Homotopy rigidity for quasitoric manifolds over a product of d-simplices

Xin Fu

BIMSA

Joint with Tseleung So, Jongbaek Song and Stephen Theriault
arXiv:2405.00932

Workshop on Polyhedral Products, Toronto

July 29 - August 2, 2024

Main result

Main Theorem (F.-So-Song-Theriault)

Let M and N be $2n$-dimensional quasitoric manifolds over $\prod_{i=1}^{\ell} \Delta^d$ for $d \geq 1$ and let $\mathcal{P}$ be the set of primes $p \leq n - d + 1$. If $H^*(M) \cong H^*(N)$, then

$$M \simeq N$$

after localizing away from $\mathcal{P}$.

Moment-angle manifolds

Let P^n be a simple polytope of dimension n. Write

$$P^n = \{\underline{x} \in \mathbb{R}^n \mid \langle \underline{a}_i, \underline{x} \rangle + b_i \geq 0 \text{ for } i = 1, \ldots, m\},$$

where $\underline{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

Construction (Buchstaber-Panov-Ray)

Let $\mathcal{Z}_P$ be the pullback

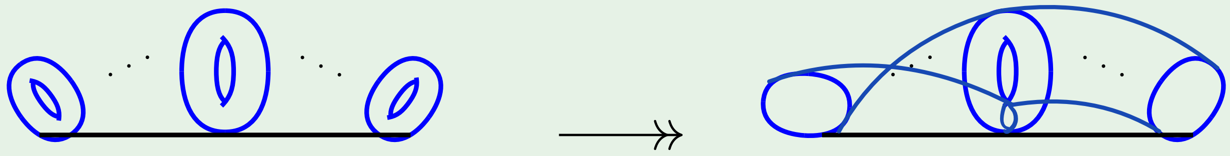
$$\begin{array}{ccc} \mathcal{Z}_P & \longrightarrow & \mathbb{C}^m \\ \downarrow & & \downarrow{\scriptstyle \mu} \\ P^n & \hookrightarrow & \mathbb{R}^m_{\geq} \end{array}$$

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$.

Identifying $\mathbb{C}^m \cong \mathbb{R}^m_{\geq} \times T^m/_{\sim}$ gives $\mathcal{Z}_P \cong P^n \times T^m/_{\sim}$.

Example

Let $P = [0, 1]$ be an interval.

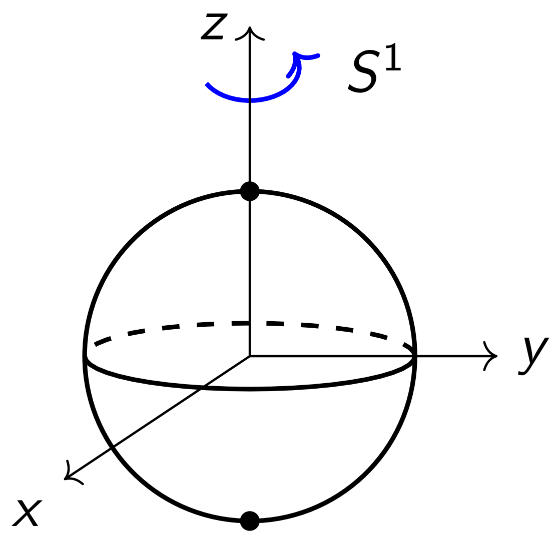


$P \times T^2 \twoheadrightarrow P \times T^2/\sim$

Quasitoric manifolds

Definition

A quasitoric manifold M is a compact $2n$-manifold with a locally standard T^n-action such that $M/T^n = P^n$.



$$M = S^2$$

Quasitoric manifolds

- Let $\lambda\colon \{F_1, \ldots, F_m\} \to \mathbb{Z}^n$ be a characteristic function.

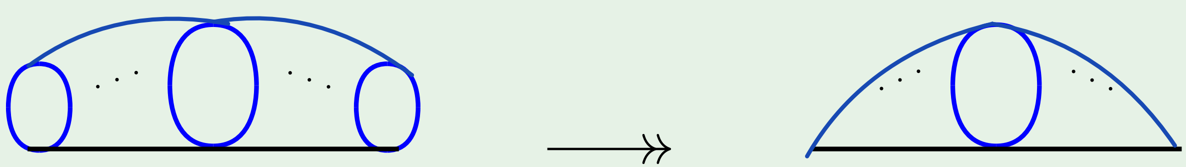
Definition

A quasitoric manifold is a quotient $M = P^n \times T^n/_{\sim_\lambda}$, where

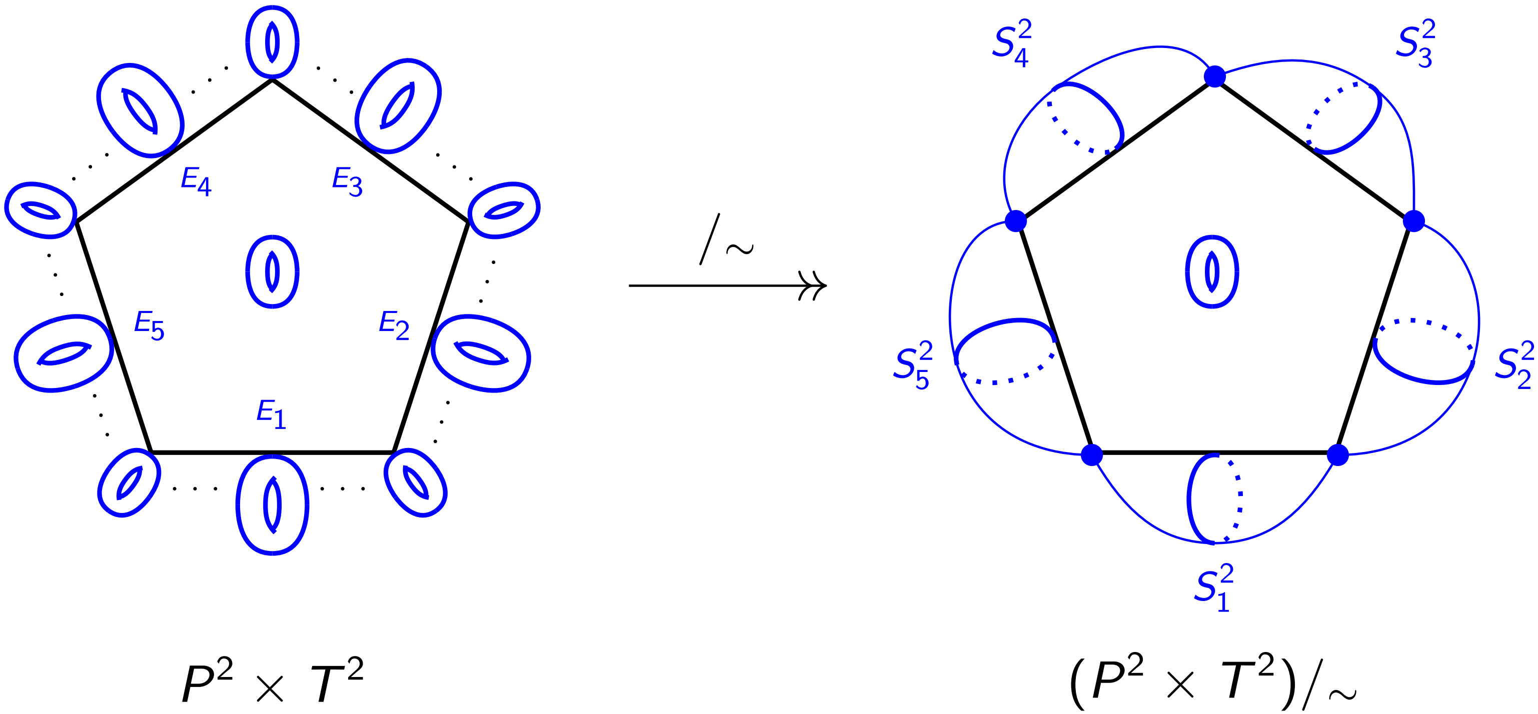
$$(x, t) \sim_\lambda (x', t') \text{ iff } x = x' \text{ and } t^{-1}t' \in T_F \text{ for } x \in \operatorname{int} F.$$

Example

Let $P = [0, 1]$ be an interval.



$P \times S^1$ $\qquad$ $P \times S^1/_\sim$



E_4
E_3
E_5
E_2
E_1
$/\sim$
S_4^2
S_3^2
S_5^2
S_2^2
S_1^2
$P^2 \times T^2$
$(P^2 \times T^2)/\sim$

Example

1. A Bott manifold B_n of height n arises from a sequence of manifolds

$$B_n \to B_{n-1} \to \cdots \to B_1 \to B_0 = \text{point},$$

where $B_k = P(\mathbb{C} \oplus \gamma_{k-1})$ is the projectivization over B_{k-1}.

2. A generalised Bott manifold B_n of height n arises from a sequence of manifolds

$$B_n \to B_{n-1} \to \cdots \to B_1 \to B_0 = \text{point},$$

where $B_k = P(\mathbb{C} \oplus \gamma_{k-1}^{(1)} \oplus \cdots \oplus \gamma_{k-1}^{(n_k-1)})$ is the projectivization over B_{k-1}.

- There is a homotopy fibration

$$M \longrightarrow ET^n \times_{T^n} M \xrightarrow{\pi} BT^n$$

Theorem (DJ)

There is an isomorphism of graded rings

$$\begin{aligned} H^*(M) &\cong \mathrm{Tor}_{H^*(BT^n)}(H_T^*(M), \mathbb{Z}) \\ &\cong H_T^*(M) / \left\langle \mathrm{Im}\left(\pi^{>0} \colon H^*(BT^n) \to H_T^*(M)\right)\right\rangle \\ &\cong \mathbb{Z}[K]/\mathcal{J}. \end{aligned}$$

Cohomology Rigidity Problem (Masuda-Suh 2008)

Given two (quasi)toric manifolds M and M',

$$H^*(M) \cong H^*(M') \quad \overset{?}{\Longrightarrow} \quad M \cong M'$$

No counterexamples are produced!

Supportive cases

- 4-diml quasitoric manifolds (DJ+Freedman)
- 6-diml quasitoric manifolds associated to Pogorelov class (Buchstaber-Erokhovets-Masuda-Panov-Park, '17)
- Bott manifolds (Choi-Hwang-Jang, '22)
- Quasitoric manifolds over $\mathbb{I}^3$ (Hasui, '15)
- Quasitoric manifolds over $\Delta^p \times \Delta^q$ (Choi-Park, 15')

Homotopy version

Given two quasitoric manifolds M and N,

$$H^*(M) \cong H^*(N) \quad \overset{?}{\Longrightarrow} \quad M \simeq N$$

- (Hasui-Kishimoto) For quasitoric manifolds with dimension $< 2p^2 - 4$,

$$H^*(M) \cong H^*(N) \quad \Longrightarrow \quad \Sigma^\infty M_{(p)} \simeq \Sigma^\infty N_{(p)}.$$

- (F.-So-Song) For four-dimensional toric orbifolds without 2-torsion,

$$H^*(M) \cong H^*(N) \quad \Longrightarrow \quad M \simeq N.$$

Main Theorem (F.-So-Song-Theriault)

Let M and N be $2n$-dimensional quasitoric manifolds over $\prod_{i=1}^{\ell} \Delta^d$ for $d \geq 1$ and let $\mathcal{P}$ be the set of primes $p \leq n - d + 1$.

$$H^*(M) \cong H^*(N) \quad \Longrightarrow \quad M \simeq N \text{ after localizing away from } \mathcal{P}.$$

Strategy

$$\mathrm{skel}_0(M) \simeq *, \ \mathrm{skel}_2(M) \simeq \bigvee^{m-n} S^2, \ \mathrm{skel}_{2n}(M) \simeq M;$$

$$\bigvee S^{2k+1} \to \mathrm{skel}_{2k}(M) \to \mathrm{skel}_{2k+2}(M), \ \text{for } 0 \leq k \leq n-1.$$

Prove $\mathrm{skel}_{2k}(M) \simeq \mathrm{skel}_{2k}(N)$ after localisation for

$$k = d; \qquad\qquad k \geq d.$$

The case $k = d$

Lemma

Let M and N be $2n$-dimensional quasitoric manifolds over $\prod_{i=1}^{\ell} \Delta^d$ for $d \geq 1$. Suppose $H^*(M) \cong H^*(N)$. Then

$$\mathrm{skel}_{2d}(M) \simeq \mathrm{skel}_{2d}(N).$$

Sketch of Proof. Consider the fibration

$$\prod_{\ell} S^{2d+1} \to M \xrightarrow{\delta} BT^{m-n}.$$

The induced map $\delta_{2d} \colon \mathrm{skel}_{2d}(M) \to \mathrm{skel}_{2d}(BT^{m-n})$ is a homotopy equivalence.

The case $k = d$

Lemma

Let M and N be $2n$-dimensional quasitoric manifolds over $\prod_{i=1}^{\ell} \Delta^d$ for $d \geq 1$. Suppose $H^*(M) \cong H^*(N)$. Then

$$\mathrm{skel}_{2d}(M) \simeq \mathrm{skel}_{2d}(N).$$

Sketch of Proof. Consider the fibration

$$\prod_{\ell} S^{2d+1} \to M \xrightarrow{\delta} BT^{m-n}.$$

The induced map $\delta_{2d} \colon \mathrm{skel}_{2d}(M) \to \mathrm{skel}_{2d}(BT^{m-n})$ is a homotopy equivalence. The composite

$$\mathrm{skel}_{2d}(M) \xrightarrow{\delta_{2d}} \mathrm{skel}_{2d}(BT^{m-n}) \to \mathrm{skel}_{2d}(BT^{m-n}) \xrightarrow{(\delta'_{2d})^{-1}} \mathrm{skel}_{2d}(N)$$

is a homotopy equivalence. □

Let X be a CW complex with even cells.

Lemma

There is a homomorphism

$$g_X \colon H_{2k+2}(X) \to \pi_{2k+1}(skel_{2k}(X)), \quad [e] \mapsto [f : \partial e \to skel_{2k}(X)].$$

such that given $h \colon X \to Y$, the diagram

$$\begin{array}{ccc} H_{2k+2}(X) & \longrightarrow & \pi_{2k+1}(skel_{2k}(X)) \\ \big\downarrow h_* & & \big\downarrow (h_{2k})_* \\ H_{2k+2}(Y) & \longrightarrow & \pi_{2k+1}(skel_{2k}(Y)) \end{array}$$

commutes.

Let X be a CW complex with even cells.

Lemma

There is a homomorphism

$$g_X \colon H_{2k+2}(X) \to \pi_{2k+1}(skel_{2k}(X)), \quad [e] \mapsto [f \colon \partial e \to skel_{2k}(X)].$$

such that given $h \colon X \to Y$, the diagram

$$\begin{array}{ccc} H_{2k+2}(X) & \longrightarrow & \pi_{2k+1}(skel_{2k}(X)) \\ \downarrow h_* & & \downarrow (h_{2k})_* \\ H_{2k+2}(Y) & \longrightarrow & \pi_{2k+1}(skel_{2k}(Y)) \end{array}$$

commutes.

For M over $\prod_\ell \Delta^d$, when $k > d$, the map g_M is an isomorphism after localising away from $\mathcal{P}$.

The case $k \geq d$

Lemma

Under the same assumption, we have $\mathrm{skel}_{2k}(M) \simeq \mathrm{skel}_{2k}(N)$ *after localising away from* $\mathcal{P}$*.*

Sketch of Proof. After localisation,

$$\begin{array}{ccc} H_{2k+2}(M) & \xrightarrow{g_M} & \pi_{2k+1}(\mathrm{skel}_{2k}(M)) \\ \downarrow & & \downarrow \\ H_{2k+2}(N) & \xrightarrow{g_N} & \pi_{2k+1}(\mathrm{skel}_{2k}(N)), \end{array}$$

The case $k \geq d$

Lemma

Under the same assumption, we have $\mathrm{skel}_{2k}(M) \simeq \mathrm{skel}_{2k}(N)$ *after localising away from* $\mathcal{P}$.

Sketch of Proof. After localisation,

$$\begin{array}{ccc} H_{2k+2}(M) & \xrightarrow{g_M} & \pi_{2k+1}(\mathrm{skel}_{2k}(M)) \\ \vdots\downarrow & & \downarrow \\ H_{2k+2}(N) & \xrightarrow{g_N} & \pi_{2k+1}(\mathrm{skel}_{2k}(N))\,, \end{array} \qquad \begin{array}{ccc} \bigvee S^{2k+1} & \xrightarrow{\vee f_e} & \mathrm{skel}_{2k}(M) \\ \downarrow & & \downarrow \simeq \\ \bigvee S^{2k+1} & \xrightarrow{\vee f'_e} & \mathrm{skel}_{2k}(N)\,. \end{array}$$

□