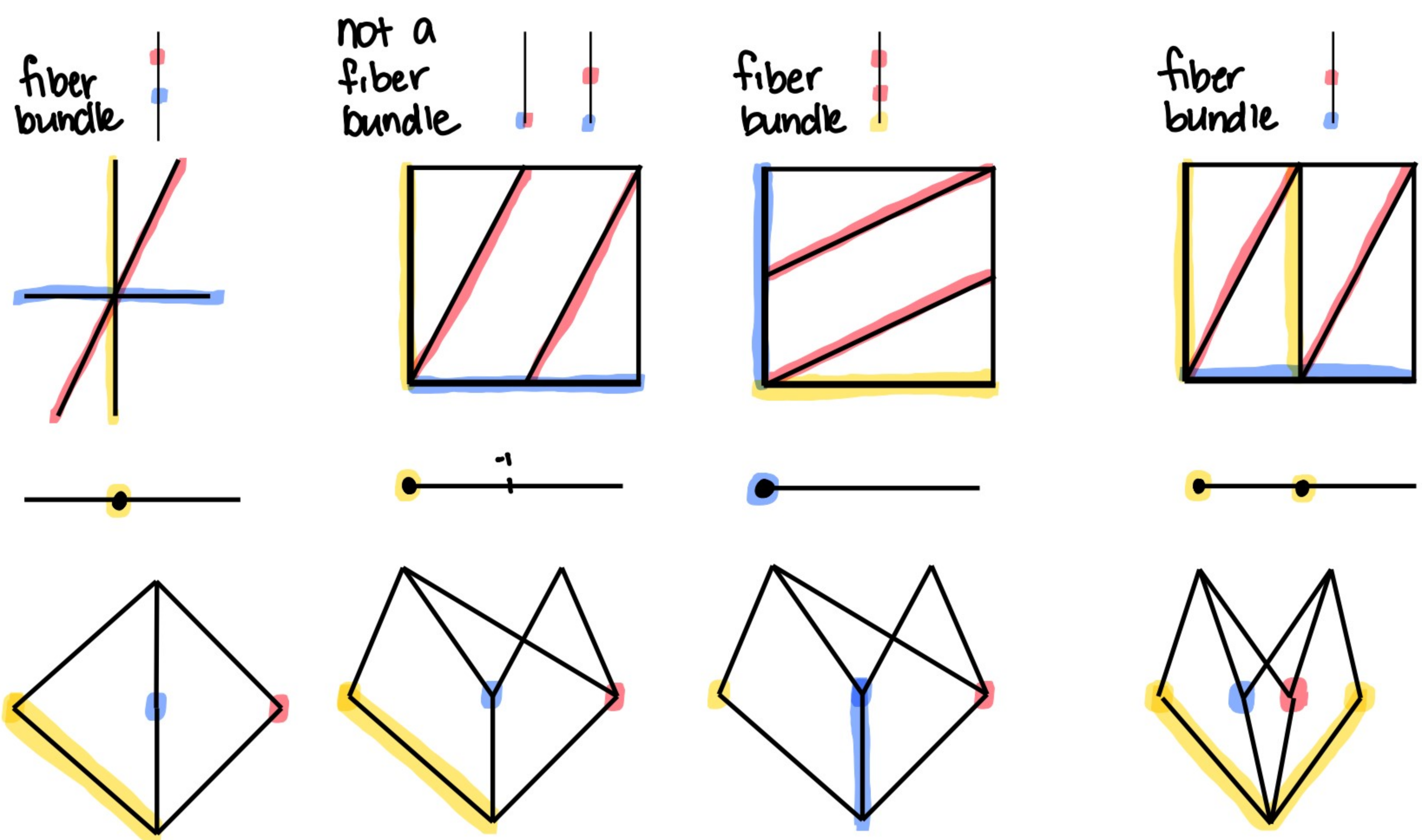
Supersolvability

[Stanley '72 for lattices]

$\mathcal{A}$ = abelian arrangement

$P = P(\mathcal{A})$ = poset of layers

atoms(P) = connected components of $H_\alpha \in \mathcal{A}$



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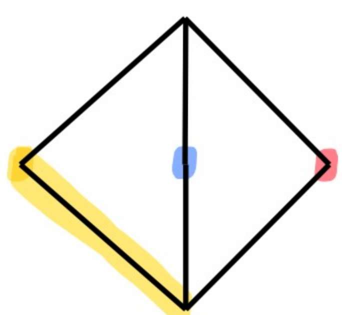
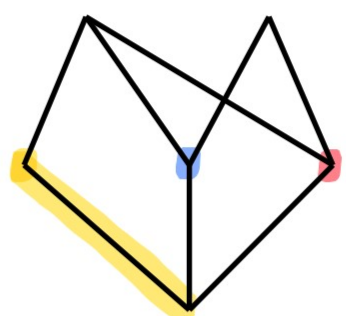
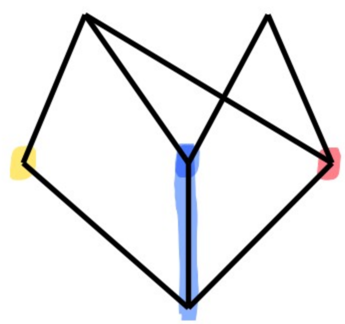


M-ideal



not an M-ideal



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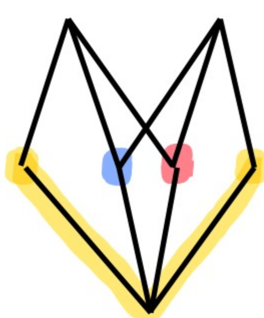

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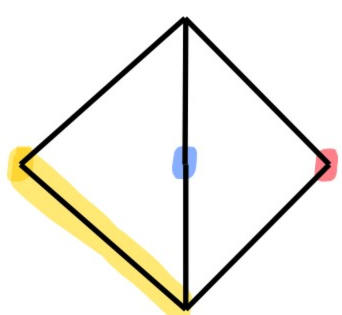
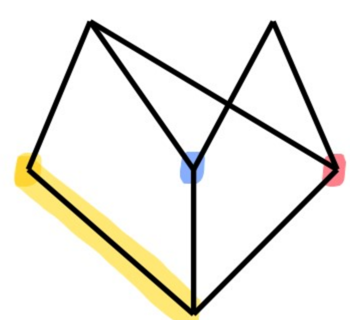
An m-ideal $Q \subseteq P$ is a **TM-ideal** if for any $H \in \text{atoms}(P) - Q$ and $y \in Q$, $H \cap y$ is connected



TM-ideal



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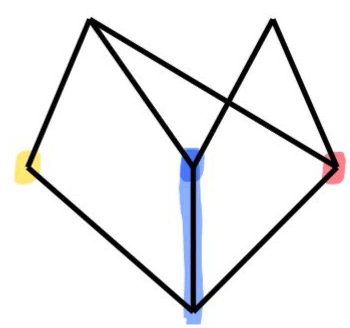
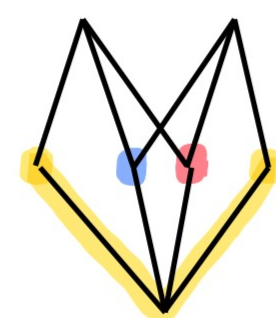


m-ideal



TM-ideal

Supersolvability

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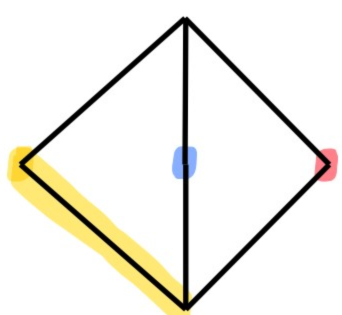
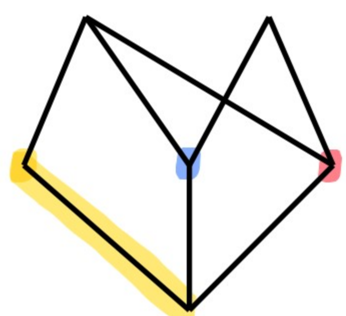
Say P is **strictly supersolvable** if there is a chain of TM-ideals

$$\{\min P\} = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_{r-1} \subsetneq Q_{r = rk(P)} = P$$



TM-ideal



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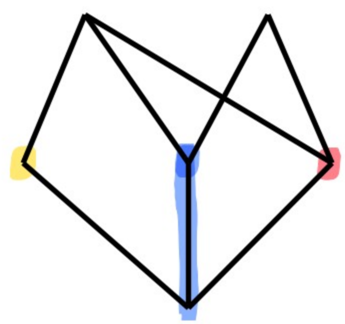
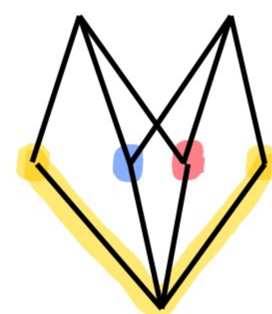
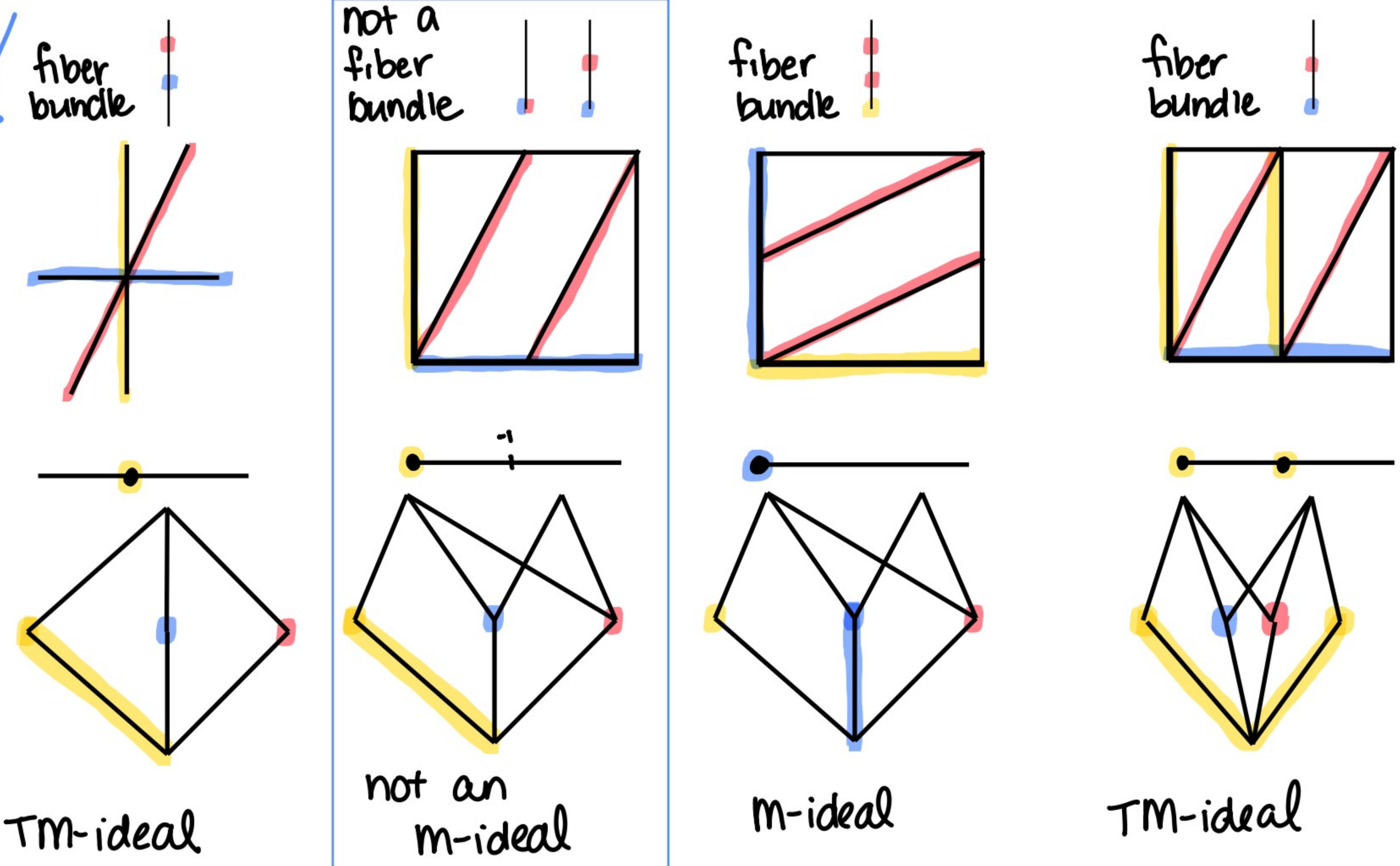


M-ideal



TM-ideal

Arrangement Bundles

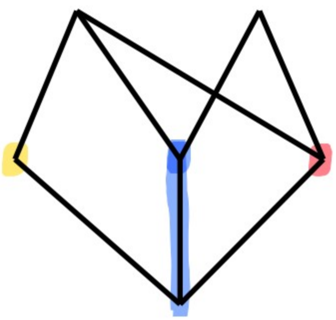


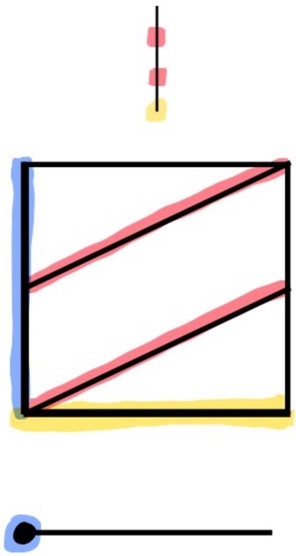
Theorem [B-Delucchi '22, Terao '86 case $X=\mathbb{C}$]

Let $\mathcal{A}$ be an abelian arrangement. There is a choice of coordinates so that $X^{n+1}\to X^n$ restricts to a fiber bundle $M(\mathcal{A})\to M(\mathcal{B})$ if and only if $P(\mathcal{A})$ has a corank-one M-ideal $(Q\cong P(\mathcal{B}))$

* Fiber is X with k points removed
* tower of fibrations $\longleftrightarrow$ supersolvable

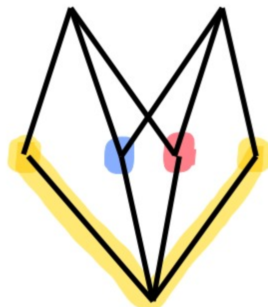
Pullback FN-bundles



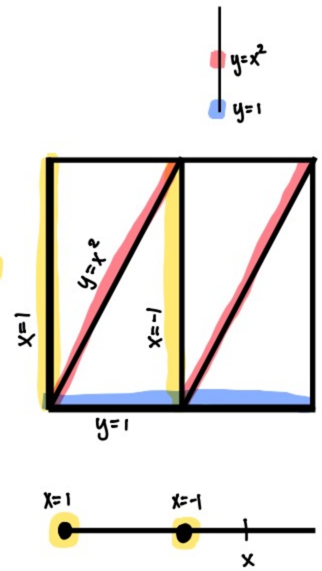


m-ideal

not TM-ideal ⇝ nontrivial monodromy



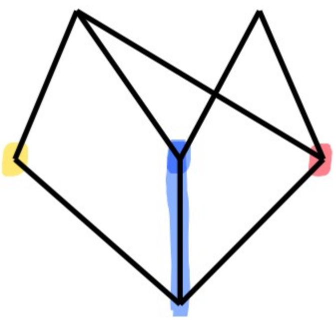
TM-ideal



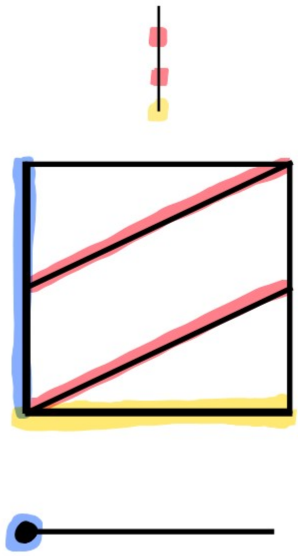
$$\begin{tikzcd} M(A) \arrow[d] & \\ M(B) \arrow[r] & \mathrm{Conf}_2(\mathbb{C}^\times) \end{tikzcd}$$

$$x \longmapsto (1, x^2)$$

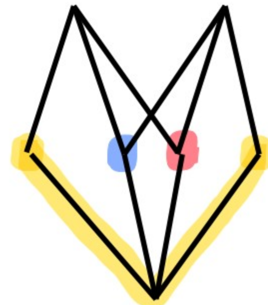
Pullback FN-bundles



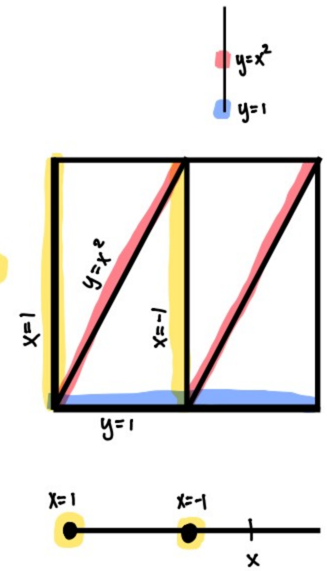
m-ideal



not TM-ideal $\rightsquigarrow$ nontrivial monodromy



TM-ideal



$$\begin{tikzcd}
M(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Conf}_3(\mathbb{C}^\times) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Conf}_4(\mathbb{C}) \arrow[d] \\
M(B) \arrow[r] & \mathrm{Conf}_2(\mathbb{C}^\times) \arrow[r] & \mathrm{Conf}_3(\mathbb{C})
\end{tikzcd}$$

$$x \longmapsto (1, x^2) \longmapsto (1, x^2, 0)$$

Theorem [B-Cohen-Delucchi]

If $X - \{k \text{ pts}\} \to M(A) \xrightarrow{\pi} M(B)$ is a fiber bundle associated to an m-ideal $Q \subseteq P(A)$, then we have a pullback diagram

$$\begin{tikzcd}
M(A) \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Conf}_{k+1}(X)/\Sigma_k \times \Sigma_1 \arrow[d] \\
M(B) \arrow[r] & \mathrm{Conf}_k(X)/\Sigma_k
\end{tikzcd}$$

$x \longmapsto$ punctures of $\pi^{-1}(x)$

Corollaries / $X = \mathbb{C}, \mathbb{C}^\times$, or $S^1 \times S^1$ and $P(A)$ supersolvable:

* $\pi : M(A) \to M(B)$ has a section

* $\pi_i(M(A)) = 0$ for $i > 1$ and $\pi_1(M(A)) \cong \rtimes_i F_{k_i}$

LES + induction $\cdots \to \pi_i(X - \{k \text{ pts}\}) \to \pi_i(M(A)) \to \pi_i(M(B)) \to \cdots$

Corollaries

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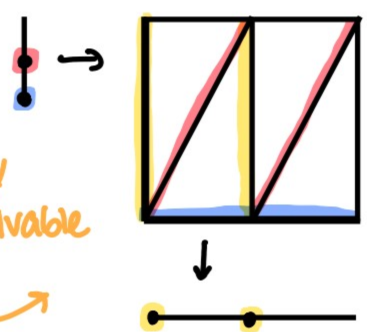
LES + induction $\cdots \to \pi_i(X - \{k\,\text{pts}\}) \to \pi_i(M(\mathcal{A})) \to \pi_i(M(\mathcal{B})) \to \cdots$

$X = \mathbb{C}$ or $\mathbb{C}^\times$ and $P(\mathcal{A})$ strictly supersolvable:

* $\pi_1(M(\mathcal{B})) \circlearrowright H^*(X - \{k\,\text{pts}\})$ trivial

* $H^*(M(\mathcal{A})) \cong H^*(M(\mathcal{B})) \otimes H^*(X - \{k\,\text{pts}\})$



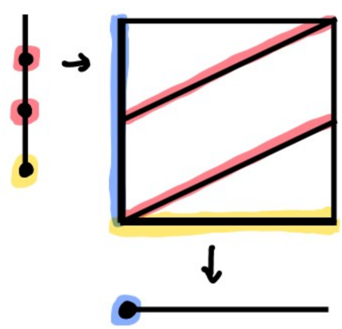
Poincaré polynomial

$(1+3t)(1+3t)$

strictly supersolvable

$= \mathbb{C} - \{0, \pm 1\}$

supersolvable but not strictly



Poincaré polynomial

$1 + 5t + 7t^2$

LCS Formula

Theorem [B-Delucchi '22]

[Falk-Randell '85 case $X=\mathbb{C}$, Kohno '85 case $\mathrm{Conf}_n(\mathbb{C})$]

$X = \mathbb{C}^\times$

chain of TM-ideals

$$\begin{array}{c} P(\mathcal{A}) \\ \rotatebox{90}{$\subsetneq$} \\ Q_{n-1} \\ \rotatebox{90}{$\subsetneq$} \\ \vdots \\ \rotatebox{90}{$\subsetneq$} \\ Q_1 \\ \rotatebox{90}{$\subsetneq$} \\ \{\hat{0}\} \end{array}$$

$$\begin{array}{ccc} \mathbb{C} - \{k_n \text{ pts}\} & \to & M(\mathcal{A}) \\ & & \downarrow \\ \mathbb{C} - \{k_{n-1} \text{ pts}\} & \to & M(\mathcal{A}_{n-1}) \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ & & M(\mathcal{A}_1) \end{array}$$

tower of fibrations

$k_i = 1 + \#\,\text{atoms}(Q_i) - Q_{i-1}$

The lower central series of $G := \pi_1(M(\mathcal{A}))$

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots \qquad \text{where} \qquad G_j = [G_{j-1}, G]$$

* Each $G(j) := G_j / G_{j+1}$ is a free abelian group and

* $$\prod_{j=1}^{\infty} (1 - t^j)^{\operatorname{rank} G(j)} = \prod_{i=1}^{n} (1 - k_i t) = \mathrm{Poin}_{M(\mathcal{A})}(-t)$$

Poincaré polynomial

Thank you