An invitation to graph braid groups

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<u>Ex</u> The configuration spaces of an aspherical surface are aspherical

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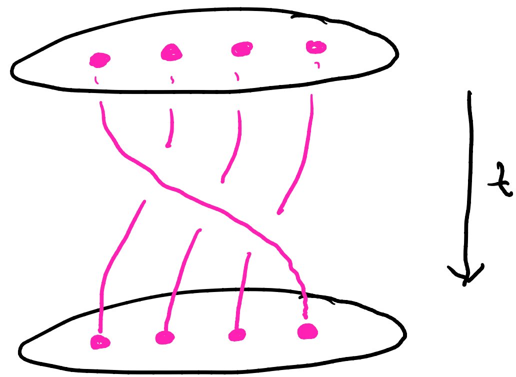

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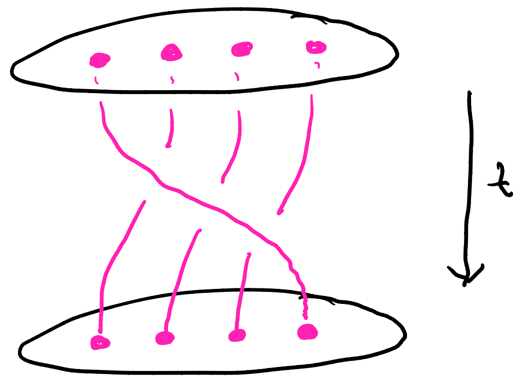

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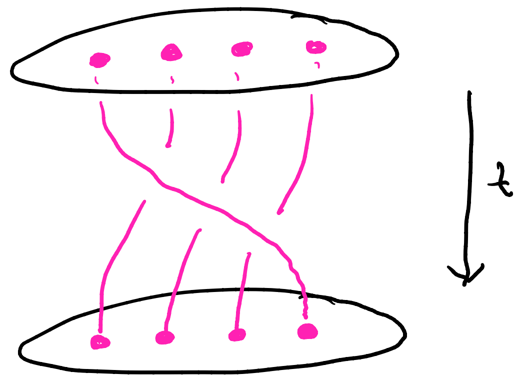

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In this talk, X will be a graph Γ.

Thm (Abrams, Świątkowski) The configuration spaces of a graph are aspherical.

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Idea

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<u>Idea</u> Abrams shows $F_k^{\square}(\Gamma)$, the largest subcomplex of Γ^k contained in $F_k(\Gamma)$

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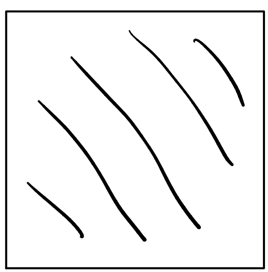

Γ^2

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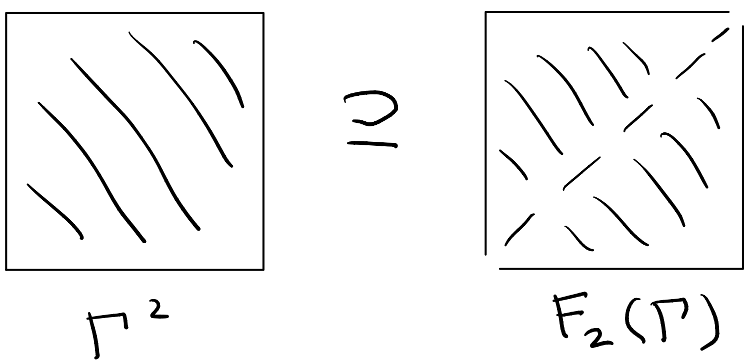

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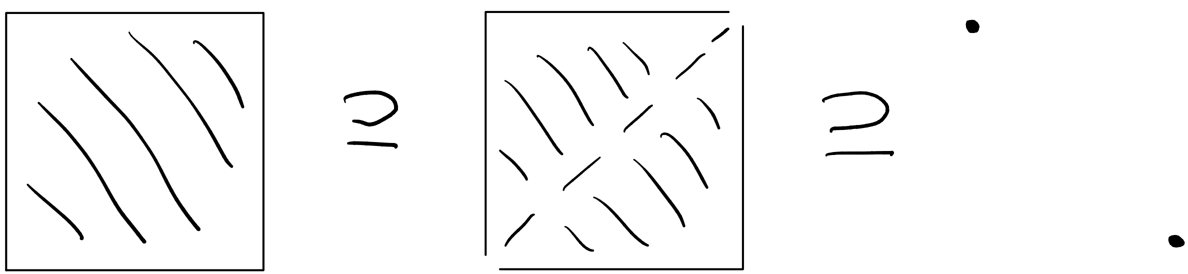
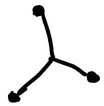
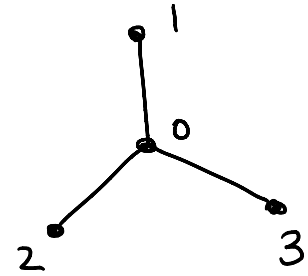
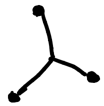
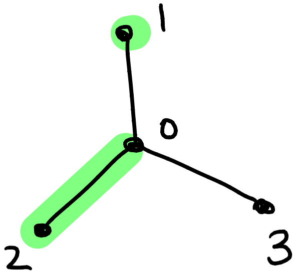


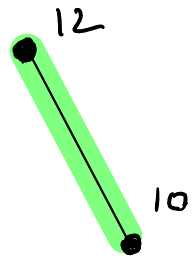
$\Gamma^2 \supseteq F_2(\Gamma) \supseteq F_2^{\square}(\Gamma)$

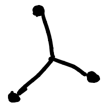
Ex $\Gamma =$ , $k=2$

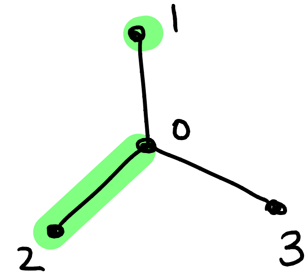


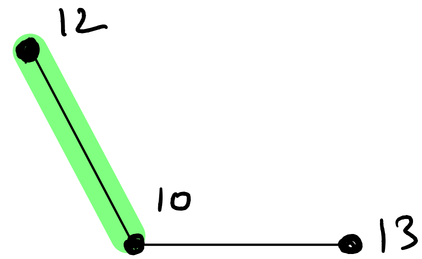
Ex $\Gamma =$ , $k=2$

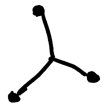


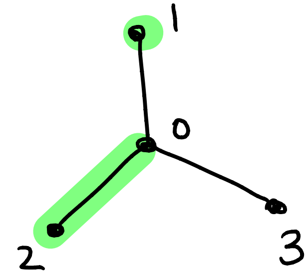


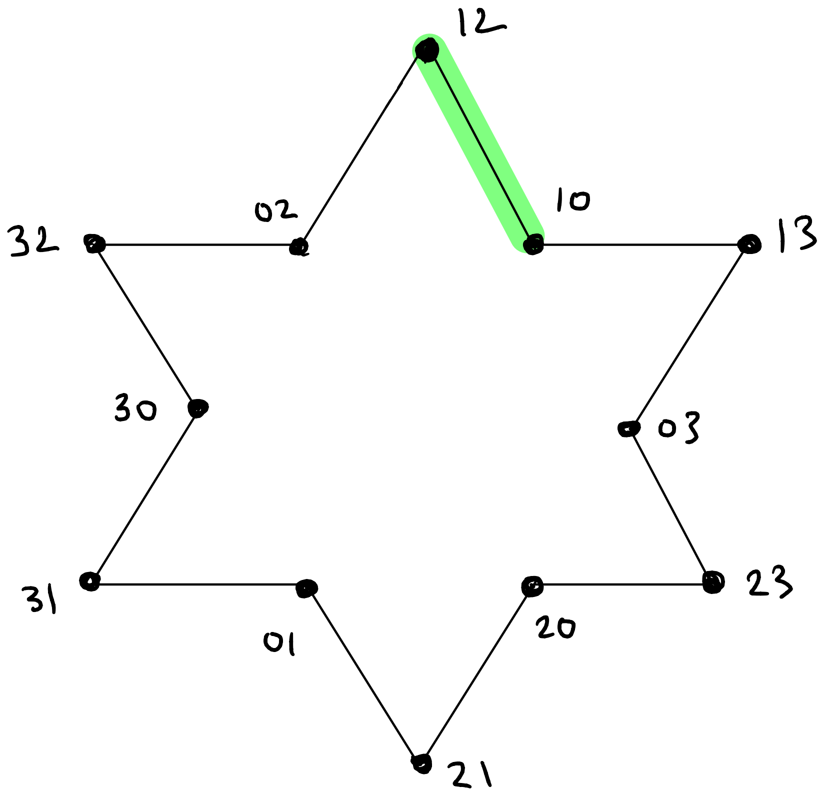
Ex $\Gamma =$  , $k=2$

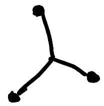


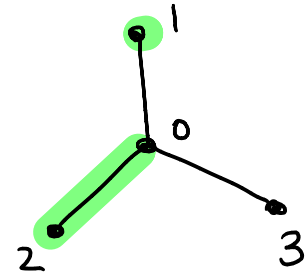


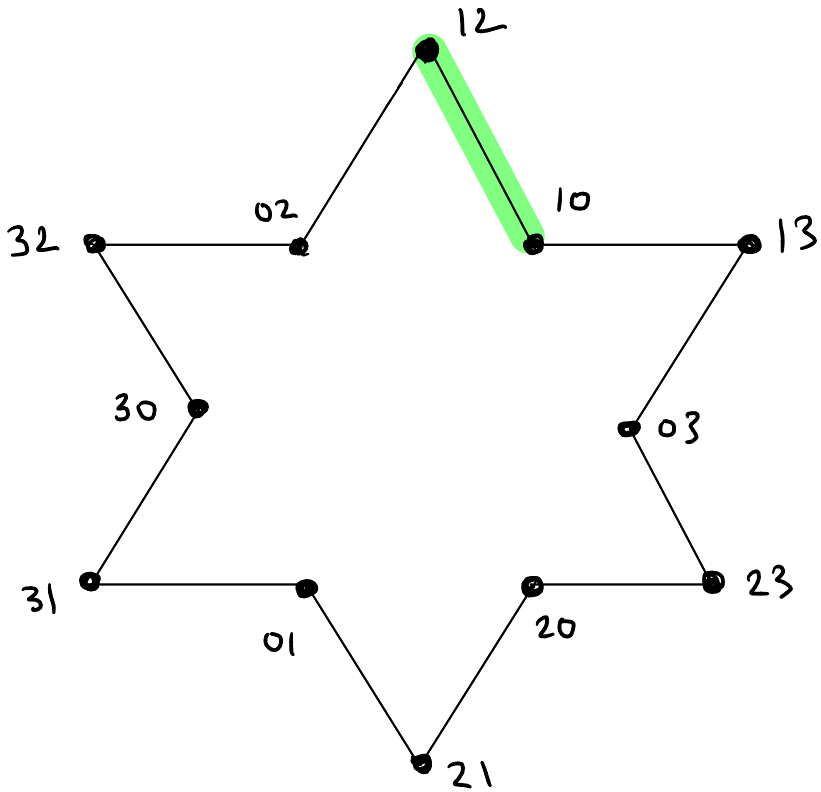
Ex $\Gamma =$  , $k=2$





Ex $\Gamma =$  , $k=2$





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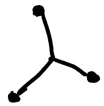
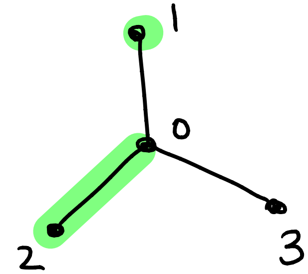
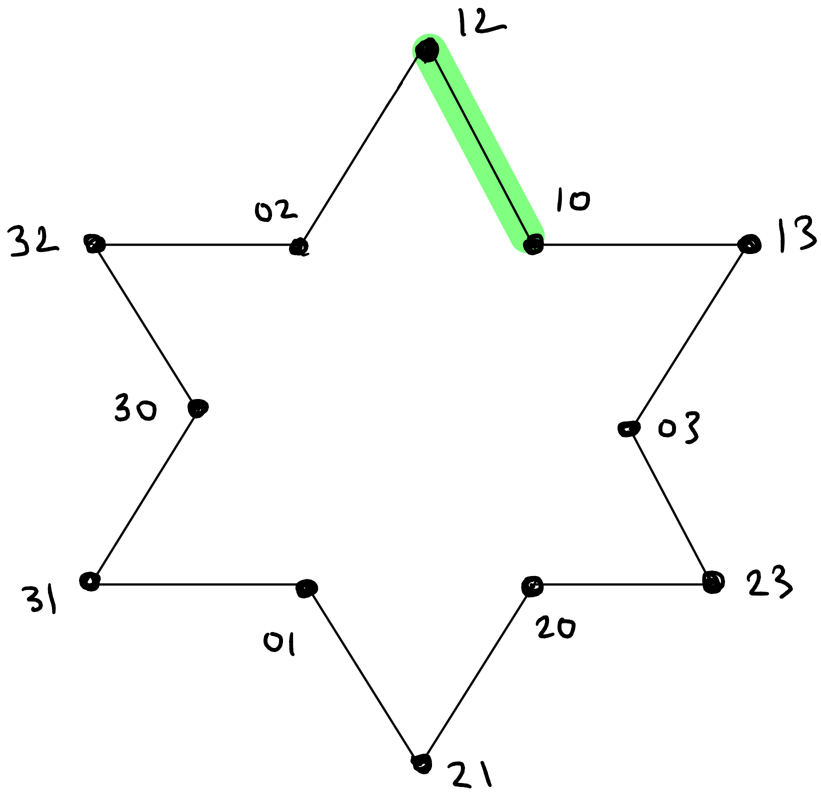
Ex $\Gamma =$ , $k=2$





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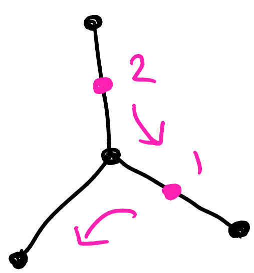
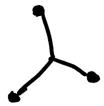
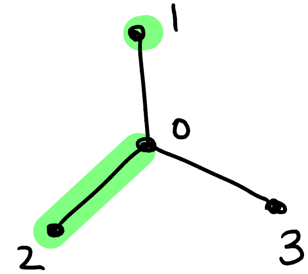
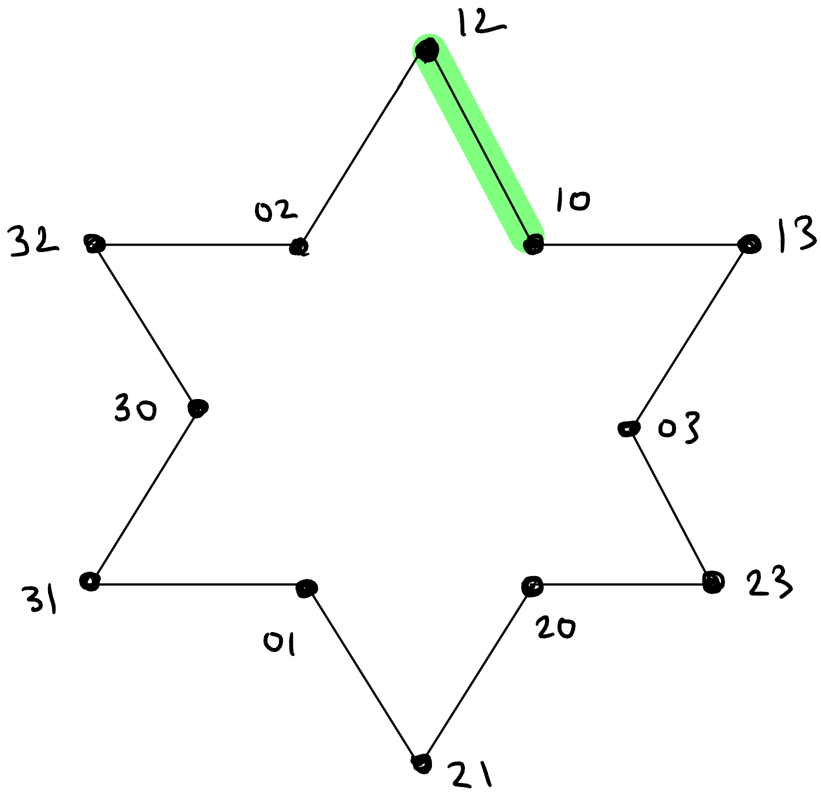


Ex $\Gamma =$ , $k=2$





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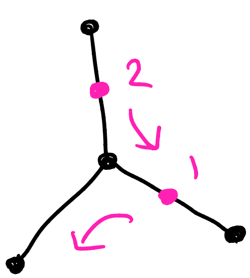
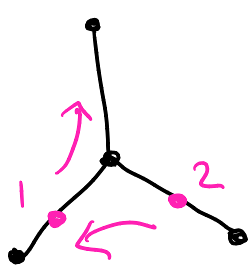
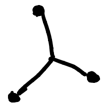
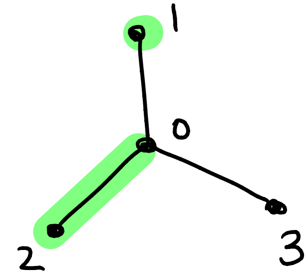
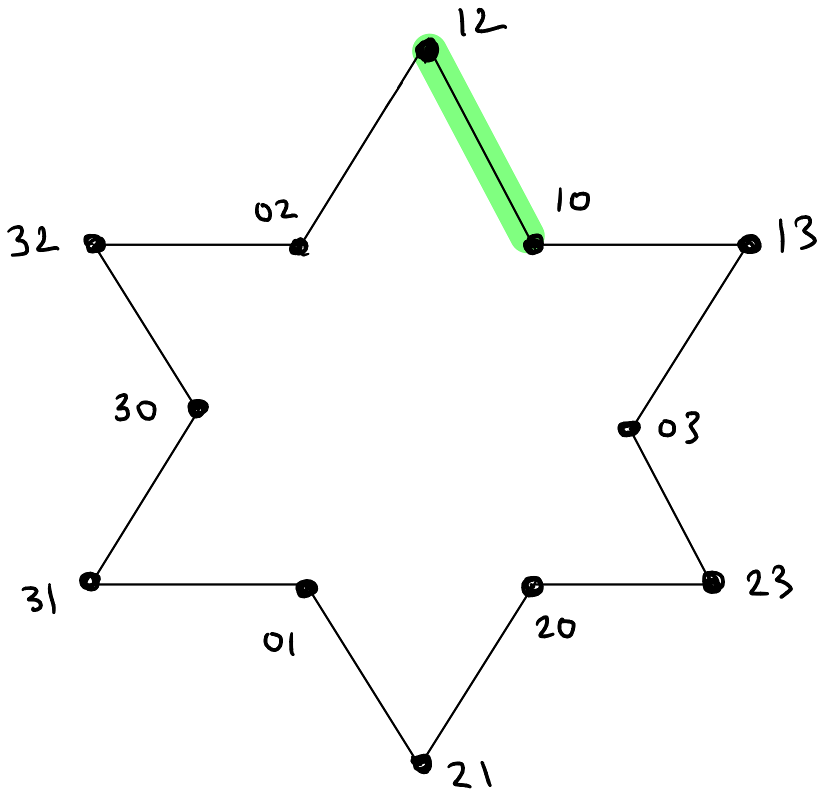




Ex $\Gamma =$ , $k=2$





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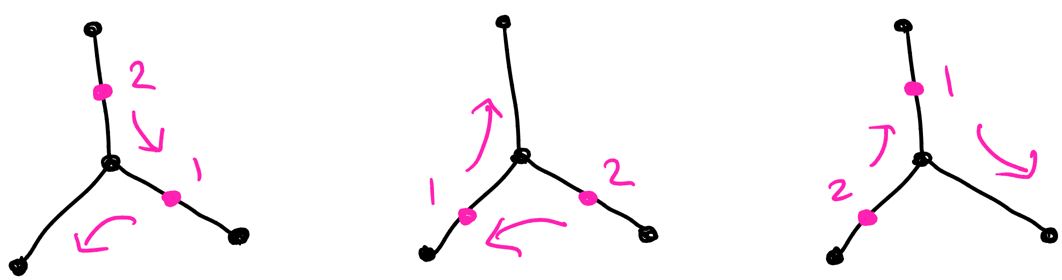
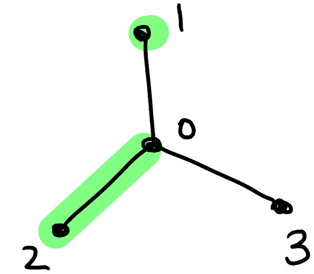
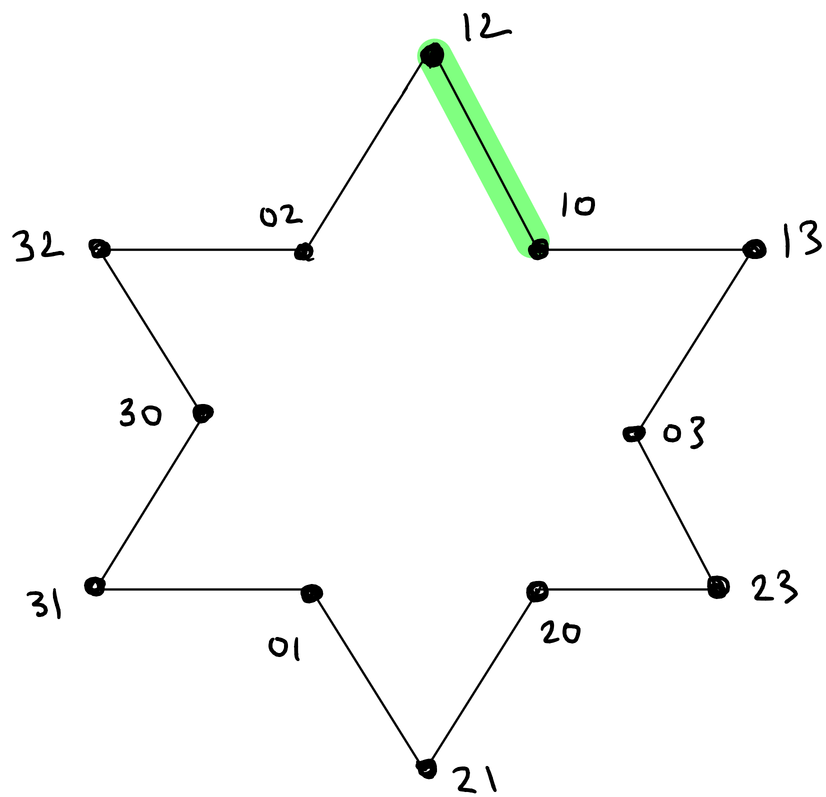


Ex $\Gamma = \lambda$, $k=2$





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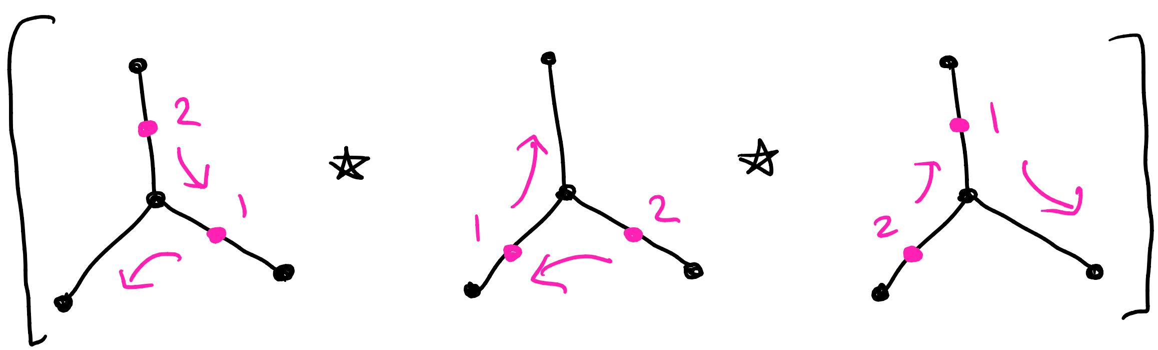
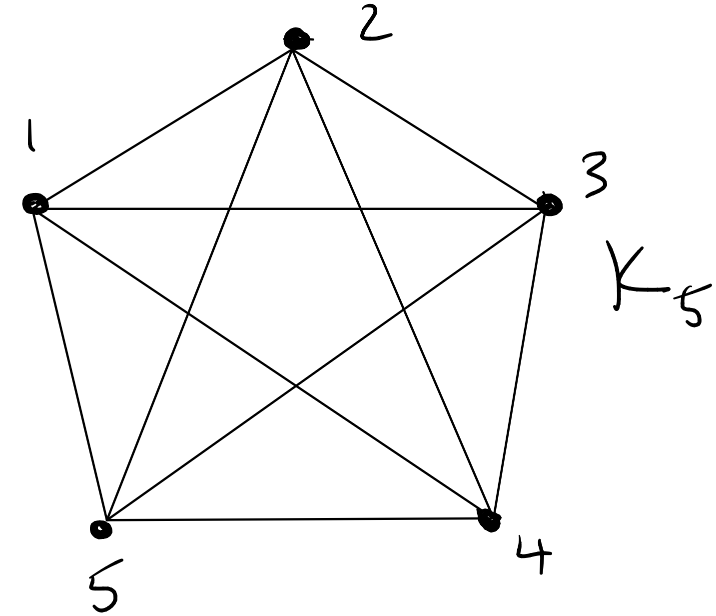
The loop



freely generates $\pi_1(F_2(\lambda))$

Ex $\Gamma = K_5$, $k = 2$



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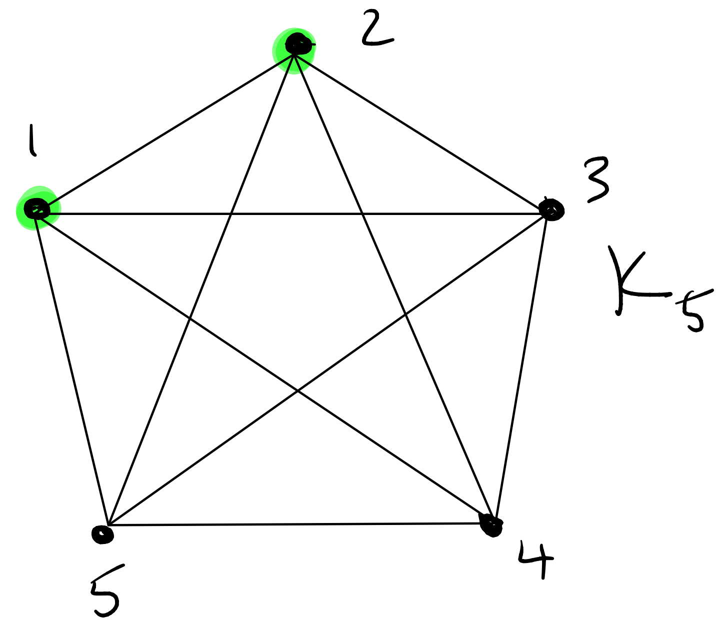
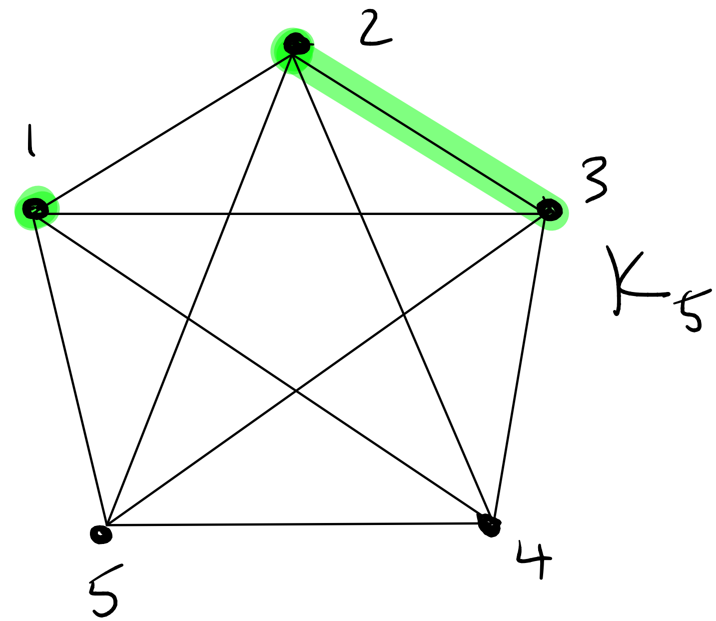
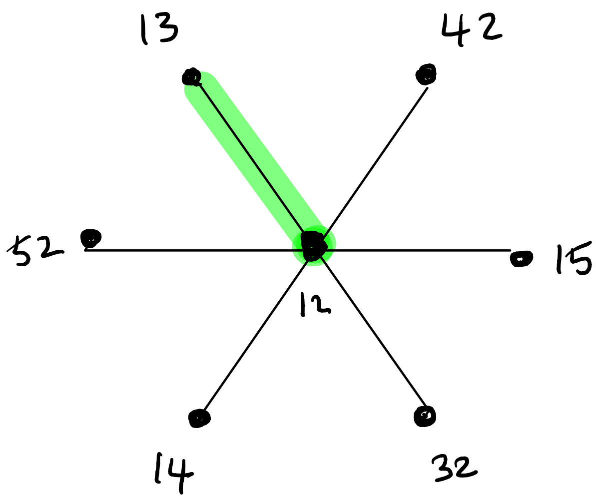


Local picture near the vertex 12:

Ex $\Gamma = K_5$, $k=2$



Local picture near the vertex 12:



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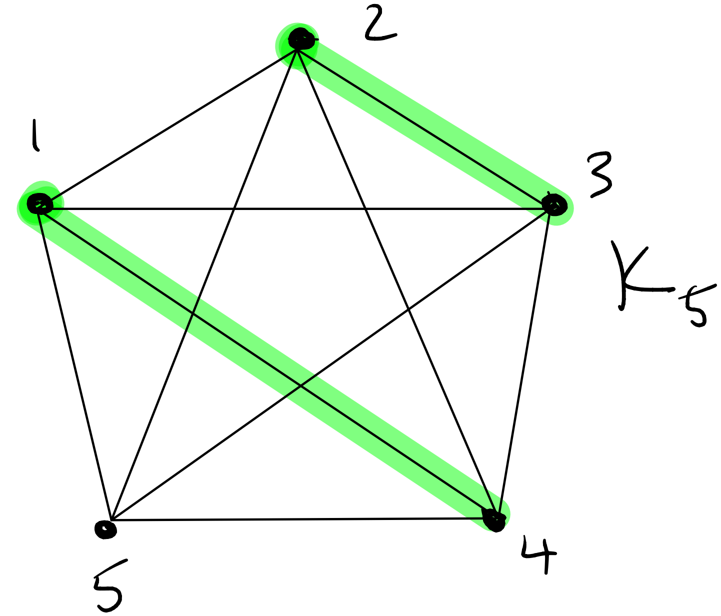

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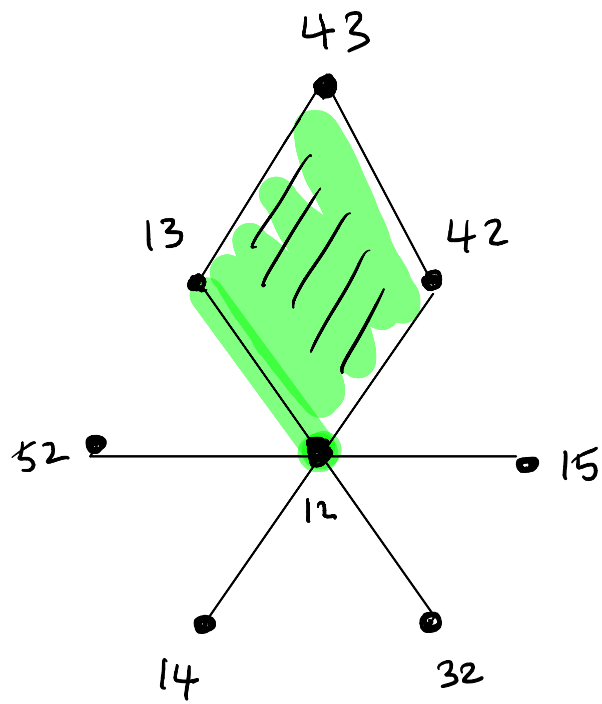

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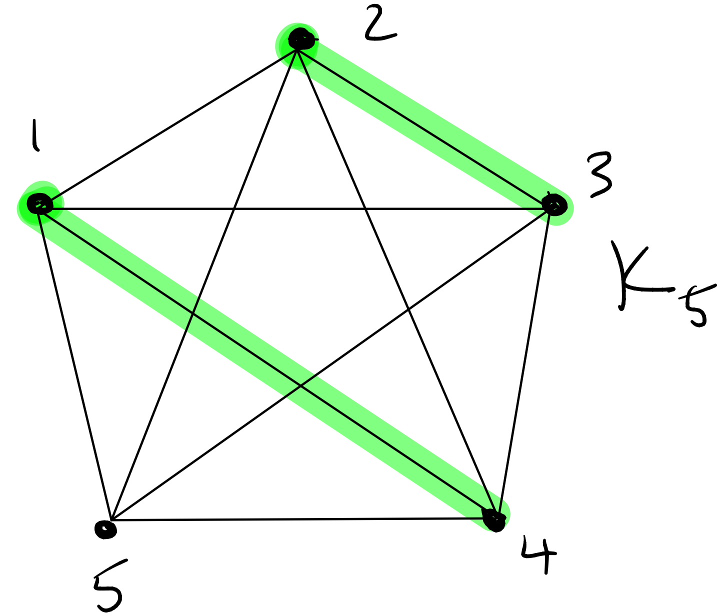

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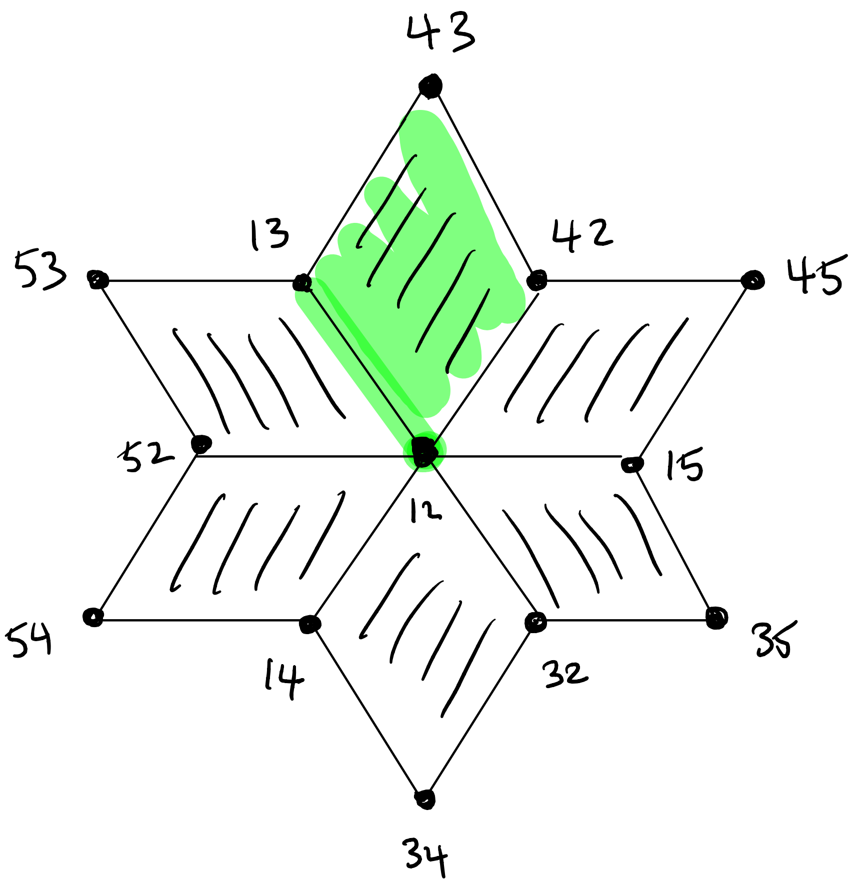

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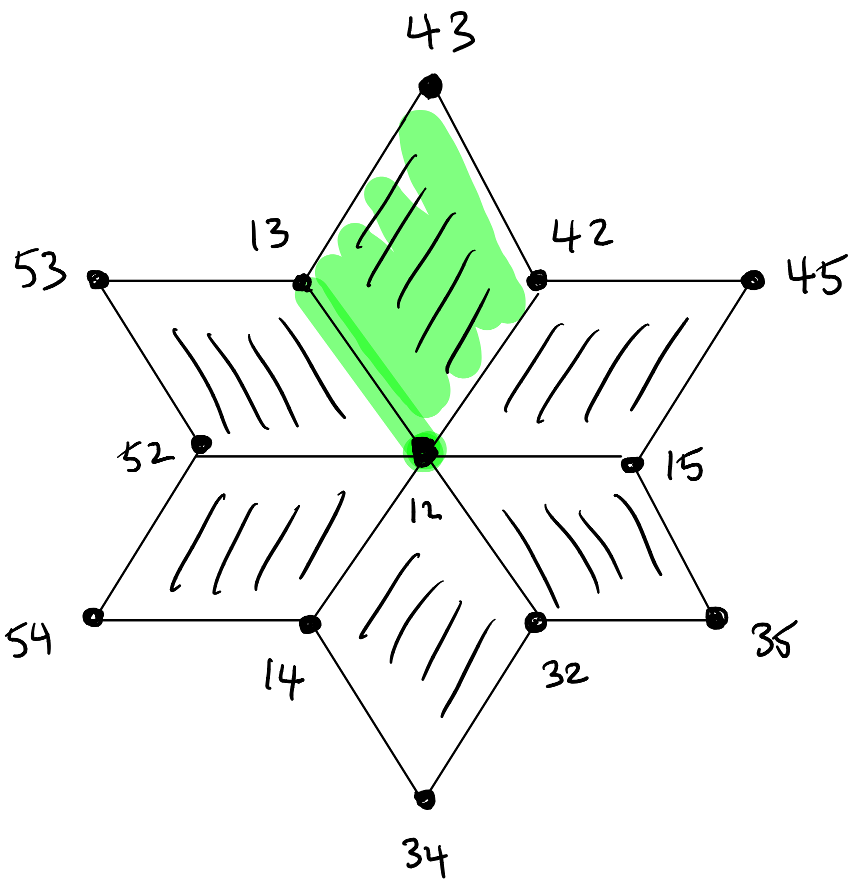

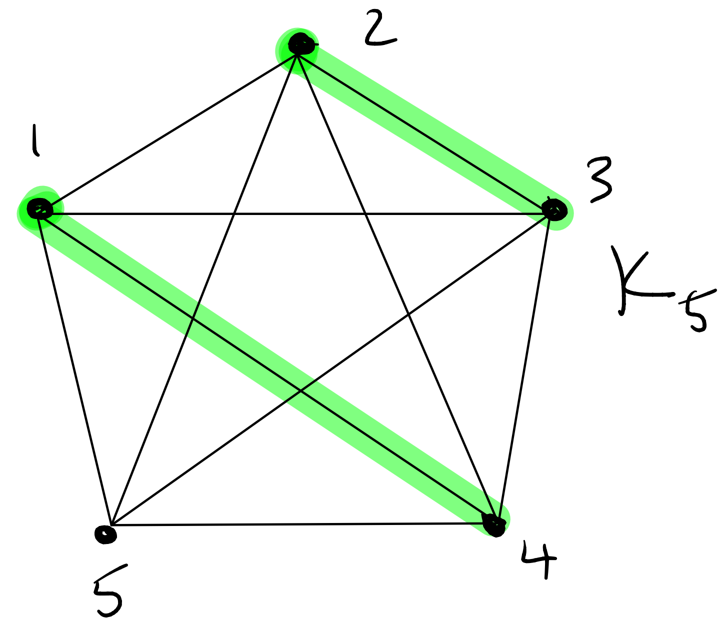

So $F_2^{\square}(K_5)$ is a surface

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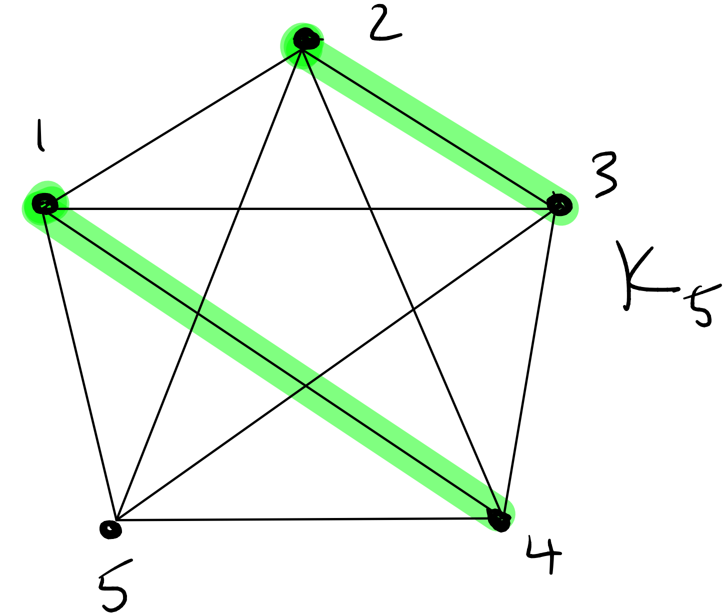

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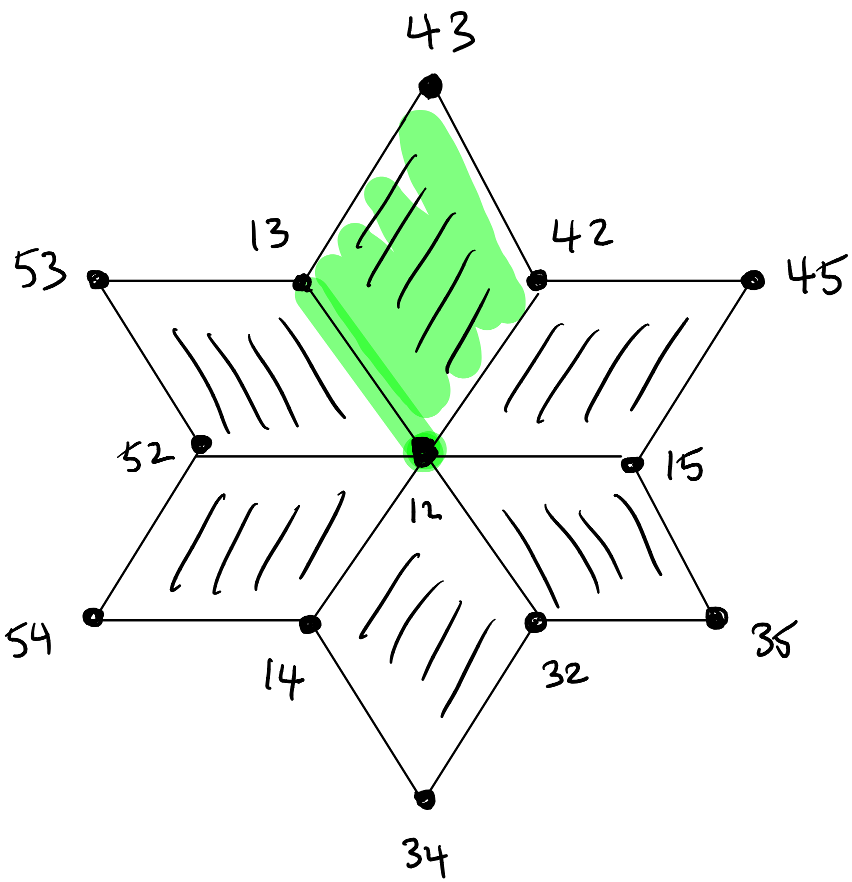

So $F_2^{\square}(K_5)$ is a surface* of Euler characteristic -10 (exercise)

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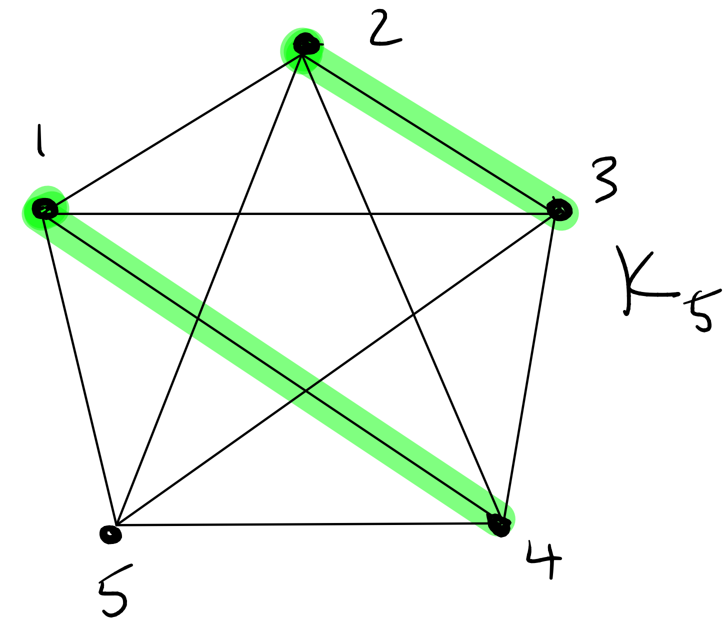

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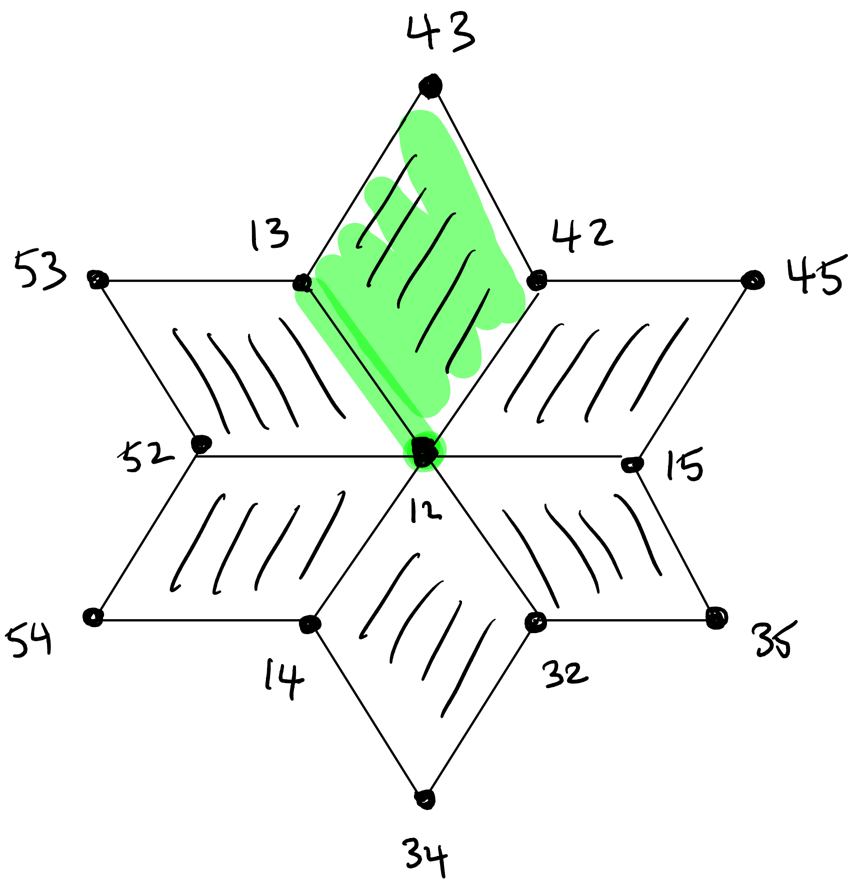

So $F_2^{\square}(K_5)$ is a surface* of Euler characteristic -10 (exercise) hence genus 6.

What do we know about these groups?

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Answer! Everything!

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Farley - Sabalka give presentations by applying DMT to Abrams' complex.

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$$\Delta_\Gamma^i = \max_{|W|=i} |\pi_0(\Gamma \setminus W)|$$

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What about homology?

Def The i^{th} Ramos number of Γ is

$$\Delta_\Gamma^i = \max_{|W|=i} |\pi_0(\Gamma \setminus W)|$$

where W is a set of essential vertices of Γ.

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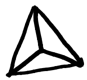
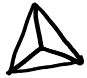
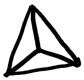
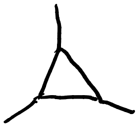
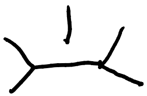
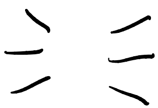
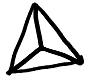
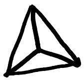
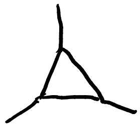
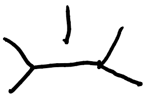
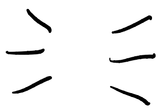
$$\frac{1}{(\Delta^i_\Gamma - 1)!} \sum_W \prod_{v \in W} (d(w)-2).$$

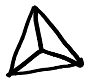
Ex ($\Gamma = K_4 =$ )

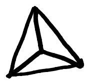
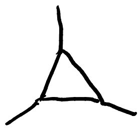
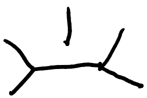
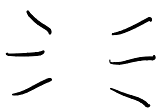
Ex ($\Gamma = K_4 =$ )

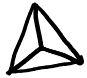
i	$K_4 \setminus W$
0	
1	
2	
3	
4	
≥ 5	?

Ex ($\Gamma = K_4 =$ )

i	$K_4 \setminus W$	$\Delta^i_{K_4}$
0		1
1		1
2		2
3		4
4		6
≥ 5	?	0

Ex ($\Gamma = K_4 =$ )

i	$K_4 \setminus W$	$\Delta^i_{K_4}$	Asymptotic Betti
0		1	1
1		1	
2		2	
3		4	
4		6	
≥ 5	?	0	

Ex ($\Gamma = K_4 =$ )

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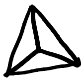
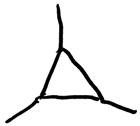
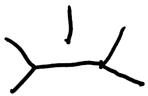
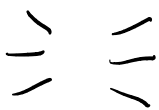
0		1	1
1		1	Ko-Park
2		2	
3		4	
4		6	
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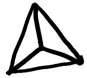
Ex ($\Gamma = K_4 =$ )

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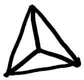
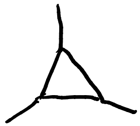
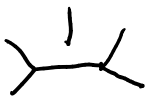
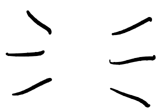
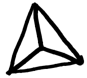
0		1	1
1		1	Ko-Park
2		2	$\binom{4}{2}\frac{1}{1!}k = 6k$
3		4	
4		6	
≥ 5	?	0	

Ex ($\Gamma = K_4 =$ )

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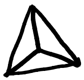
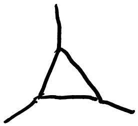
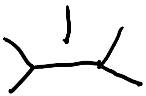
0		1	1
1		1	Ko-Park
2		2	$\binom{4}{2}\frac{1}{1!}k = 6k$
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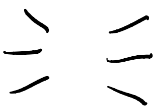
4		6	
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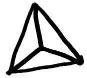
Ex ($\Gamma = K_4 =$ )

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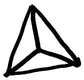
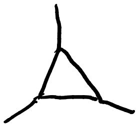
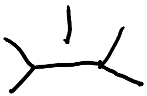
0		1	1
1		1	Ko-Park
2		2	$\binom{4}{2}\frac{1}{1!}k = 6k$
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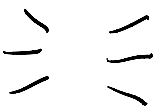
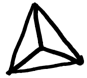
4		6	$\binom{4}{4}\frac{1}{5!}k^5 = \frac{1}{120}k^5$
≥ 5	?	0	

Ex ($\Gamma = K_4 =$ )

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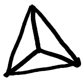
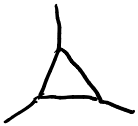
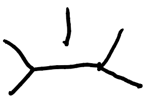
0		1	1
1		1	Ko–Park
2		2	$\binom{4}{2}\frac{1}{1!}k = 6k$
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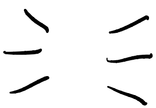
4		6	$\binom{4}{4}\frac{1}{5!}k^5 = \frac{1}{120}k^5$
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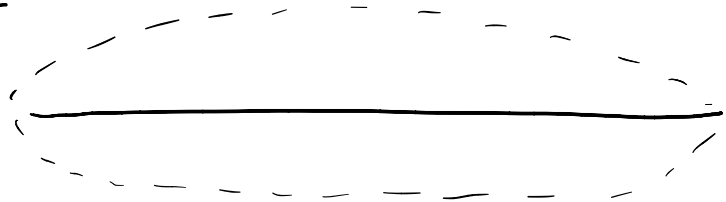

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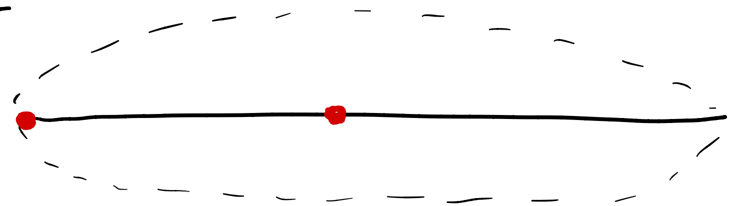

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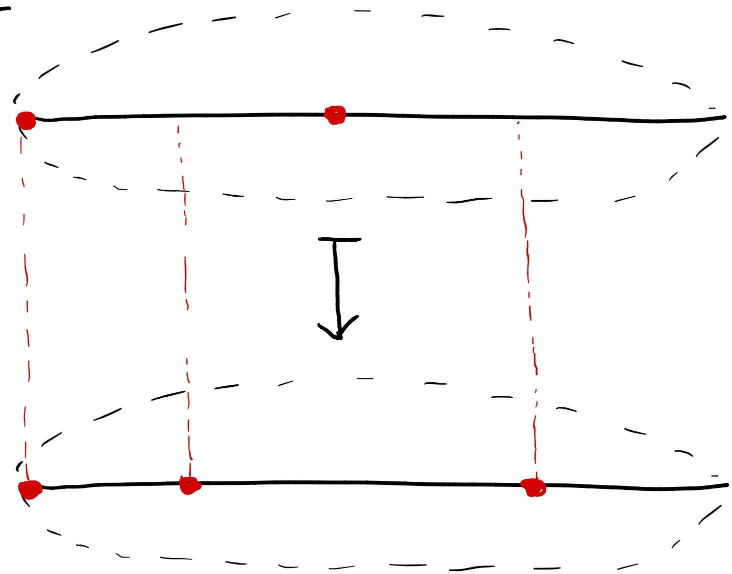

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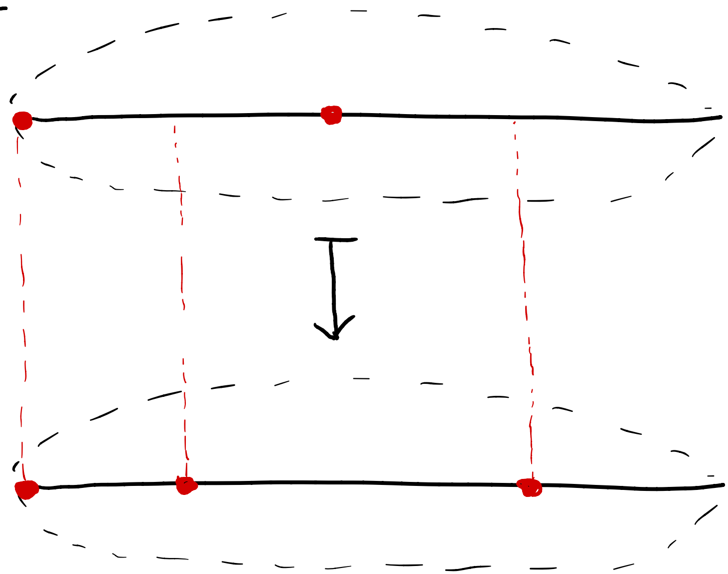

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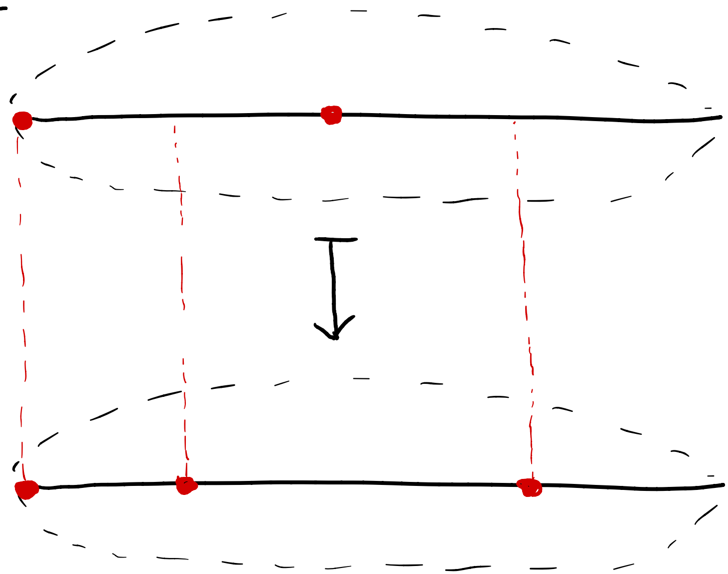
$H_*(B(\Gamma)) \circlearrowleft \mathbb{Z}[E]$

$B(\Gamma) := \coprod_{k \geq 0} B_k(\Gamma)$



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Perspective Homological stability

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Space	Stable homology
$B_k(M)$	constant

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Observation The action of $\mathbb{Z}[E]$ on a W-torus α

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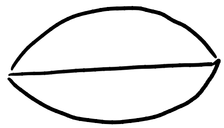
---

essential vertex $v \in \Gamma$ $\rightsquigarrow$ $\lambda \hookrightarrow \Gamma$ $\rightsquigarrow$ "star class" in $H_1(B_2(\Gamma))$

set W of essential vertices $\rightsquigarrow$ $\coprod_W \lambda \hookrightarrow \Gamma$ $\rightsquigarrow$ "W-torus" in $H_{|W|}(B_{2|W|}(\Gamma))$

Observation The action of $\mathbb{Z}[E]$ on a W-torus α factors through $\mathbb{Z}[E] \twoheadrightarrow \mathbb{Z}[\pi_0(\Gamma \setminus W)]$.

Ex



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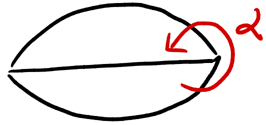

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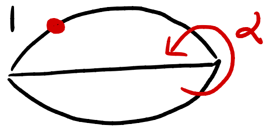

e, α

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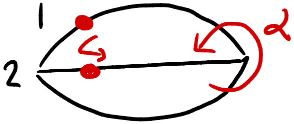

$e_1\alpha = e_2\alpha$

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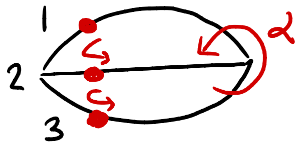
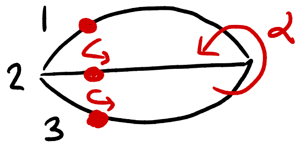


$$e_1\alpha = e_2\alpha = e_3\alpha$$

Ex 

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Sometimes the action factors further.

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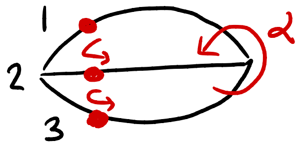

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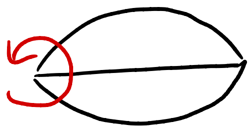

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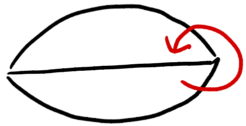

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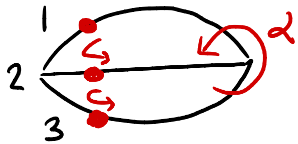

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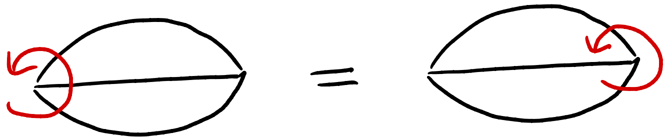
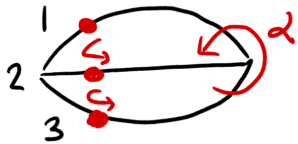


"θ-relation"

Ex 

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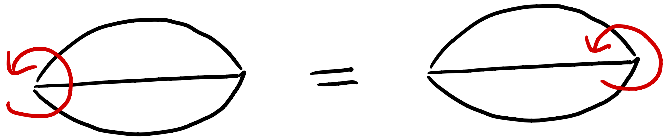
Ex 

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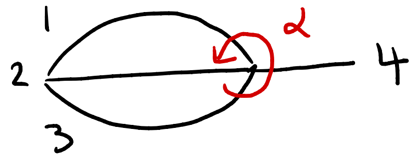
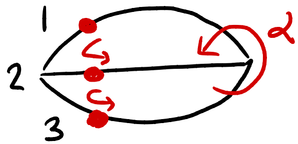


Ex 

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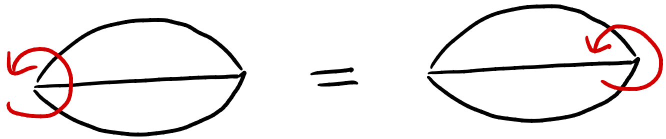
Ex 

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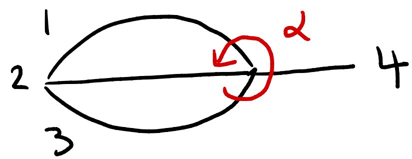

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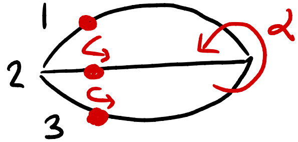

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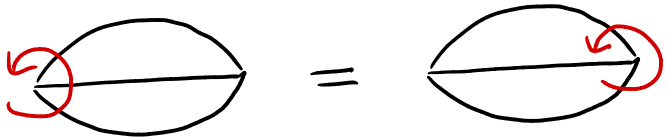

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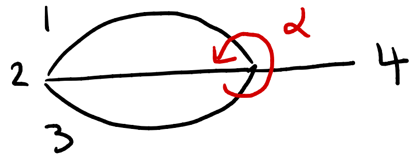

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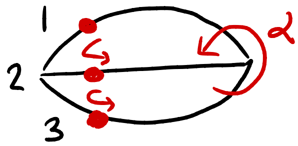

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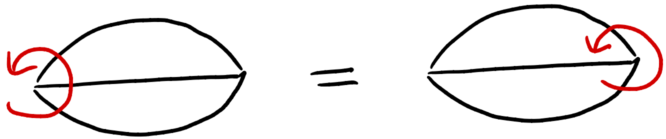

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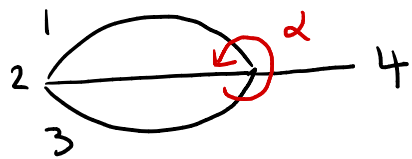

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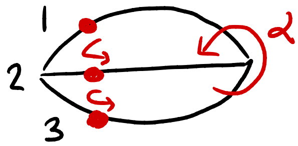
Ex 

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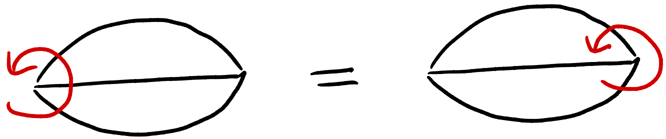
Ex 

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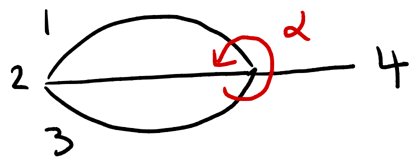

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$$\operatorname{rk} H_i(B(\Gamma)) \geq \operatorname{rk} \mathbb{Z}[\pi_0(\Gamma\setminus W)] \sim \frac{1}{(\Delta^i_\Gamma - 1)!} k^{\Delta^i_\Gamma - 1}.$$

<u>Perspective</u> Generators and relations

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Asymptotically, the only generator is $\curlywedge$.

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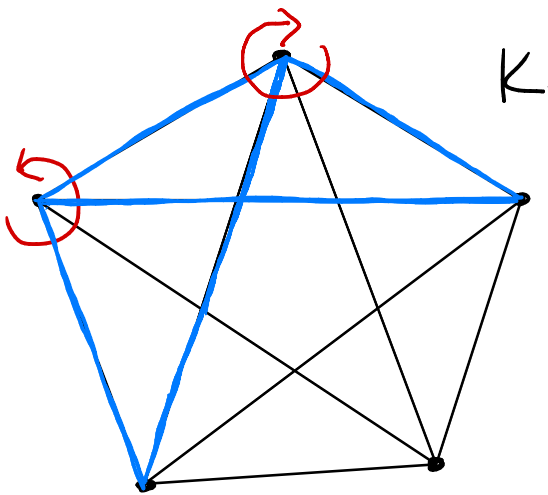

K_5

Θ-relation

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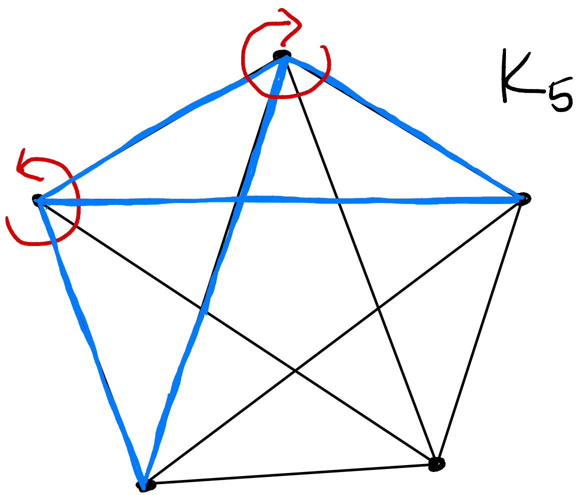

Θ-relation $\Longrightarrow \alpha = (-1)^5 \alpha$

No other torsion is known.

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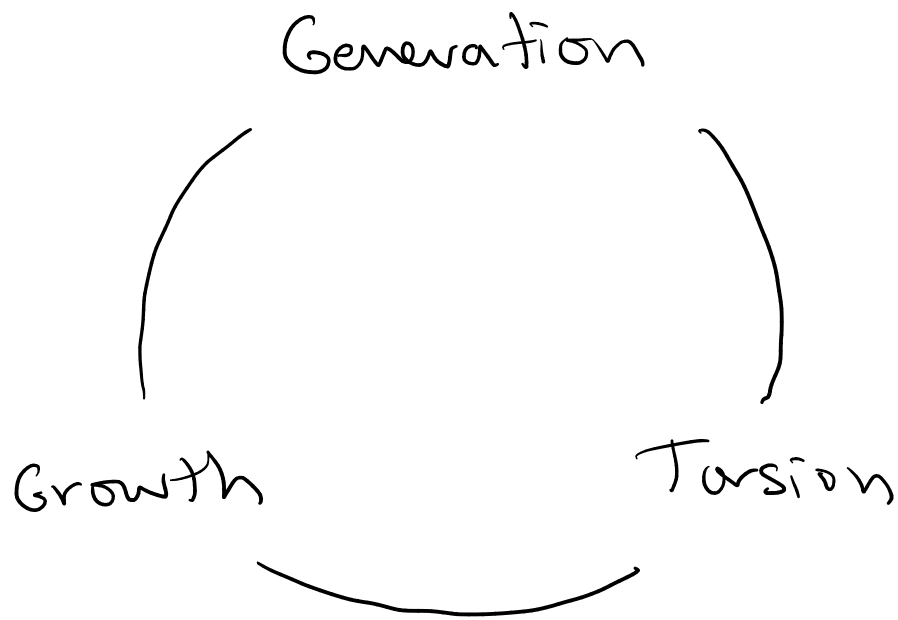
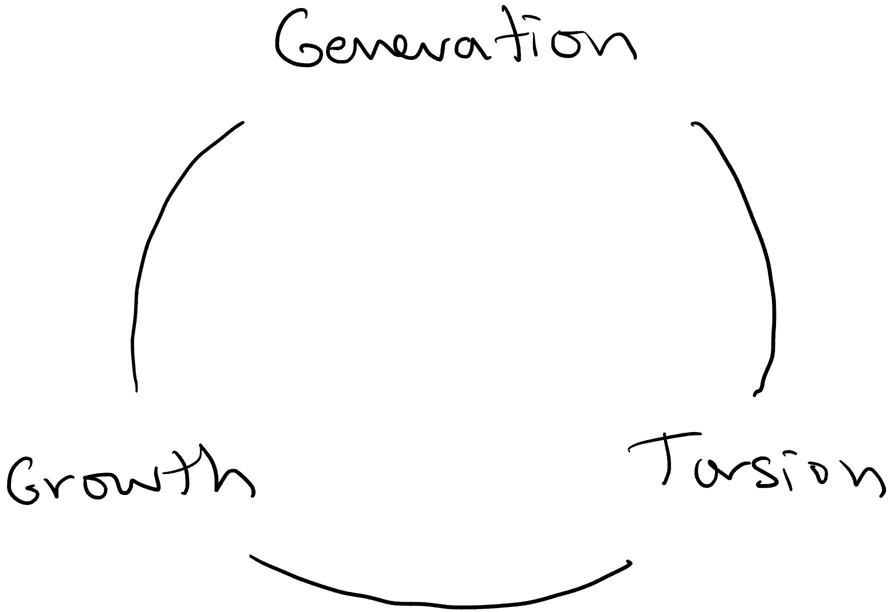
__Conjecture__ (K) If $\alpha \in H^*(B_k(\mathbb{R}^2))$ is torsion, then $\varphi^*\alpha = 0$ for any embedding $\varphi: \Gamma \to \mathbb{R}^2$.


Generation
Growth
Torsion



<u>Problem</u> Calculate the secondary asymptotics of $H_i(B_k(\Gamma);\mathbb{F})$.

Thank you!