Geometric quantization of toric manifolds

Quantization

In the previous lecture we reviewed how a toric manifold is obtained as a symplectic quotient of a linear action of a torus on a vector space. The main result in this lecture is the proof that the integer lattice points in the image of the moment map are in bijective correspondence with the holomorphic sections of the prequantum line bundle. This construction is due to Susan Tolman. I refer to the exposition by Mark Hamilton [Hamilton].

Let

$$\Delta = \{x \in \mathbf{R}^n | < x, v_j > \geq \lambda_j, \quad 1 \leq j \leq N\}.$$

Define a map $\pi : \mathbf{R}^N \to \mathbf{R}^n$ by Also $\pi(e_j) = v_j$ where v_j are N vectors in $\mathbf{R}^n$. $\pi : \mathbf{Z}^N \to \mathbf{Z}^n$.

So defining $T = \mathbf{R}/\mathbf{Z}$, we get $\pi : T^N \to T^n$ and let $K = \mathrm{Ker}(\pi)$ and $i : K \to T^N$. We also use $i : k \to \mathbf{R}^N$ and $i^* := L : (\mathbf{R}^N)^* \to k^*$. Let $\nu \in k^*$ be a regular value of the moment map for the K action.

Claim 1: $\pi^* - \lambda$ is a bijective map from Δ to Δ' where $\Delta' = L^{-1}(\nu) \cap (\mathbf{R}_+)^N$ so that the integer lattice points in Δ correspond to $\Delta' \cap (\mathbf{Z}_+)^N$.

Proof:

$\pi^* - \lambda$ maps $\mathbf{Z}^n$ into $\mathbf{Z}^N$. So if a point of Δ' has integer coefficients, then it is in the image of $\pi^* - \lambda$ of a point in $\mathbf{Z}^n$. To see this, we need to show that if $\pi^*(x) = y \in \mathbf{Z}^N$ then $x \in \mathbf{Z}^n$.

This boils down to

$$< (\pi^* - \lambda)(x), v_j > \quad \geq 0 \ \ \forall j = 1, \ldots, N$$

iff

$$< x, v_j > \quad \geq \quad \lambda_j \ \ \forall j = 1, \ldots, N$$

which is the condition for x to be in Δ. Since $\pi^* - \lambda$ is an affine injection from $\mathbf{R}^n$ into $\mathbf{R}^N$, it is a bijection onto its image, so onto the image of $(\pi^* - \lambda) \cap (\mathbf{R}_+)^N$.

The map π^* can be written $y = Vx$ where $y \in \mathbf{R}^N$ and $x \in \mathbf{R}^n$ and V is the $N \times n$ matrix whose rows are the vectors v_j.

• n of the v_j corresponding to one vertex of the moment polytope form a $\mathbf{Z}$ basis of $\mathbf{Z}^n$. WLOG suppose $v_1, \ldots, v_n$ form such a basis.

Let V by the $n \times n$ matrix whose rows are $v_1, \ldots, v_n$.

So $y = Vx$ defines $y_1, \ldots, y_n$ from the x (where the column vector Y is the transpose vector of $(y_1, \ldots, y_n)$.

Let $\bar{V}$ be the $n \times n$ matrix whose rows are $v_1, \ldots, v_n$ so that $Y = \bar{V}x$ defines $y_1, \ldots, y_n$ from the x where Y is the column vector of $y_1, \ldots, y_n$. If $v_1, \ldots, v_n$ form a $\mathbf{Z}$ basis for $\mathbf{Z}^n$, the determinant of $\bar{V}$ is ± 1, so $\bar{V}$ is invertible and its inverse has integer entries.

• So given a Y with integer entries, $x = \bar{V}^{-1}Y$ will also have integer entries, and so integer lattice points in the image of π^* come from integer points in $\mathbf{R}^n$.

Complex construction:

Let F_j be the facets of Δ.

• Define a family $\mathcal{F}$ of subsets of Δ by

$$\emptyset \in \mathcal{F}$$

and $I \in \mathcal{F}$ if and only if

$$\cap_{i \in I} F_i \neq \emptyset.$$

• Define the zero-index set of a point $z \in \mathbf{C}^N$ by

$$I_z = \{j | z_j = 0\}.$$

Define $U_{\mathcal{F}}$ by the set of $z \in \mathbf{C}^N$ whose zero-index sets are in $\mathcal{F}$.

Note that M is prequantizable if λ is in $\mathbf{Z}^N$.

Also the prequantum line bundle of $M = U_{\mathcal{F}} / K_C$ is

$$L = U_{\mathcal{F}} \times_{K_C} \mathbf{C}.$$

Here $K_{\mathbf{C}}$ acts on $\mathbf{C}$ with the weight $\nu = L(-\lambda)$.

Theorem: The dimension of the space of holomorphic sections of the prequantum line bundle is the number of integer lattice points in Δ.

Proof: A holomorphic section is a $K_{\mathbf{C}}$-equivariant holomorphic function $s : U_{\mathcal{F}} \to \mathbf{C}$.

Since $\mathbf{C}^N \setminus U_{\mathcal{F}}$ is the union of submanifolds of codimension greater than or equal to 2, s extends to a holomorphic function on $\mathbf{C}^N$ (by Hartogs' theorem).

So we want to count the $K_{\mathbf{C}}$-equivariant holomorphic functions $s : \mathbf{C}^N \to \mathbf{C}$ where $K_{\mathbf{C}}$ acts on $\mathbf{C}$ by weight ν and the action on $\mathbf{C}^N$ is via the inclusion $i : K_{\mathbf{C}} \to T^N_{\mathbf{C}}$ and the standard action of $T^N_{\mathbf{C}}$ on $\mathbf{C}^N$.

Write such a s as its Taylor series:

$$s = \sum_{I \in \mathbf{Z}^+_N} a_I z^I.$$

Consider each term z^I separately.

(2) Action of T^N

$(t \cdot z)^I = t^I z^I$ for $t \in T^N_{\mathbf{C}}$.

Let $k \in K_{\mathbf{C}}$, and recall that $\nu \in Hom(K, U(1)) = Hom(K_{\mathbf{C}}, \mathbf{C}^*)$ is a weight. So $k^\nu \in \mathbf{C}^*$.

$s(k \cdot z) = k \cdot s(z)$ But also $k \cdot s(z) = k^\nu z^I$.

So $s(k \cdot z) = k \cdot s(z)$ whenever $i^*(I) = \nu$ or $L(I) = \nu$.

So a basis for the equivariant sections is $(Z_+)^N \cap L^{-1}(\nu)$, which corresponds precisely to the set of integer lattice points in the moment polytope.

Example:

Let the polytope Δ be the triangle in $\mathbf{R}^2$ with vertices $(0,0), (0,m), (m,0)$

(a right triangle with the edges of length m, m, $\sqrt{2}m$).

The three vectors normal to the edges are $(0,1)$, $(1,0)$, $(-1,-1)$.

The λ is $(0,0,-m)$.

So the map π is the 2×3 matrix

$\pi(x,y,z) = (y-z, x-z)$.

The kernel of π is

$$\{(t,t,t)|t \in \mathbf{R}\}.$$

The map on tori is

$$(e^{2\pi ix}, e^{2\pi iy}, e^{2\pi iz}) :\mapsto (e^{2\pi i(y-z)}, e^{2\pi i(x-z)})$$

with kernel

$$\{(e^{2\pi it}, e^{2\pi it}, e^{2\pi it})\}.$$

This is S^1 embedded in T^3 as the diagonal subtorus.

$\pi^*(a,b) = (a,b,-a-b)$.

$L = i^*$ is $L(x,y,z) = x+y+z$.

So

$$\nu = L(-(0,0,-m)) = m.$$

So the affine space $L^{-1}(\nu)$ is the space $\{x+y+z=m\}$ lying in $\mathbf{R}^3$.

The intersection with $\mathbf{R}^3$ is a triangle which identifies with Δ.

We pull $L^{-1}(\nu)\cap\mathbf{R}^3$ back under $\phi:\mathbf{C}^3\to\mathbf{R}^3$ to give

$$\mu^{-1}(\nu)=\{z\in\mathbf{C}^3|\pi(|z_1|^2+|z_2|^2+|z_3|^2)=m\}\cong S^5.$$

The integer points in Δ' are the set

$$\{(z,y,z)\in\mathbf{Z}^3|x,y,z\geq 0, x+y+z=m\}.$$

This is in bijective correspondence with

$$\{(x,y)\in\mathbf{Z}^2|x,y\geq 0, x+y\leq m\}$$

This is the set of integer points in Δ.

For the complex construction,

$$\mathcal{F}=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\}\}$$

So $U_{\mathcal{F}}$ is the set of points in $\mathbf{C}^2$ with either 0, 1 or 2 coordinates 0, in other words

$$U_{\mathcal{F}}=\mathbf{C}^3\setminus\{0\}.$$

The complex torus is $K_{\mathbf{C}} = \mathbf{C}^*$ acting on $\mathbf{C}^3$ by the diagonal action.

The quotient of $\mathbf{C}^3 \setminus \{0\}$ by the diagonal action of $\mathbf{C}^*$ is $\mathbf{C}P^2$.

The prequantum line bundle is $L = U_{\mathcal{F}} \times_{K_C} \mathbf{C}$ where $K_{\mathbf{C}}$ acts on $\mathbf{C}$ with weight m.

The sections are

$$s(z) = (z_1)^{j_1}(z_2)^{j_2}(z_3)^{j_3}.$$

$$s(k \cdot z) = k^{j_1+j_2+j_3}(z_1)^{j_1}(z_2)^{j_2}(z_3)^{j_3}$$

This equals $k \cdot s(z)$ iff $j_1 + j_2 + j_3 = m$, in other words $(j_1, j_2, j_3) \in \Delta'$.

The number of such points is

$$(m+1) + m + \dots 1 = m(m+1)/2.$$

So this is the dimension of the quantization.

Danilov's theorem

Let M be a toric manifold which is obtained as the symplectic quotient of $\mathbf{C}^N$ by the action of a torus $T = U(1)^n$.

Theorem (Danilov) The cohomology of M is

$$H^*(M; \mathbf{Q}) \cong \mathbf{Q}[x_1, \ldots, x_N]/(\mathcal{I}, J).$$

Here the ideals $\mathcal{I}$ and J will be defined below.

Proof (Tolman-Weitsman):

Let G be a torus with Lie algebra . Define

$$0 \to \xrightarrow{i} \mathbf{R}^N \xrightarrow{\pi} \to 0$$

The dual sequence is

$$0 \to^* \xrightarrow{\pi^*} (\mathbf{R}^N)^* \to \xrightarrow{i^*}{}^* \to 0.$$

The cohomology of a toric manifold M obtained as a symplectic quotient as above is as follows. Define the ideal J as the image of π^* in $\mathbf{R}^{N^*}$ which is a subset of the polynomial ring on $\mathbf{R}^N$. This is

$$J = \sum_i \alpha_i x_i$$

where $\alpha_i \in \mathrm{Im}(\pi^*)$.

We also define the ideal

$$\mathcal{I}$$

which is the subset $\prod_{i \in I} x_i$ for all subsets $I \subset \{1, \ldots, N\}$ for which the facets of the moment polytope corresponding to any two $i \in I$ do not intersect.

Step 1: The G equivariant cohomology of $\mathbf{C}^N$ is the quotient of the polynomial ring on N variables by the ideal J.

Step 2: By the Kirwan surjectivity theorem, the cohomology of a symplectic quotient of $\mathbf{C}^N$ at a regular value of the moment map is isomorphic to the quotient of the equivariant cohomology of $\mathbf{C}^N$ by the kernel of the Kirwan map (the restriction map from $\mathbf{C}^N$ to the zero locus of the moment map).

Define M_ξ as the subset where the $< \phi(m), \xi >$ is ≤ 0. Define K_ξ as the subset of $\alpha \in H_T^*(M; \mathbf{Q})$ where α vanishes when restricted to M_ξ. Define $K = \sum_{\xi \in} K_\xi$.

Then

$$0 \to K \to H_T^*(M; \mathbf{Q}) \xrightarrow{\kappa} H^*(M_{\text{red}}; \mathbf{Q}) \to 0.$$

In other words, K is the kernel of the Kirwan map.

The Kirwan map is the restriction from equivariant cohomology of M to the equivariant cohomology of a regular level set of the moment map, which is isomorphic to the ordinary cohomology of the symplectic quotient.

Step 3: The kernel of the Kirwan map is the ideal $\mathcal{I}$ which is the subset $\prod_{i \in I} x_i$ for all subsets $I \subset \{1, \cdots, N\}$ for which the facets of the moment polytope corresponding to any two i in I do not intersect.

It follows that the ordinary cohomology of the quotient is the quotient of the equivariant cohomology by the kernel of the Kirwan map , which is $\mathbf{Q}[x_1, \ldots, x_N]/ < \mathcal{I}, J >$. This is Danilov's result.

Reference:

[Hamilton] M. Hamilton, The quantization of a toric manifold is given by the integer lattice points in the moment polytope. *Toric Topology* (Contemporary Math. **460**) (2008).

[Tolman-Weitsman] S. Tolman, J. Weitsman, The cohomology rings of symplectic quotients. *Commun. in Analysis and Geometry* **11** (2003) 751–774.