Golod and tight manifold triangulations

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Based on joint work with Kouyemon Iriye

Toronto; July 2024

Plan

1. Golod complex
2. Tight complex
3. Main result

1. Golod complex

Golodness

- Let $\mathbb{F}$ be a field.
- Let $S = \mathbb{F}[x_1, \ldots, x_n]$, where $|x_i| = 2$.
- Let $R = S/I$, where I is a homogeneous ideal with $I \subset (x_1, \ldots, x_n)^2$.

Proposition

There is a coefficient-wise inequality

$$P(\mathrm{Tor}^R(\mathbb{F}, \mathbb{F}); t) \leq \frac{(1 + t^2)^n}{1 - t(P(\mathrm{Tor}^S(R, \mathbb{F}); t) - 1)}$$

where $P(V; t)$ denotes the Poincaré series of a graded vector space V.

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Definition

We say that R is Golod if the equality holds.

Koszul homology

- Let $\mathcal{K}(R)$ denote the Koszul complex of R (with respect to the presentation $R = S/I$).
- There is an isomorphism $H_*(\mathcal{K}(R)) \cong \mathrm{Tor}^S(R, \mathbb{F})$.

Theorem (Golod '62)

The ring R is Golod if and only if all products and (higher) Massey products in $H_*(\mathcal{K}(R))$ vanish.

Stanley-Reisner ring

- Let K be a simplicial complex with vertex set $[m] = \{1, 2, \ldots, m\}$.

Definition

The Stanley-Reisner ring of K over $\mathbb{F}$ is defined by

$$\mathbb{F}[K] = \mathbb{F}[x_1, \ldots, x_m]/(x_{i_1} \cdots x_{i_k} \mid \{i_1, \ldots, i_k\} \notin K)$$

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Problem

Given a property of noetherian ring, find a characterization of it for $\mathbb{F}[K]$ in terms of K.

Golod complex

Definition

A simplicial complex K is $\mathbb{F}$-Golod if $\mathbb{F}[K]$ is Golod.

Example

The Alexander dual of a Cohen-Macaulay complex is Golod.

Example

The square graph C_4 is not $\mathbb{F}$-Golod for any field $\mathbb{F}$ as the Koszul homology of $\mathbb{F}[C_4]$ has non-trivial products.

Example

There is a triangulation of $\mathbb{R}P^2$ which is $\mathbb{Q}$-Golod but not $\mathbb{F}_2$-Golod.

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Problem

Find a combinatorial characterization of $\mathbb{F}$-Golod complexes.

Moment-angle complex

Definition

The moment-angle complex for K is defined by

$$Z_K = \bigcup_{\sigma \in K} Z(\sigma),$$

where $Z(\sigma) = X_1 \times \cdots \times X_m$ such that $X_i = D^2$ for $i \in \sigma$ and $X_i = S^1$ for $i \notin \sigma$.

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Theorem (Baskakov, Buchstaber, and Panov '04)

The cellular cochain complex of Z_K is homotopy equivalent to the Koszul complex of $\mathbb{F}[K]$.

Corollary

If Z_K is a suspension, then K is $\mathbb{F}$-Golod for any field $\mathbb{F}$.

Fat-wedge filtration

Definition

For $i = 0, 1, \ldots, m$, let

$$Z_K^i = \{(x_1, \ldots, x_m) \in Z_K \mid \text{at least } m - i \text{ of } x_j \text{ are the basepoint}\}.$$

The filtration

$$* = Z_K^0 \subset Z_K^1 \subset \cdots \subset Z_K^{m-1} \subset Z_K^m = Z_K$$

is called the fat-wedge filtration.

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Theorem (Iriye and K '19)

The fat-wedge filtration is a cone decomposition, where the attaching maps are explicitly described in terms of K.

Theorem (Iriye and K '19)

The moment-angle complex Z_K is a suspension iff the fat-wedge filtration is trivial.

Desuspending moment-angle complexes

- A sequentially Cohen-Macaulay complex is a generalization of a Cohen-Macaulay complex.
- A totally homology fillable complex is a generalization of the Alexander dual of a sequentially Cohen-Macaulay complex.

Theorem (Iriye and K '19)

If K is a totally homology fillable complex, then Z_K is a suspension.

Desuspending moment-angle complexes

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Theorem (Iriye and K '19)

If K is a totally homology fillable complex, then Z_K is a suspension.

- We say that K is k-neighborly if any $\sigma \subset [m]$ with $|\sigma| = k + 1$ is a simplex of K.

Theorem (Iriye and K '19)

If K is $\lceil \frac{\dim K}{2} \rceil$-neighborly, then Z_K is a suspension.

1-dimension

Theorem

If K is a graph, then TFAE:

1. *K is Golod;*
2. *K is chordal, that is, minimal cycles in K are triangles;*
3. *Z_K is a suspension.*

Corollary

A triangulation of a circle is minimal if and only if it is a triangle which is the minimal triangulation.

2-dimension

Theorem (Iriye and K '18)

If K is a triangulation of a closed $\mathbb{F}$-oriented surface, then TFAE:

1. *K is Golod;*
2. *K is 1-neighborly;*
3. *Z_K is a suspension.*

Remark

1. Most 1-neighborly triangulation of surfaces are minimal, so in most cases, triangulations of a surface is Golod if and only if it is minimal.
2. General 2-dimensional Golod complexes are characterized by 1-neighborliness and other condition.

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Question

Is Golodness of manifold triangulations related to minimal triangulations?

2. Tight complex

Tight embedding

- Let M be a closed connected manifold.

Theorem (Chern and Lashof '57)

The total absolute curvature of an embedding $M \to \mathbb{R}^N$ is bounded below by the number of critical point of some Morse function on M.

Corollary

The total absolute curvature of an embedding $M \to \mathbb{R}^N$ is bounded below by the sum of the Betti numbers of M.

Definition

We say that an embedding $M \to \mathbb{R}^N$ is $\mathbb{F}$-tight if its total absolute curvature equals the sum of the Betti numbers of M over $\mathbb{F}$.

Theorem

An embedding $f\colon M \to \mathbb{R}^N$ is $\mathbb{F}$-tight if and only if the natural map

$$H_*(f(M)\cap; \mathbb{F}) \to H_*(f(M); \mathbb{F})$$

is injective for almost all half spaces H.

Tight complex

- The full subcomplex of K over $\emptyset \neq I \subset [m]$ is defined by

$$K_I = \{\sigma \in K \mid \sigma \subset I\}.$$

Definition

A connected simplicial complex K is called $\mathbb{F}$-tight if the natural map

$$H_*(K_I; \mathbb{F}) \to H_*(K; \mathbb{F})$$

is injective for all $\emptyset \neq I \subset [m]$.

Example

Observe that if K is $\mathbb{F}$-tight, then it is 1-neighborly. Hence a triangulation of a circle is $\mathbb{F}$-tight if and only if it is a triangle graph.

Minimal triangulation

Definition

A triangulation K of a space X is strongly minimal if for any k, it has less or equal k-simplices than any other triangulations of X.

Conjecture (Kühnel and Lutz '99)

Every $\mathbb{F}$-tight manifold triangulation is strongly minimal.

Remark

The conjecture was verified by Kühnel and Lutz '99 in dimension 2 and by Baguchi, Datta, and Spreer '17 in dimension 3.

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Question

Is there a relation between Golodness and tightness for manifold triangulations?

Tight-neighborly manifold triangulation

Definition

A d-manifold triangulation K is called tight-neighborly if

$$\binom{m-d-1}{2} = \binom{d+2}{2} b_1(K;\mathbb{F}).$$

Remark

Tight-neighborliness does not depend on the ground field $\mathbb{F}$, and is much easier to check than tightness.

Proposition

Tight-neighborly manifold triangulations are vertex minimal.

Theorem (Bagchi, Datta and Spreer '17)

For a d-manifold triangulation K with $d \geq 3$, we consider the conditions:

1. *K is $\mathbb{F}$-tight;*
2. *K is tight-neighborly;*

Then 1 *implies* 2*. Moreover, if $d = 3$, then* 2 *implies* 1*.*

3-dimension

Theorem (K and Iriye '23)

Let K be a triangulation of an $\mathbb{F}$-oriented d-manifold with $d \geq 3$, and consider the conditions:

1. *K is $\mathbb{F}$-Golod;*
2. *K is $\mathbb{F}$-tight;*
3. *K is tight-neighborly;*
4. *Z_K is a suspension.*

Then there are implications

$$1 \implies 2 \impliedby 3 \implies 4 \implies 1.$$

Moreover if $d = 3$, then

$$2 \implies 3$$

so all conditions are equivalent.

Golod and tight manifold triangulations

Question

Are Golodness and tightness equivalent for manifold triangulations?

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- K is weakly $\mathbb{F}$-Golod if products in the Koszul homology of $\mathbb{F}[K]$ vanish.

Proposition (Iriye and K '23)

Let K be a triangulation of an $\mathbb{F}$-oriented manifold. If K is weakly $\mathbb{F}$-Golod, then it is $\mathbb{F}$-tight.

Remark

In dim ≤ 3, all (higher) Massey products in the Koszul homology of $\mathbb{F}[K]$ are trivial, but in dim ≥ 4, there may be nontrivial ones.

Remark

There is a simplicial complex which is weakly $\mathbb{F}$-Golod but not $\mathbb{F}$-Golod (Katthän '17).

3. Main result

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Theorem

If K is an $\mathbb{F}$-tight complex, then it is an $\mathbb{F}$-Golod complex.

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If K is an $\mathbb{F}$-tight complex, then it is an $\mathbb{F}$-Golod complex.

Corollary

Let K be a triangulation of an $\mathbb{F}$-orientable manifold. Then K is $\mathbb{F}$-Golod if and only if it is $\mathbb{F}$-tight.

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Let K be a triangulation of an $\mathbb{F}$-orientable manifold. Then K is $\mathbb{F}$-Golod if and only if it is weakly $\mathbb{F}$-Golod.

The dga $\mathcal{C}^*(K)$

Definition

We define the dga $\mathcal{C}^*(K)$ by $\mathcal{C}^0(K) = \mathbb{F}$ and

$$\mathcal{C}^p(K) = \bigoplus_{\emptyset \neq I \subset [m]} \widetilde{C}^{p-|I|-1}(K_I)$$

for $p > 0$ such that the differential is induced from those of $C^*(K_I)$ and the product is given by

$$\widetilde{C}^{p-|I|-1}(K_I) \otimes \widetilde{C}^{q-|J|-1}(K_I) \to \widetilde{C}^{p+q-|I\cup J|-1}(K_{I\cup J})$$

for $I \cap J = \emptyset$ which is induced from the inclusion $K_{I\cup J} \to K_I * K_J$.

Remark

The dga $\mathcal{C}^*(K)$ is motivated by the cellular cochain complex of Z_K.

Proposition

The dga $\mathcal{C}^*(K)$ *is homotopy equivalent to the Koszul complex of* $\mathbb{F}[K]$.

- Then we need to show that if K is $\mathbb{F}$-tight, then all (higher) Massey products and products in $H^*(\mathcal{C}^*(K))$ vanish.
- To this end, we need to consider higher homotopies among simplicial complexes.

Higher prism operator

- We may choose a total order on the vertex set of a simplicial complexe.
- Then for simplicial complexes K and L, we can define a simplicial complex $K \otimes L$ which triangulates $|K| \times |L|$.

Theorem

Given a simplicial map $H \colon K \otimes \Delta^q \to L$*, there is a map*

$$P_H \colon C^*(L) \to C^{*-q}(K)$$

satisfying

$$\partial P_H + (-1)^{q-1} P_H \partial = \sum_{i=0}^{q} (-1)^i P_{H \circ (1 \otimes d^i)}$$

where $d^i \colon \Delta^{q-1} \to \Delta^q$ *denotes the* i*-th coface operator.*

- The higher prism operator P_H is defined by counting all lattice paths.
- If $q = 0$, then $P_H = H$.
- If $q = 1$, then P_H is the usual prism operator as

$$\begin{aligned}\partial P_H + P_H \partial &= P_{H\circ(1\otimes d^0)} - P_{H\circ(1\otimes d^1)} \\ &= H|_{K\otimes 1} - H|_{K\otimes 0}.\end{aligned}$$

- The proof of the main theorem is done by applying higher prism operators to the complex $\mathcal{C}^*(K)$.
- Tightness is used to lift a cocycle in $\widetilde{C}^*(K_I) \subset \mathcal{C}^*(K)$ to that in $\widetilde{C}^*(K)$, which proves that products in $H^*(\mathcal{C}^*(K))$ vanich.

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Thank you!