Toric varieties arising from polygon dissections

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Workshop on Toric Topology
The Fields Institute
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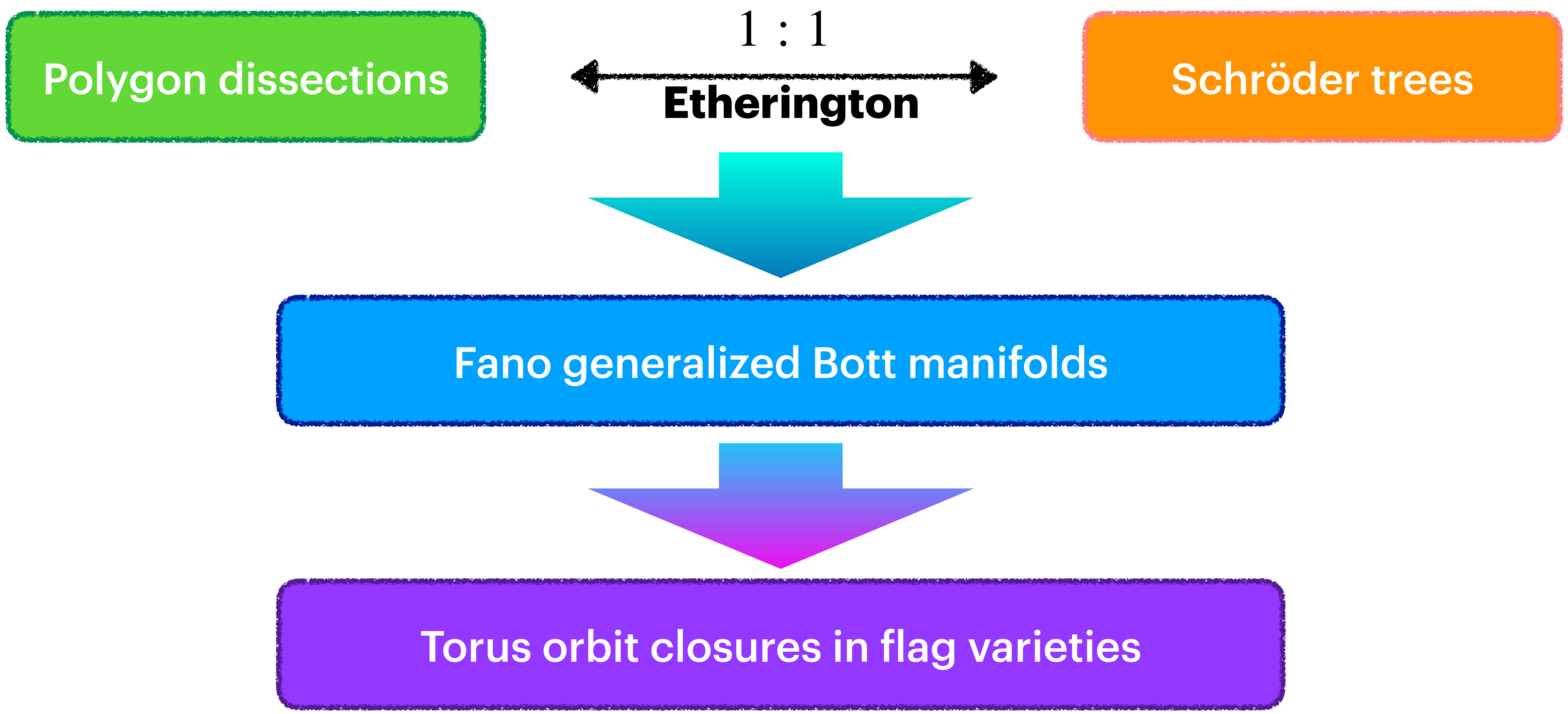
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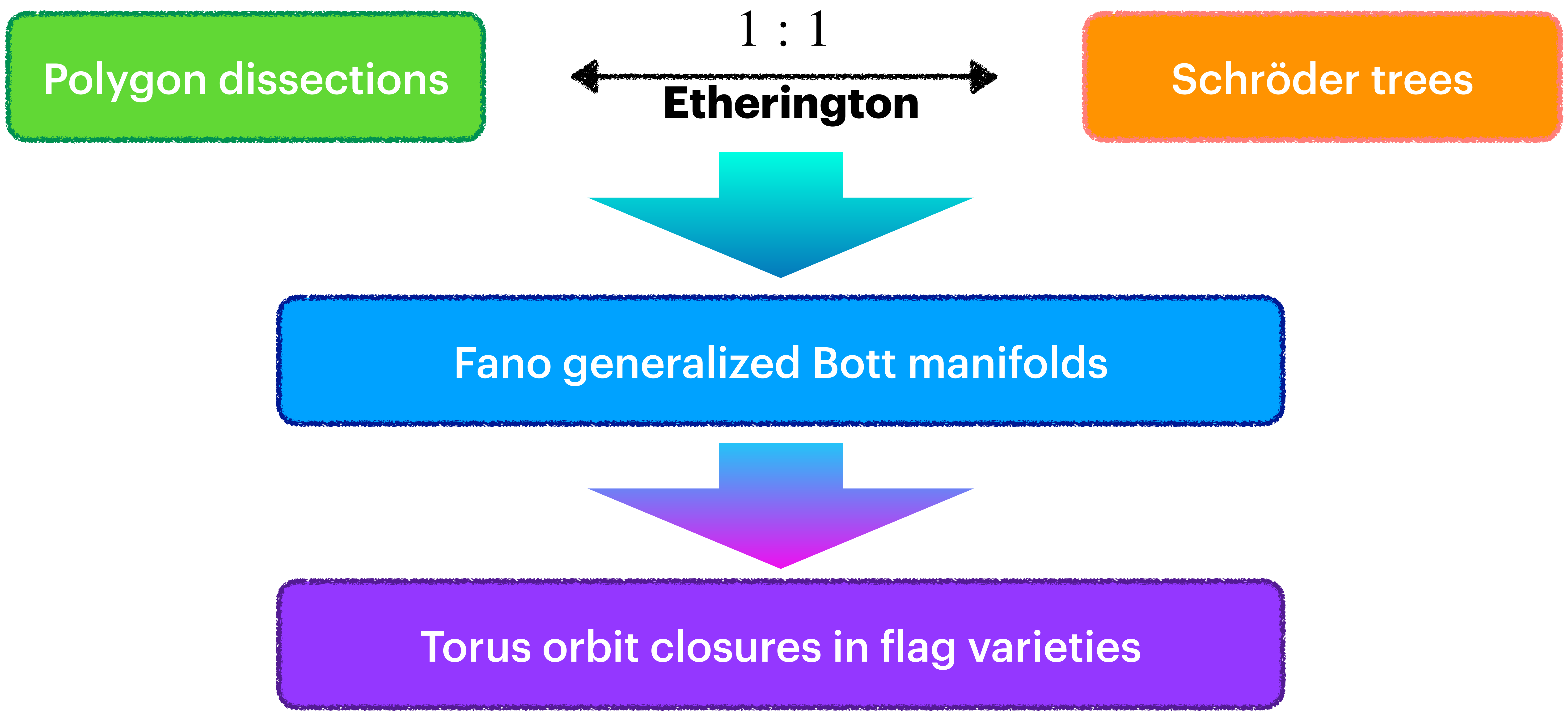

This talk is based on the following two papers with some new results.
- (With Masuda and Lee) Toric Richardson varieties of Catalan type and Wedderburn-Etherington numbers (2023)
- (With Huh) Toric varieties of Schröder type (2022).

Etherington's bijection:
Polygon dissections and Schröder trees

Polygon dissections

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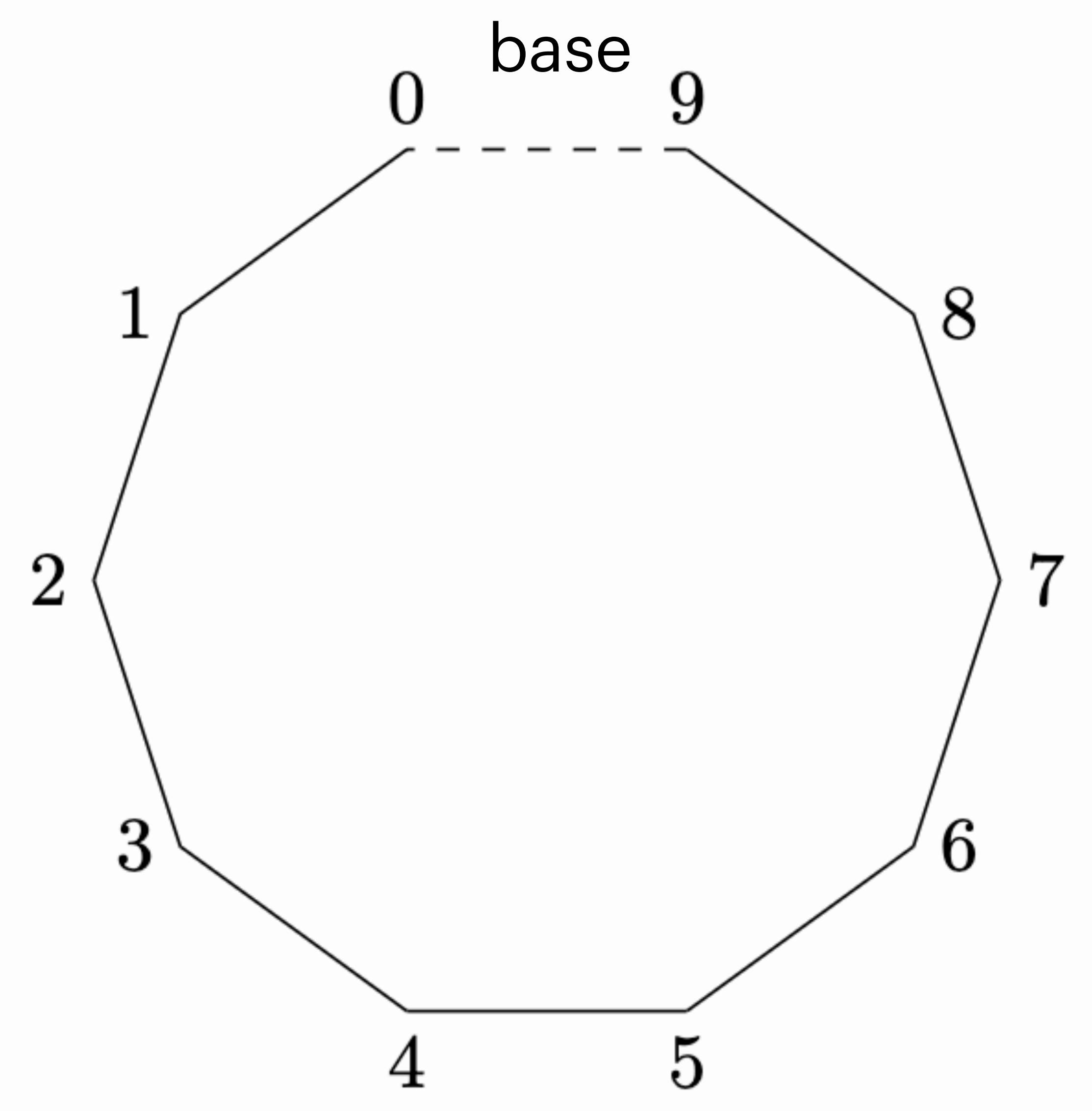

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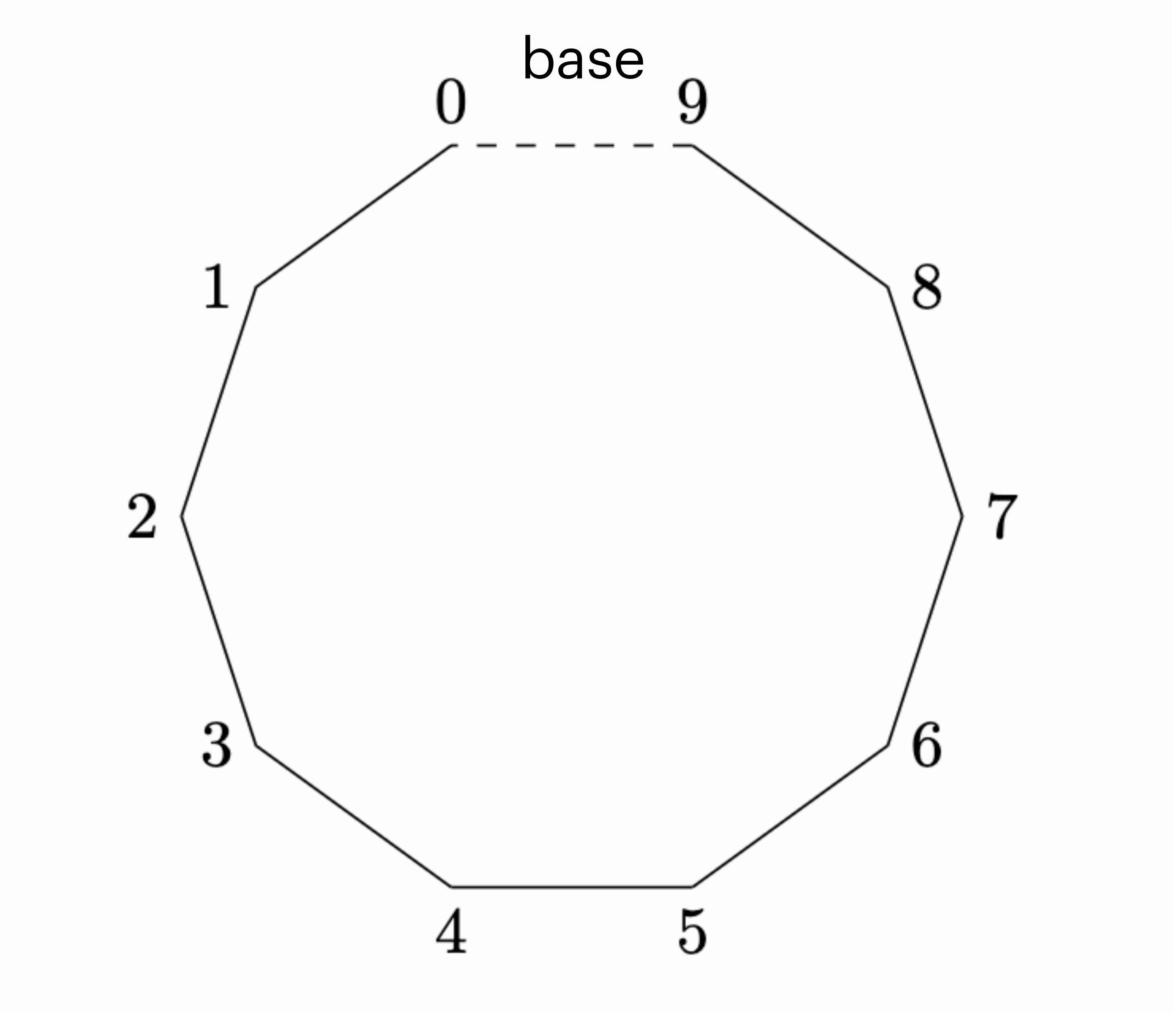

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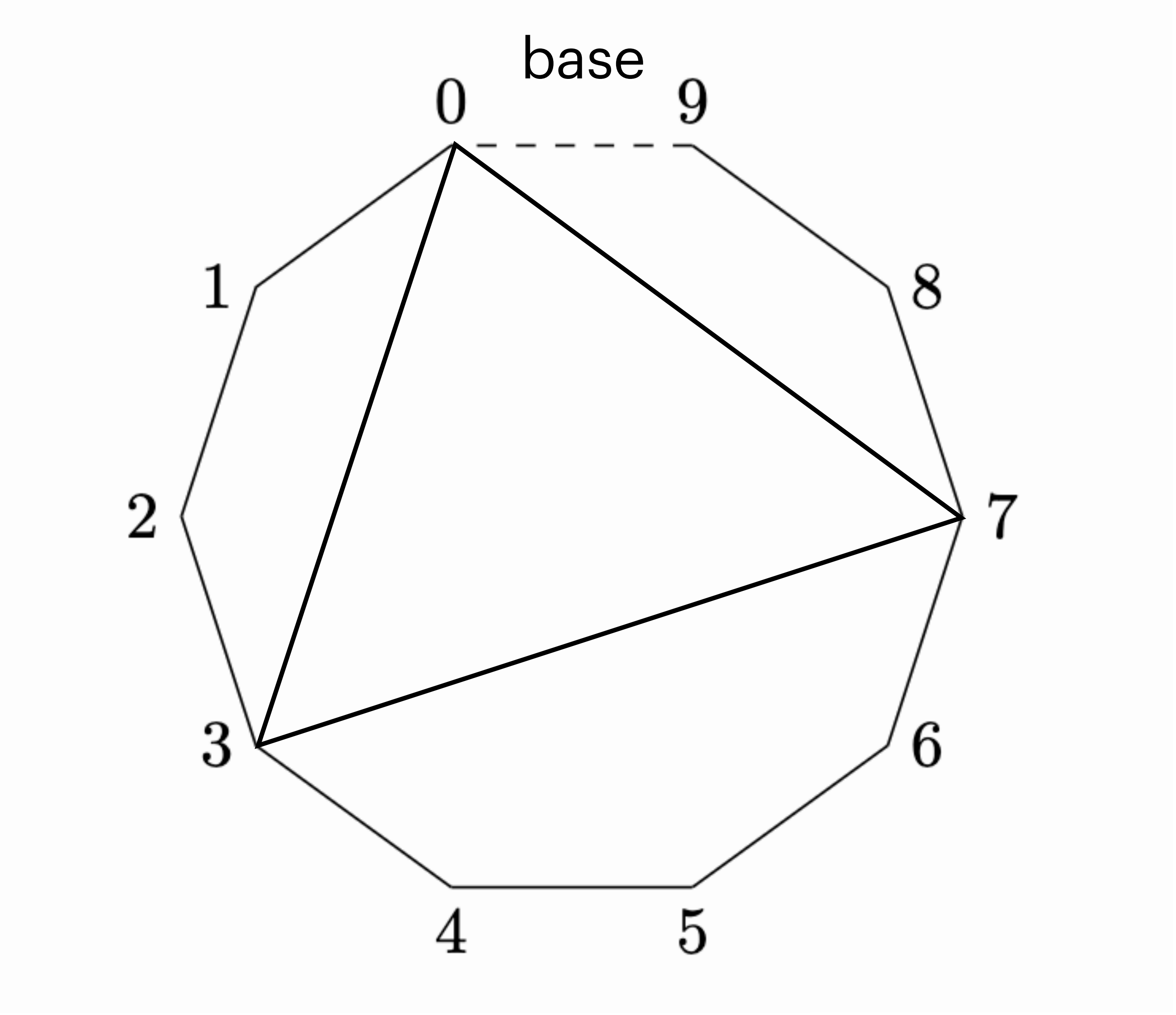

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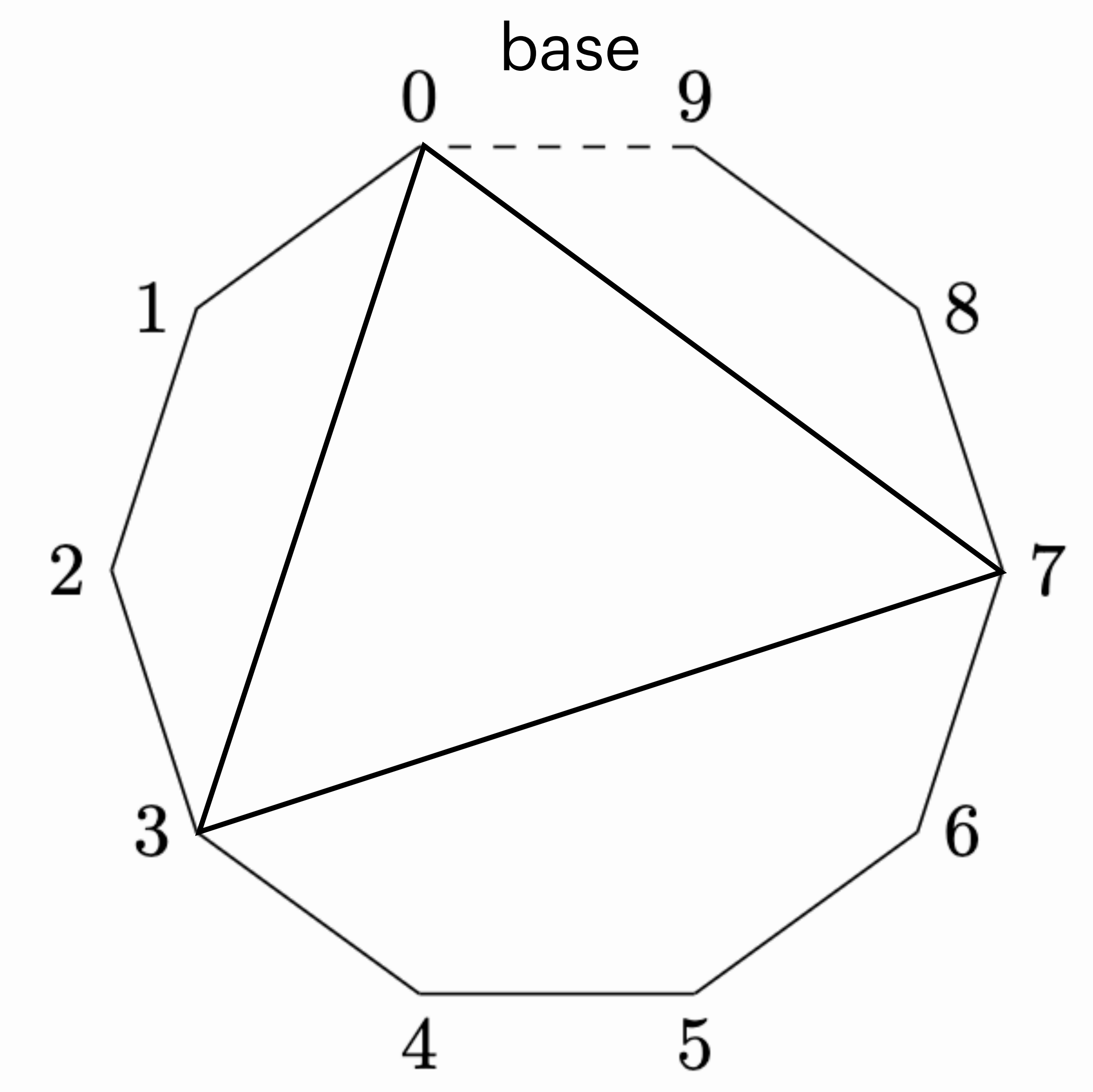

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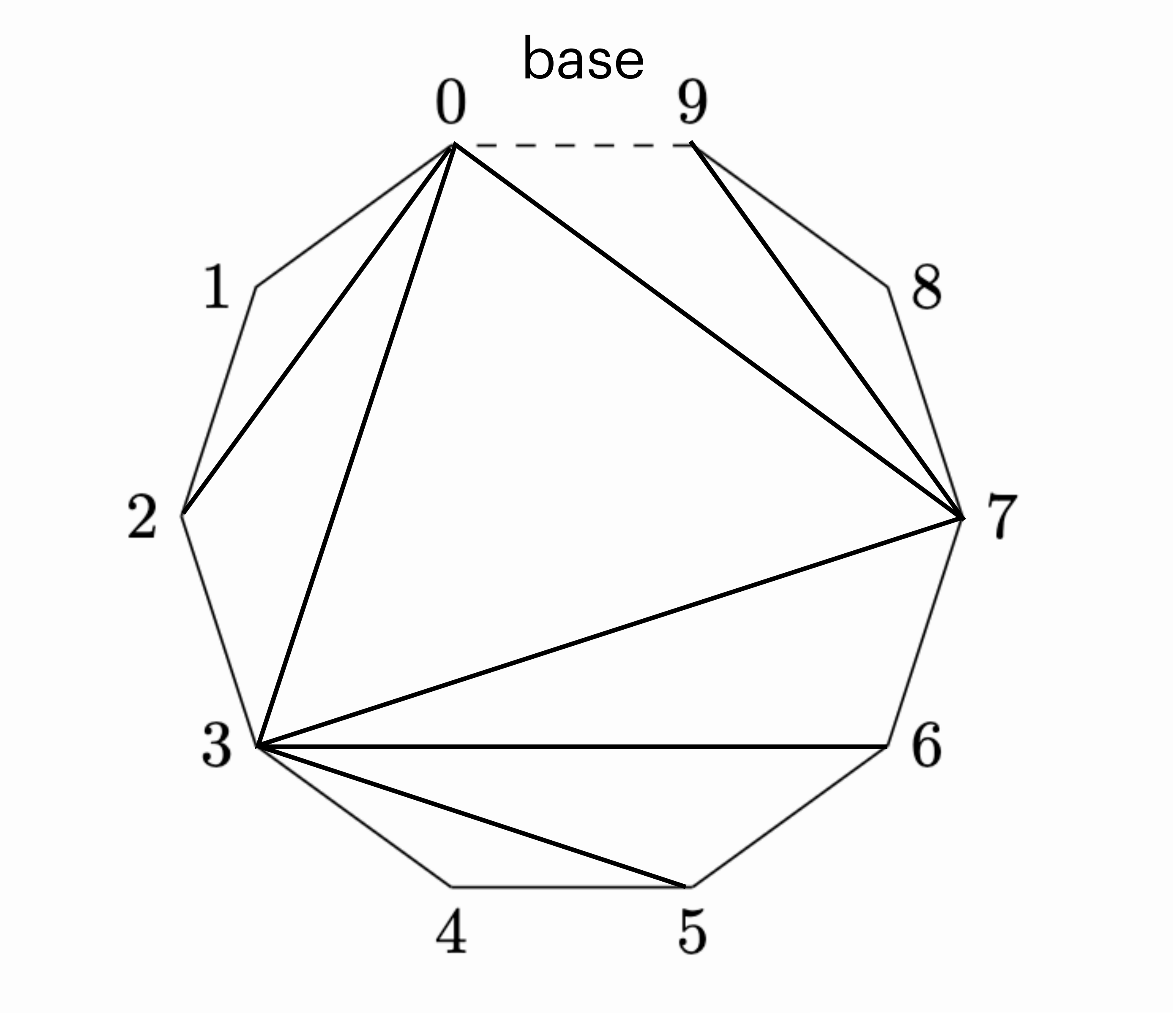

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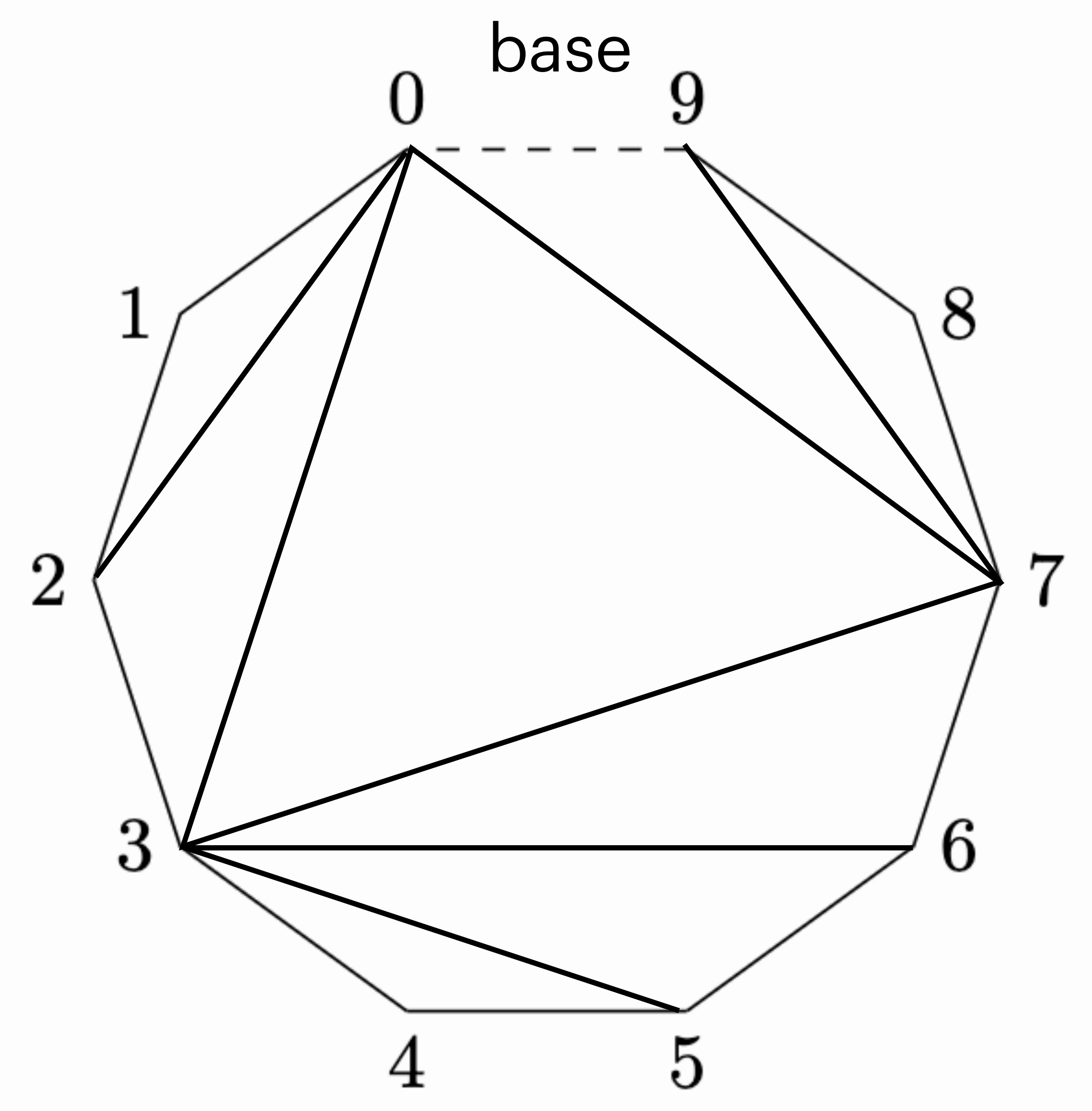

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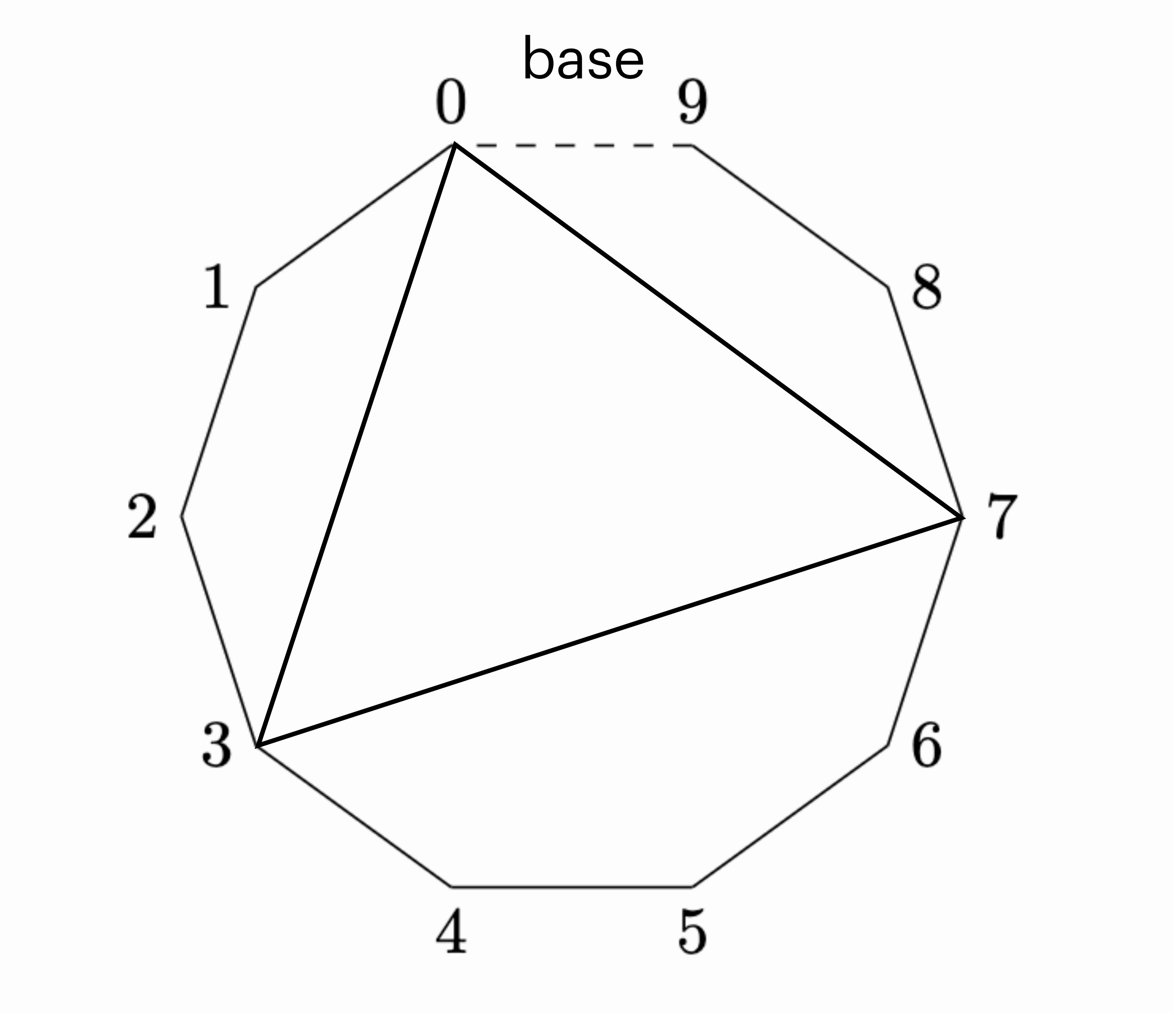

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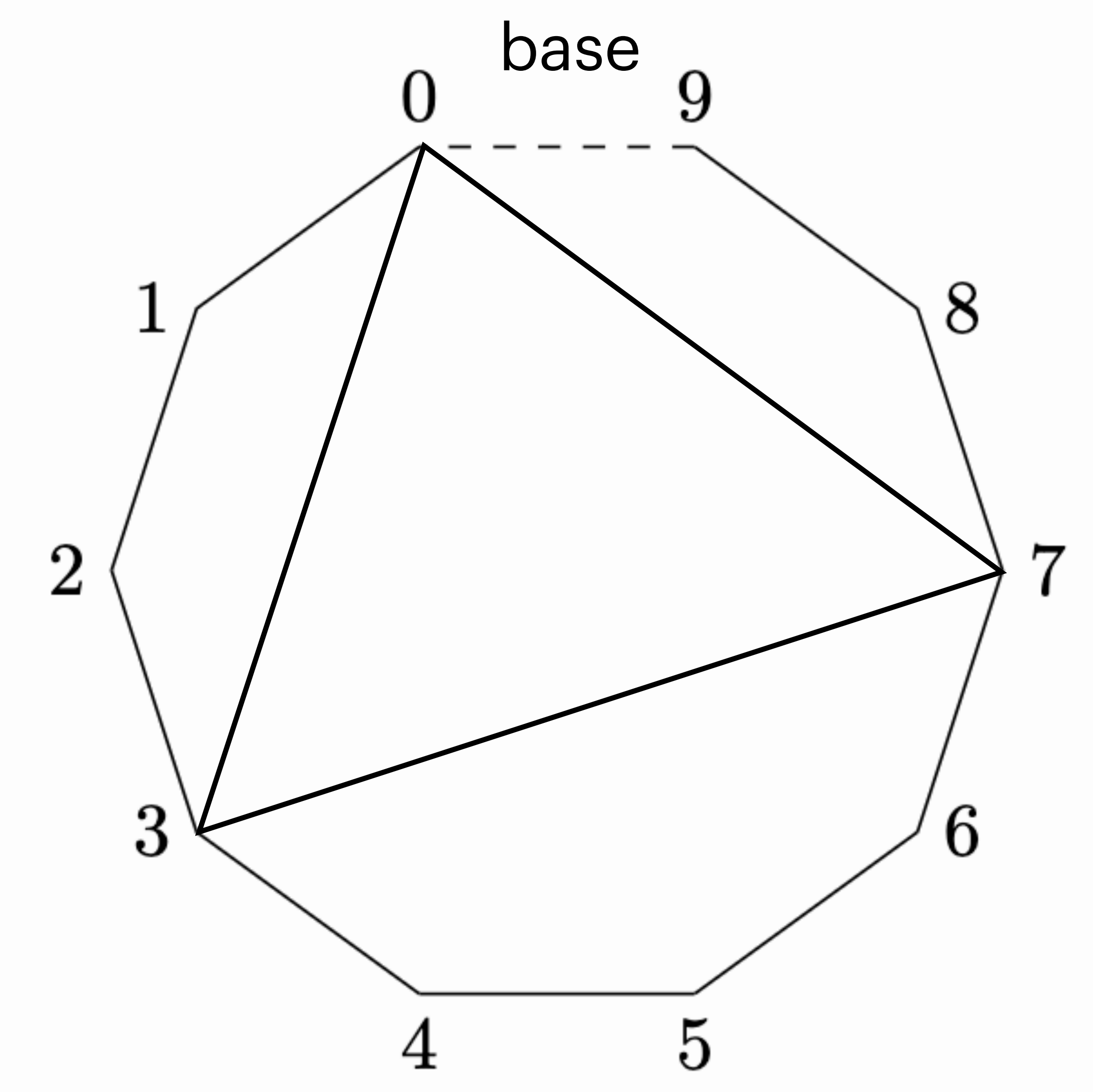

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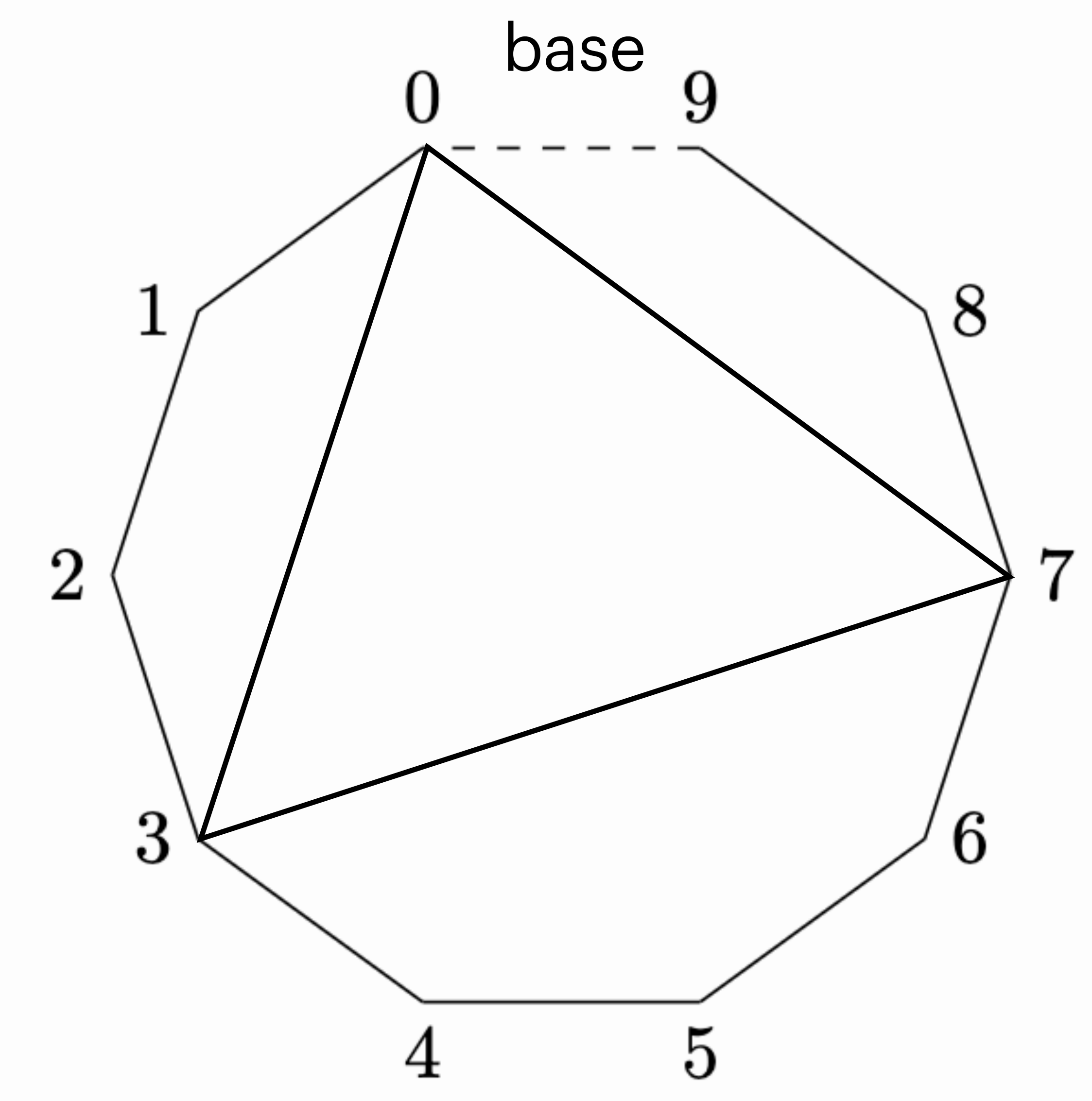

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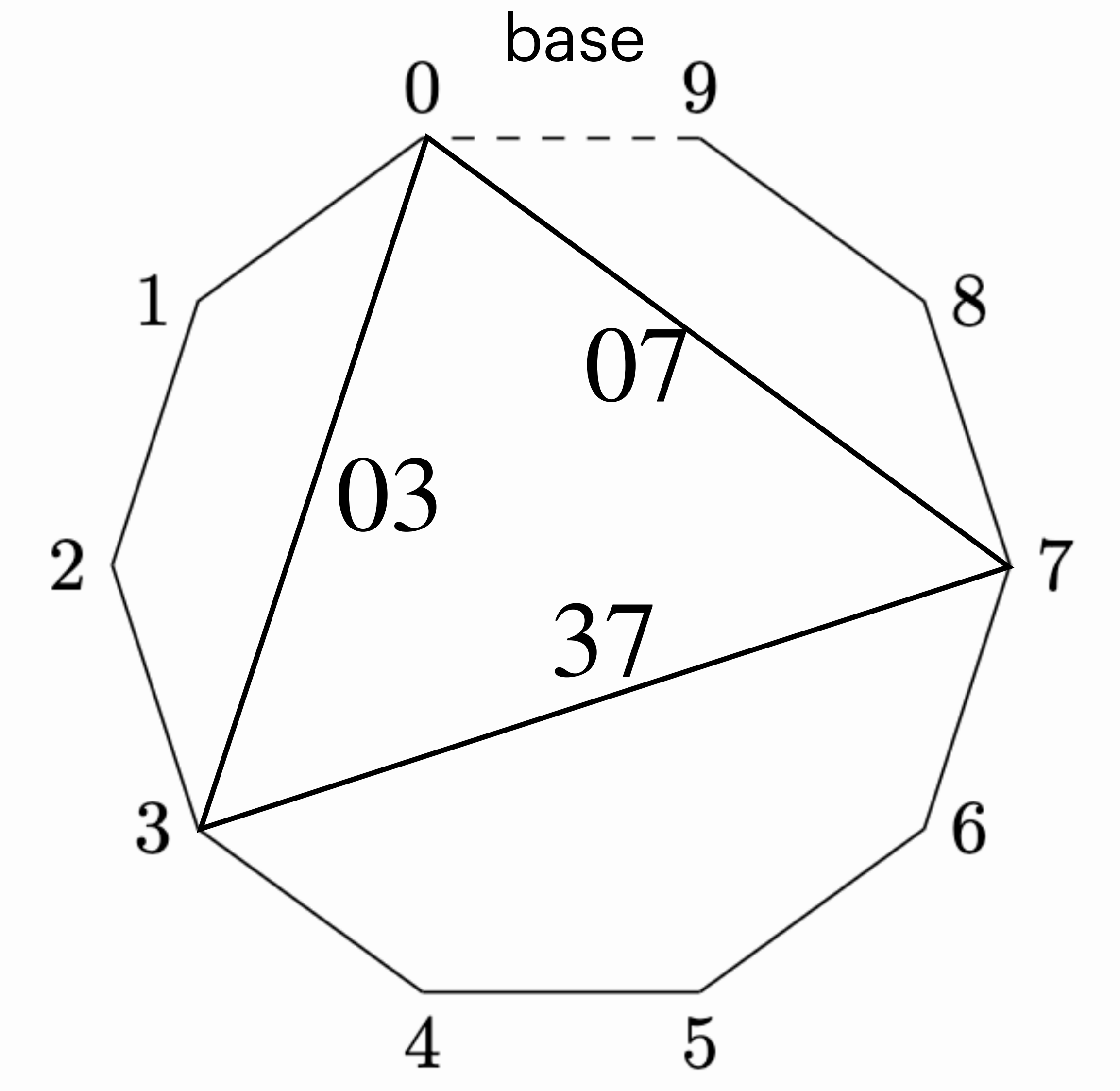

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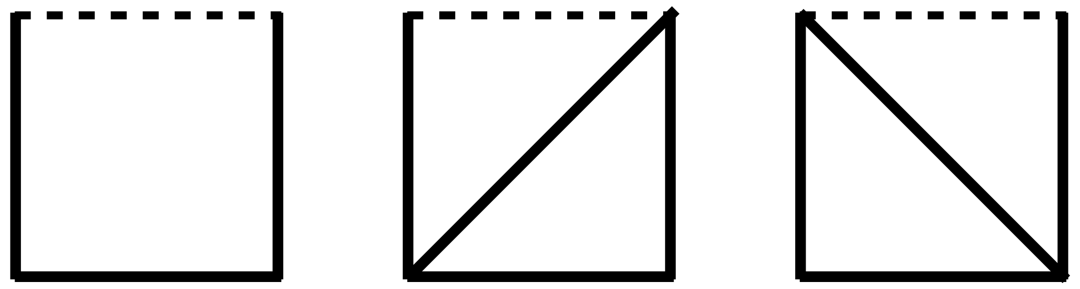

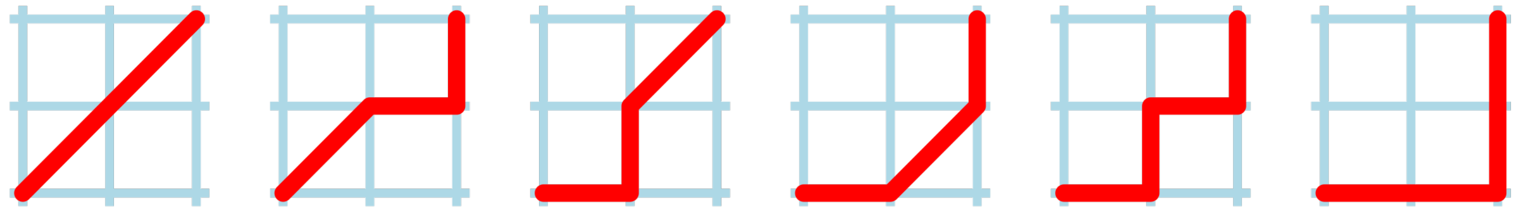

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Schröder trees

The small Schröder number s_{n+1} also describes the number of Schröder trees with $n+1$ leaves.

A Schröder tree is an ordered rooted tree whose non-leaf vertices have at least two children.

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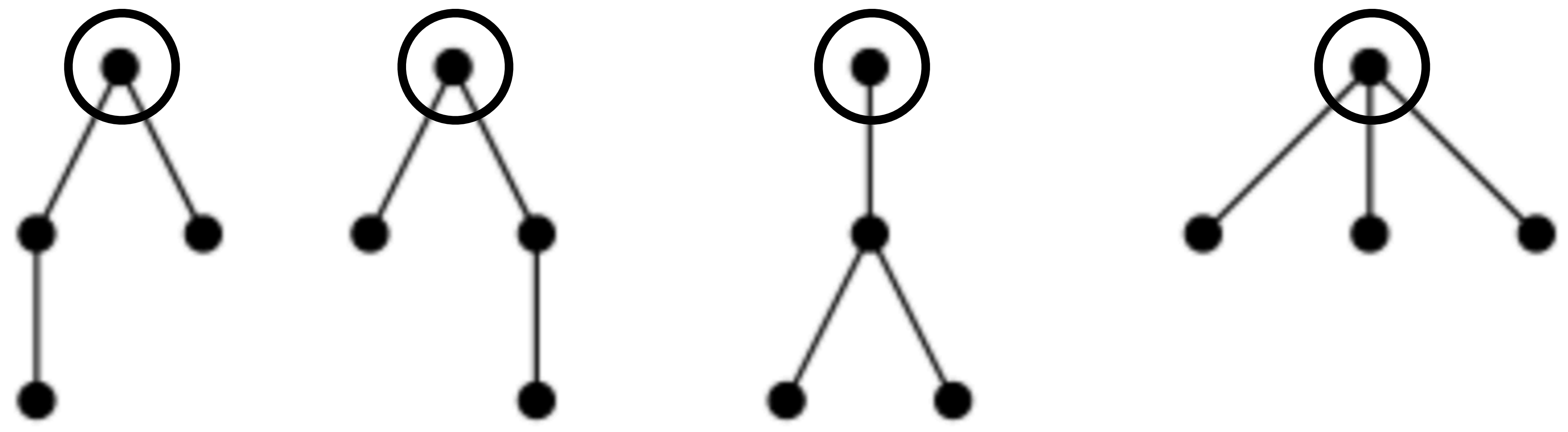

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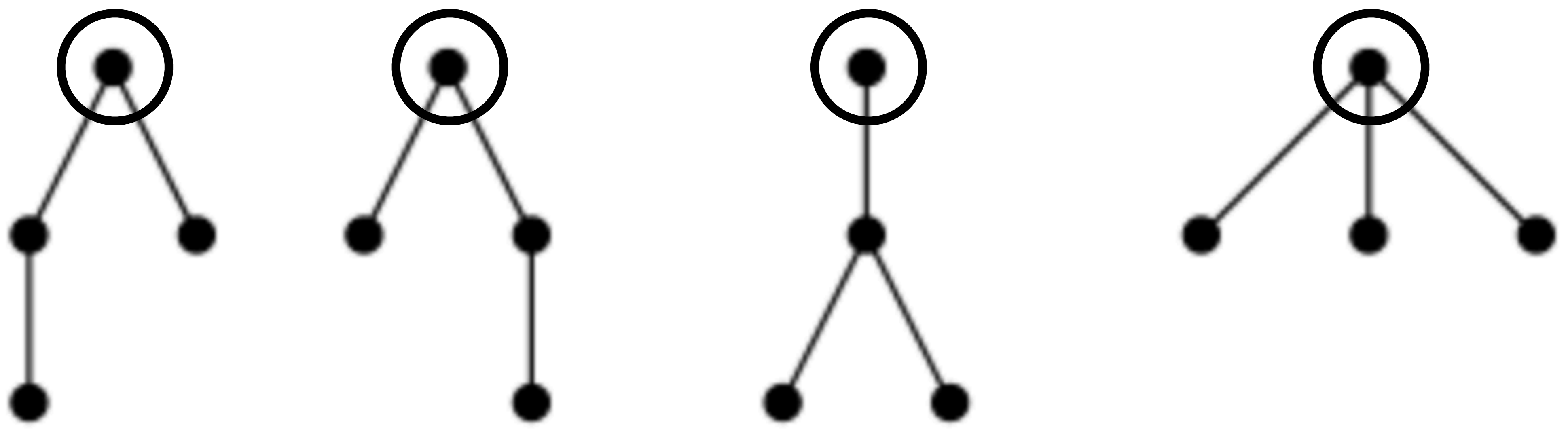

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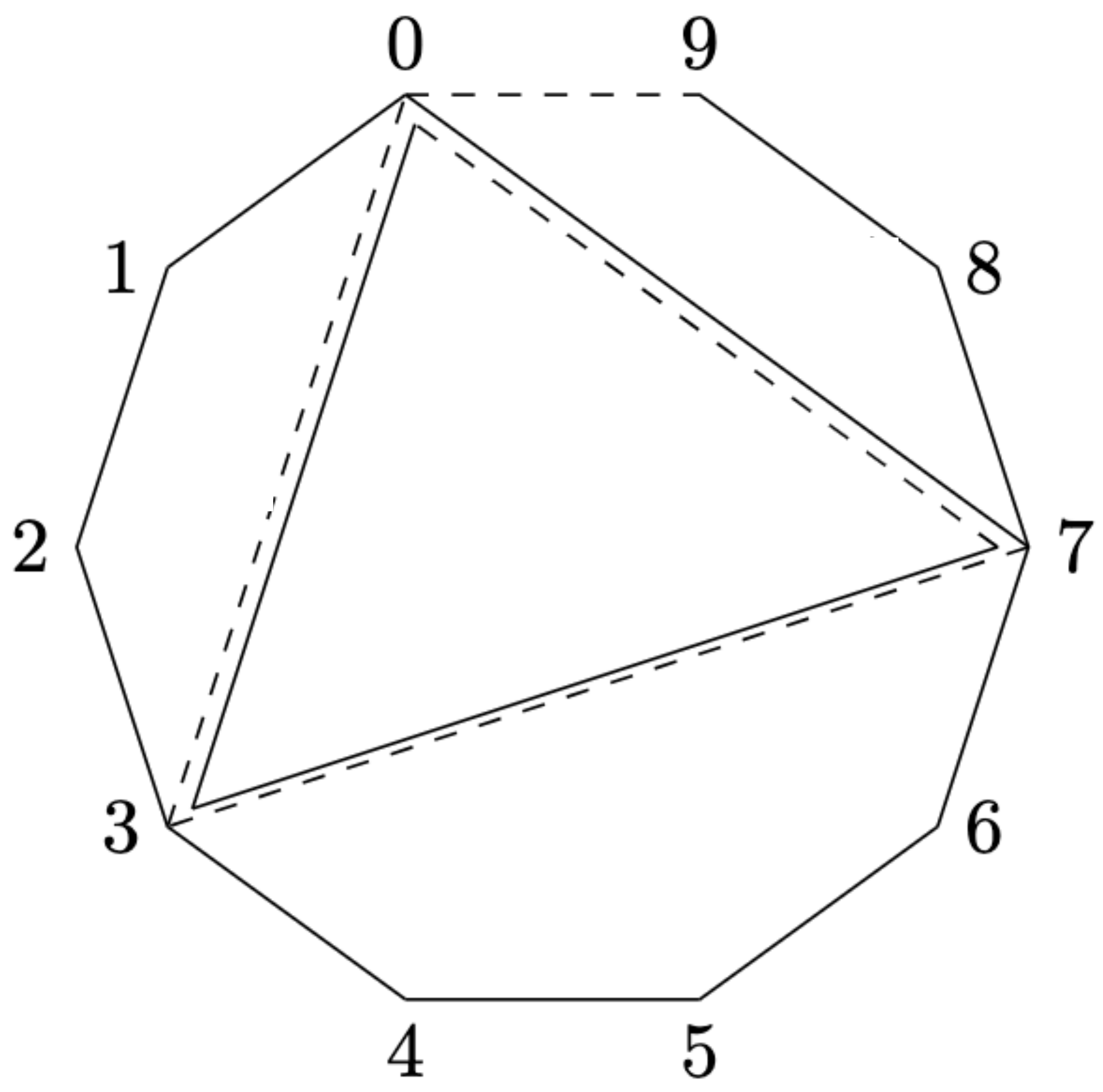
A base of a polygon is the edge connecting the smallest vertex and the largest vertex. A dissection D of P_{n+2} produces a set of small polygons whose base is the base of P_{n+2} or a diagonal in D.



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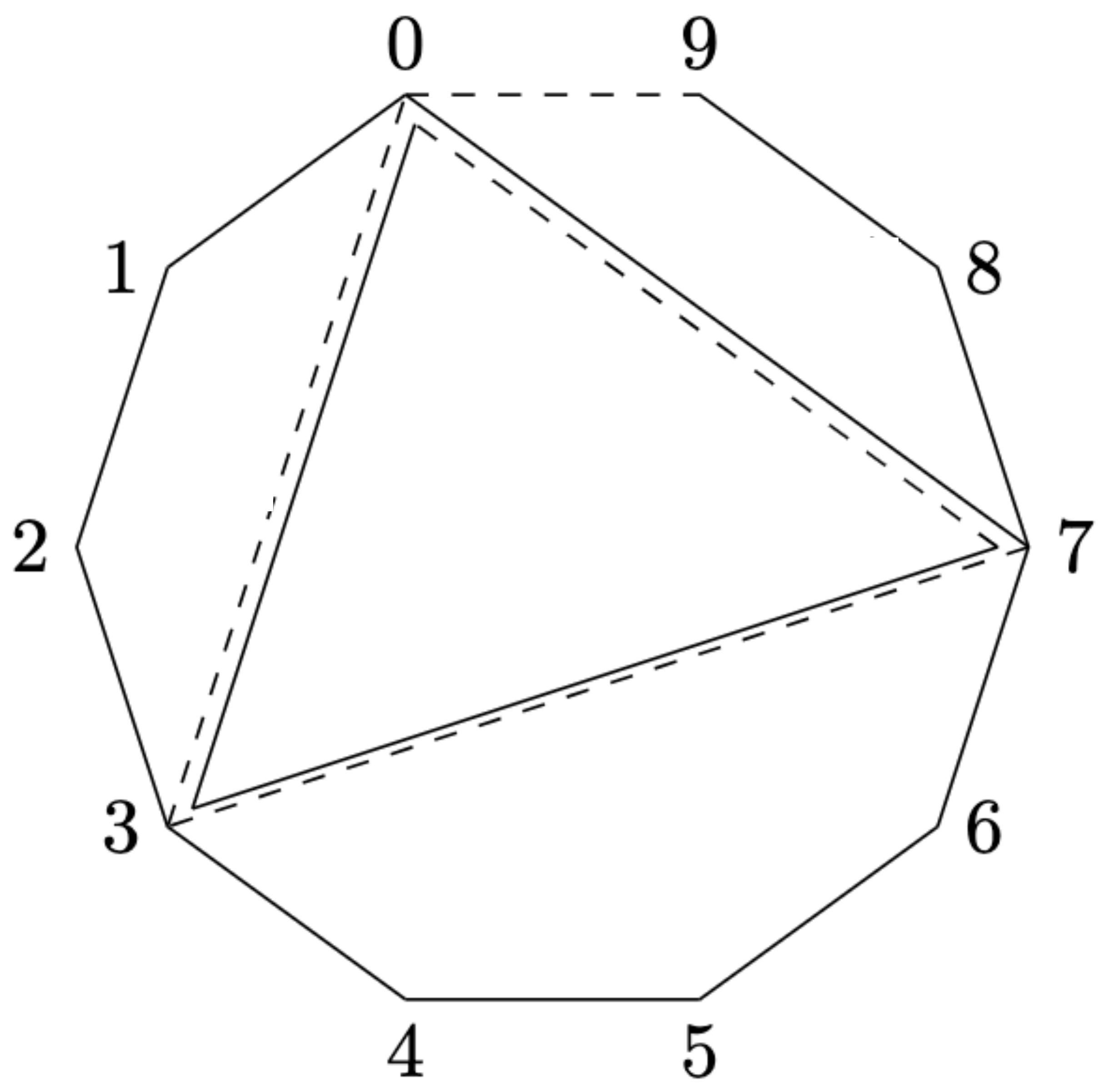
We label the small polygons in D using their bases.



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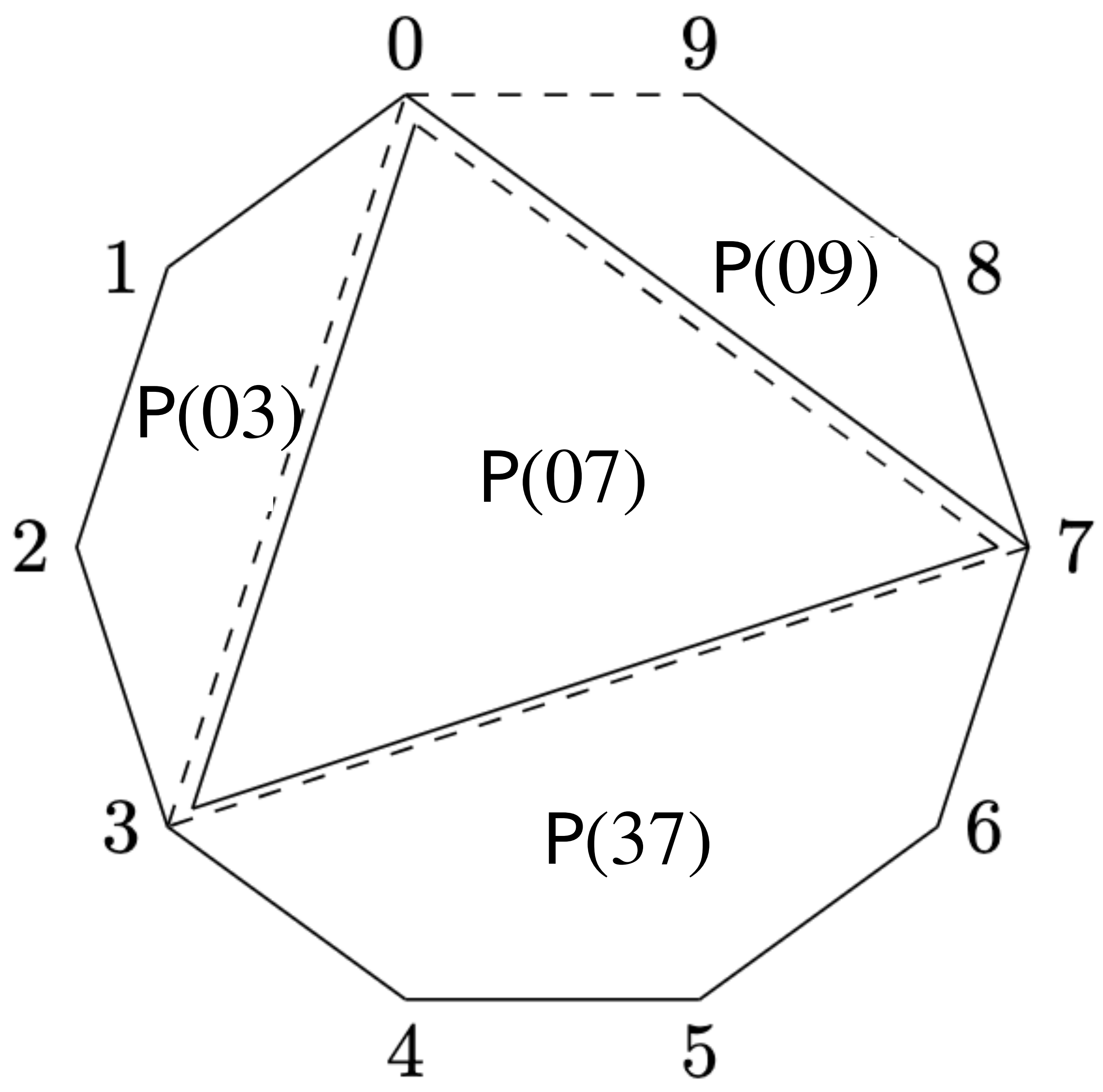

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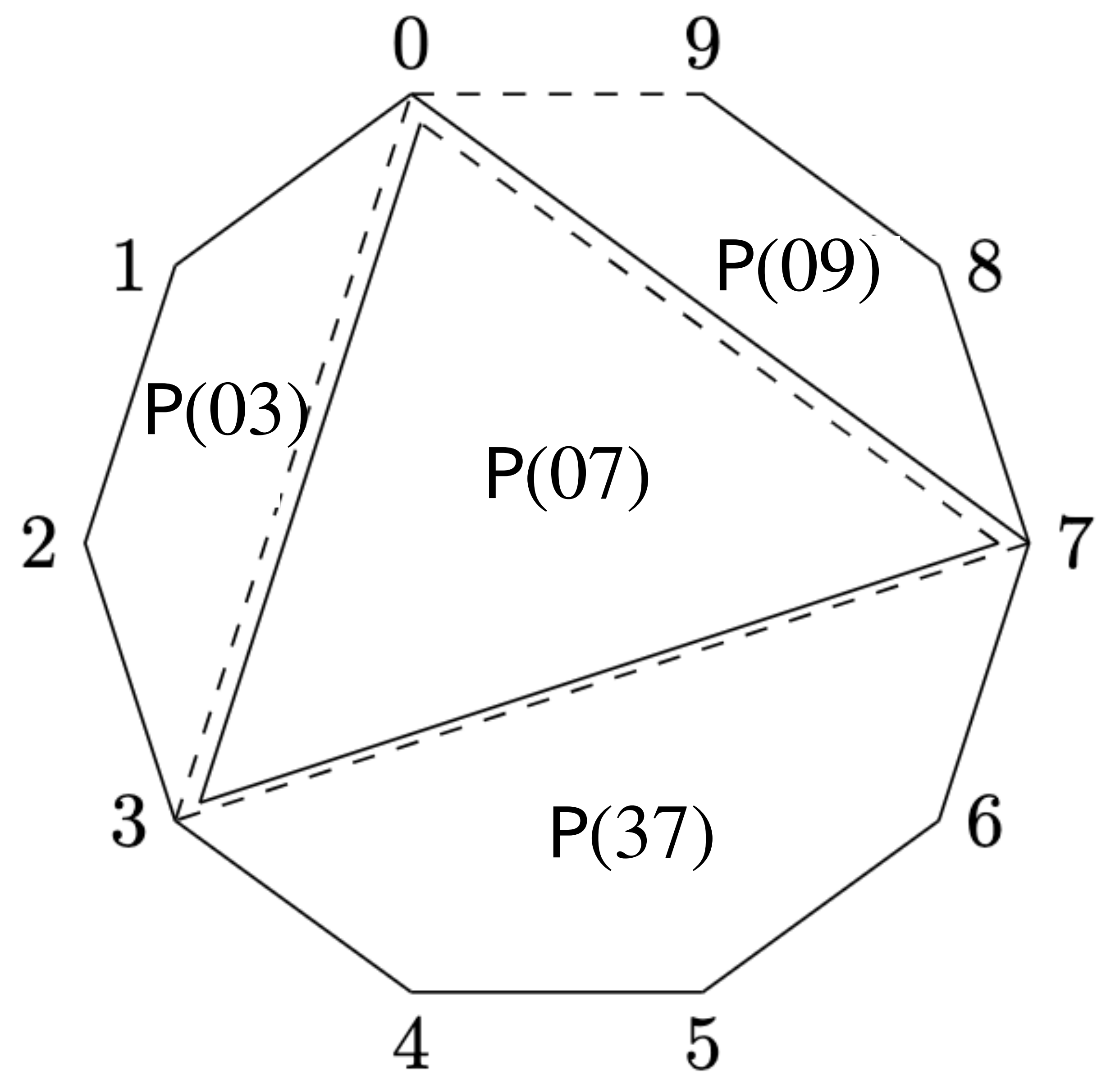
Let $\mathscr{E}_0(D)$ be the set of edges of D except the base $(0,n+1)$.

Then $\mathscr{E}_0(D)$ recovers the dissection D.



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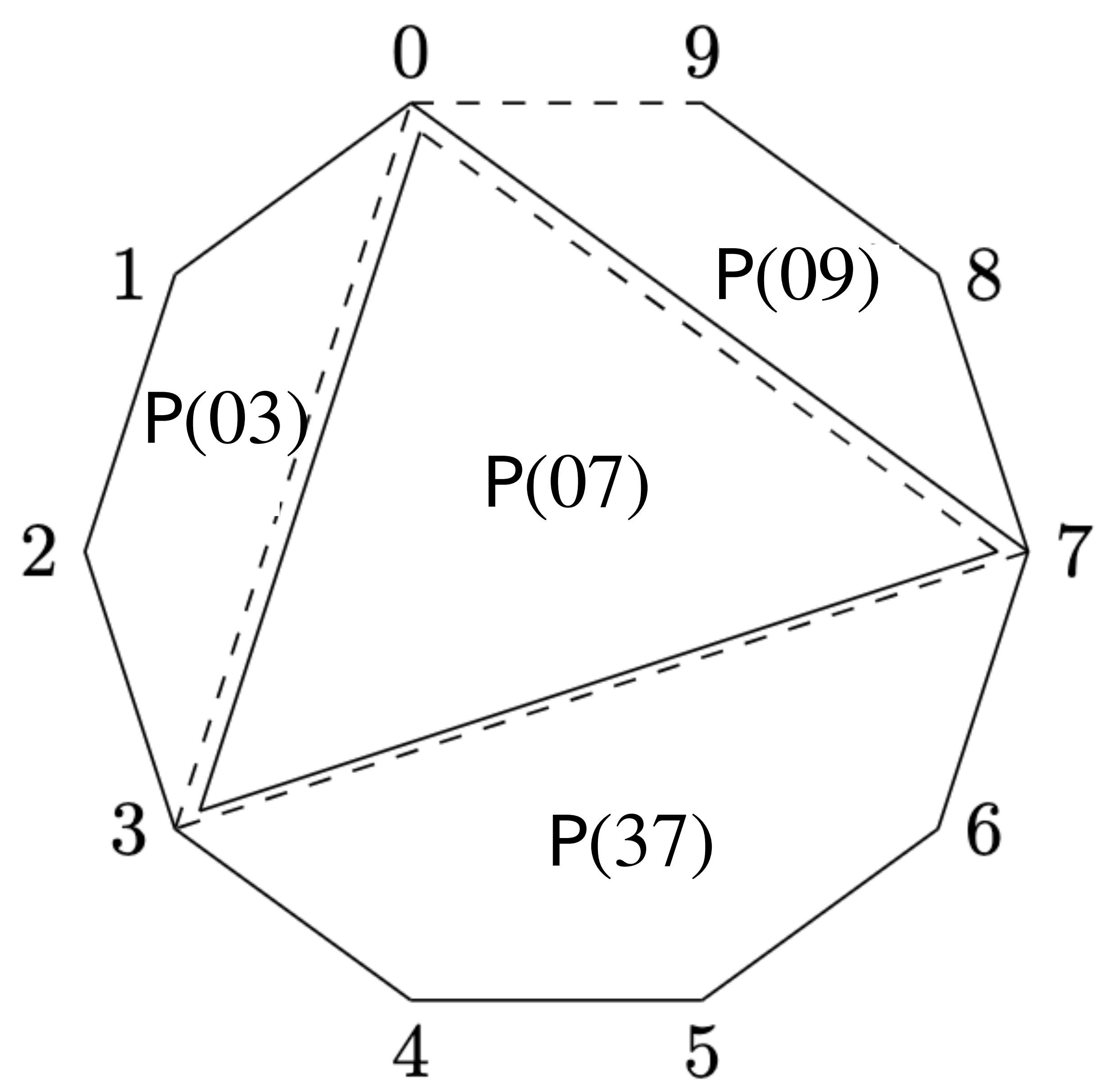

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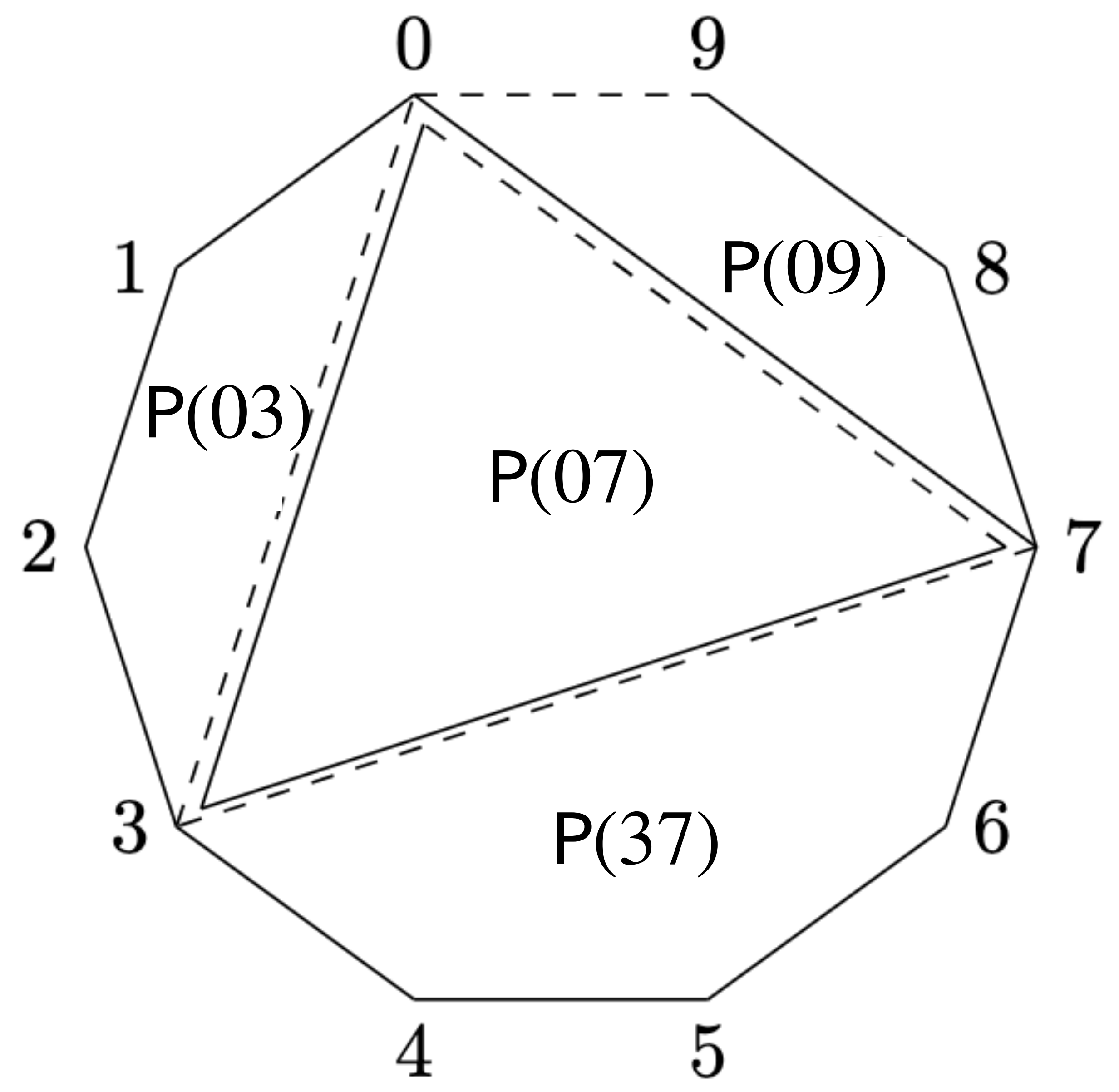

Set $(i_0, j_0) = (0,n+1)$ and denote by $\mathscr{E}_0(\mathsf{P}(i_q, j_q))$ the set of edges in the small polygon $\mathsf{P}(i_q, j_q)$ except (i_q, j_q) for $q = 0,1,\ldots,k$. Then $\mathscr{E}_0(D) = \coprod_{q=0}^{k} \mathscr{E}_0(\mathsf{P}(i_q, j_q))$.

Etherington's bijection

Theorem. (Etherington 1940)

There is a one-to-one correspondence between the set of polygon dissections of P_{n+2} and the set of Schröder trees of $n + 1$ leaves.

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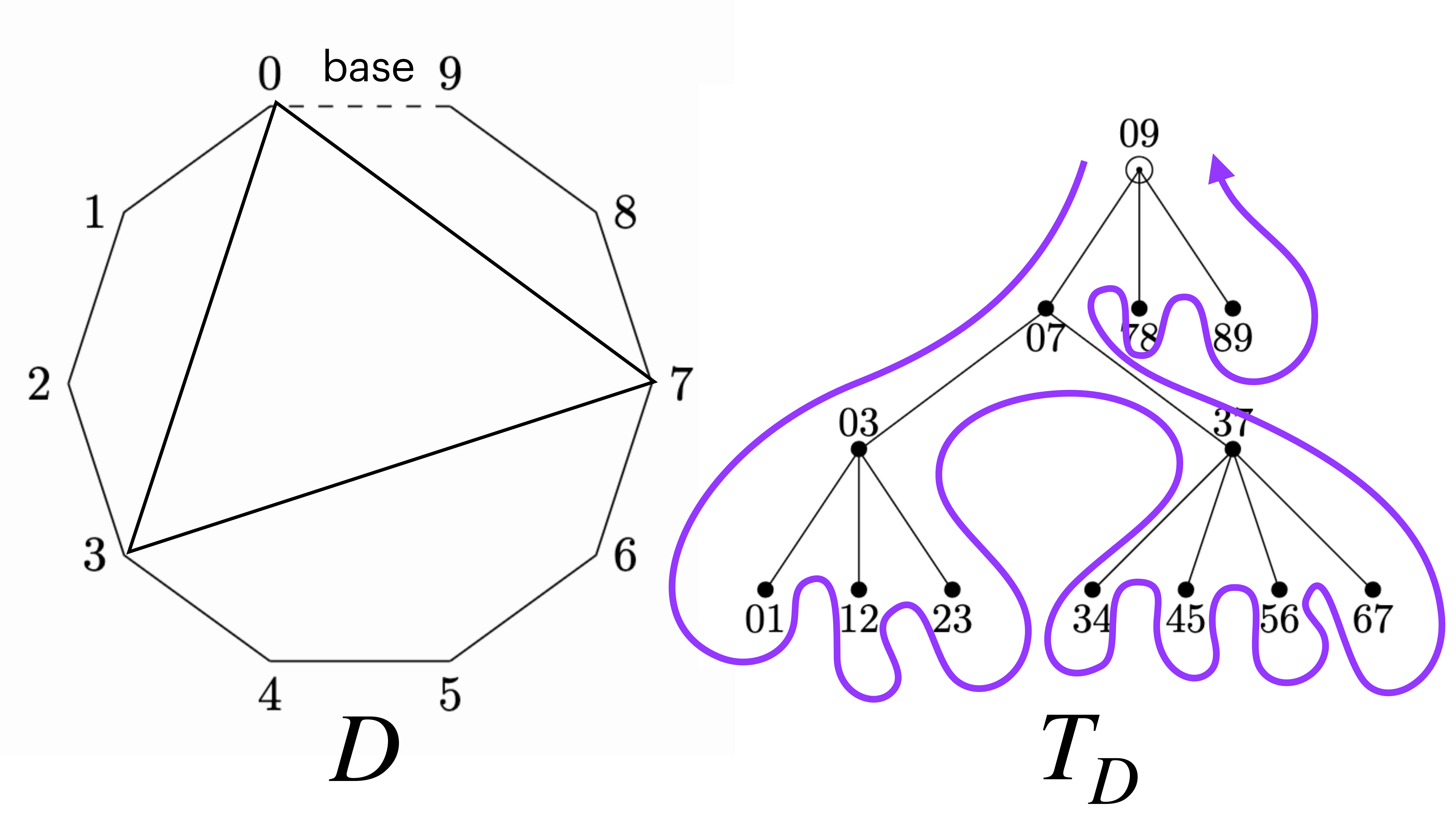


$$\phi(v) = \begin{cases} (0,n+1) & \text{if } v \text{ is the root,} \\ (i-1,i) & \text{if } v \text{ is the } i\text{th leaf in the preorder listing of } T, \text{ and} \\ (i,j) & \text{if } v \text{ is an internal vertex whose left-most and right-most} \\ & \text{children are labeled by } (i,\bullet) \text{ and } (\bullet,j), \text{ respectively.} \end{cases}$$

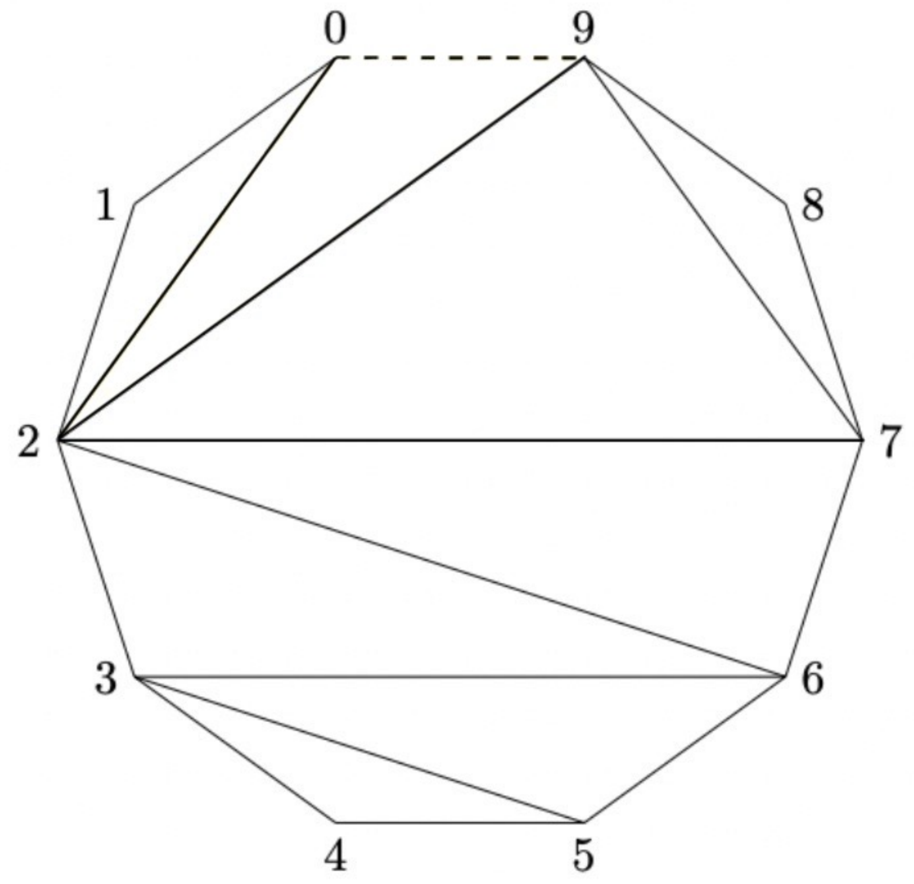
Etherington's bijection

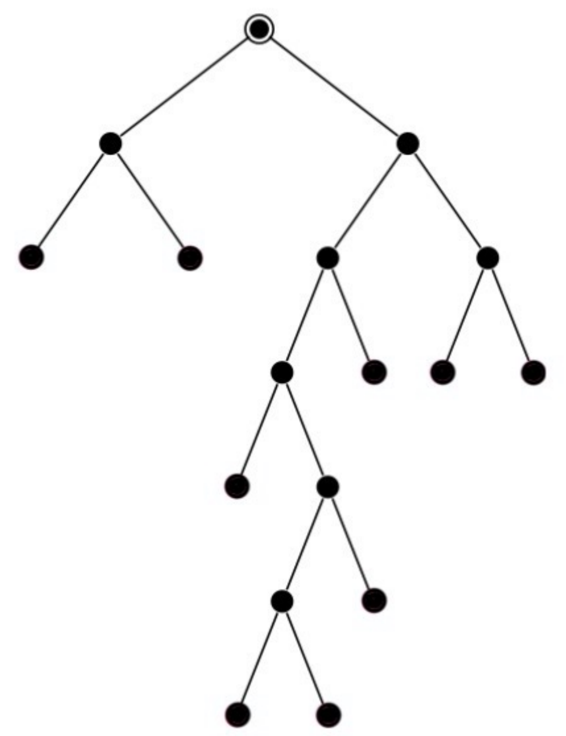
1. The base $(0, n+1)$ corresponds to the root, and the diagonals correspond to the non-leaf vertices, not the root.

2. There is a one-to-one correspondence between the small polygons in a dissection D and the non-leaf vertices of the Schröder tree T_D.

3. Each $\mathscr{E}_0(\mathsf{P}(i_q, j_q))$ corresponds to the set of children of the vertex (i_q, j_q) in T_D.

4. There is a one-to-one correspondence between the triangulations of P_{n+2} and the full binary rooted trees with $n+1$ leaves.

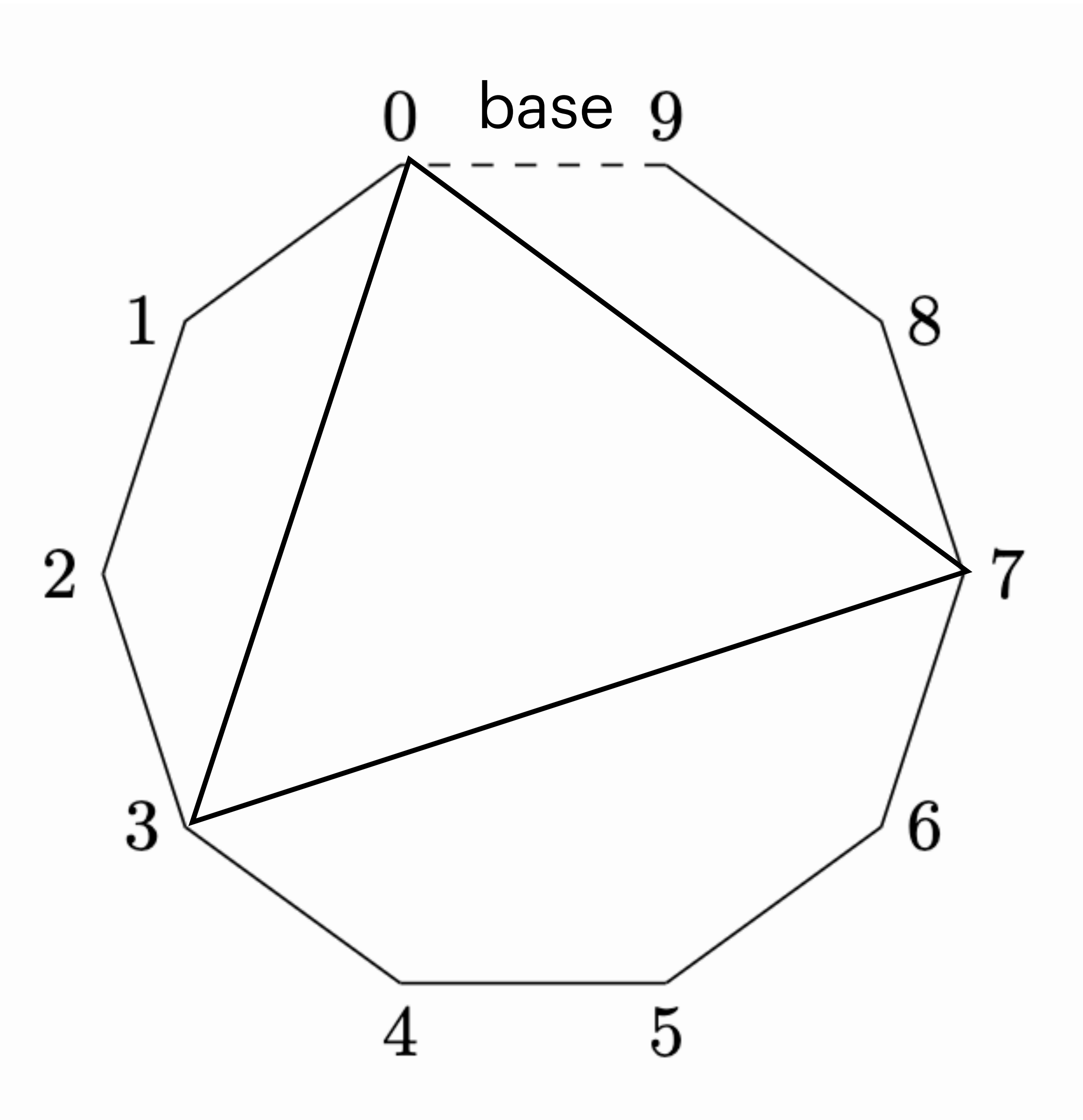




Order on the diagonals in D

For a polygon dissection D of P_{n+2} with diagonals $(i_1, j_1), \ldots, (i_k, j_k)$, we give an order on the diagonals as follows:

$$(i, j) \prec (i', j') \quad \text{if} \quad \text{(i) } i < i' \quad \text{or} \quad \text{(ii) } i = i' \text{ and } j > j'.$$



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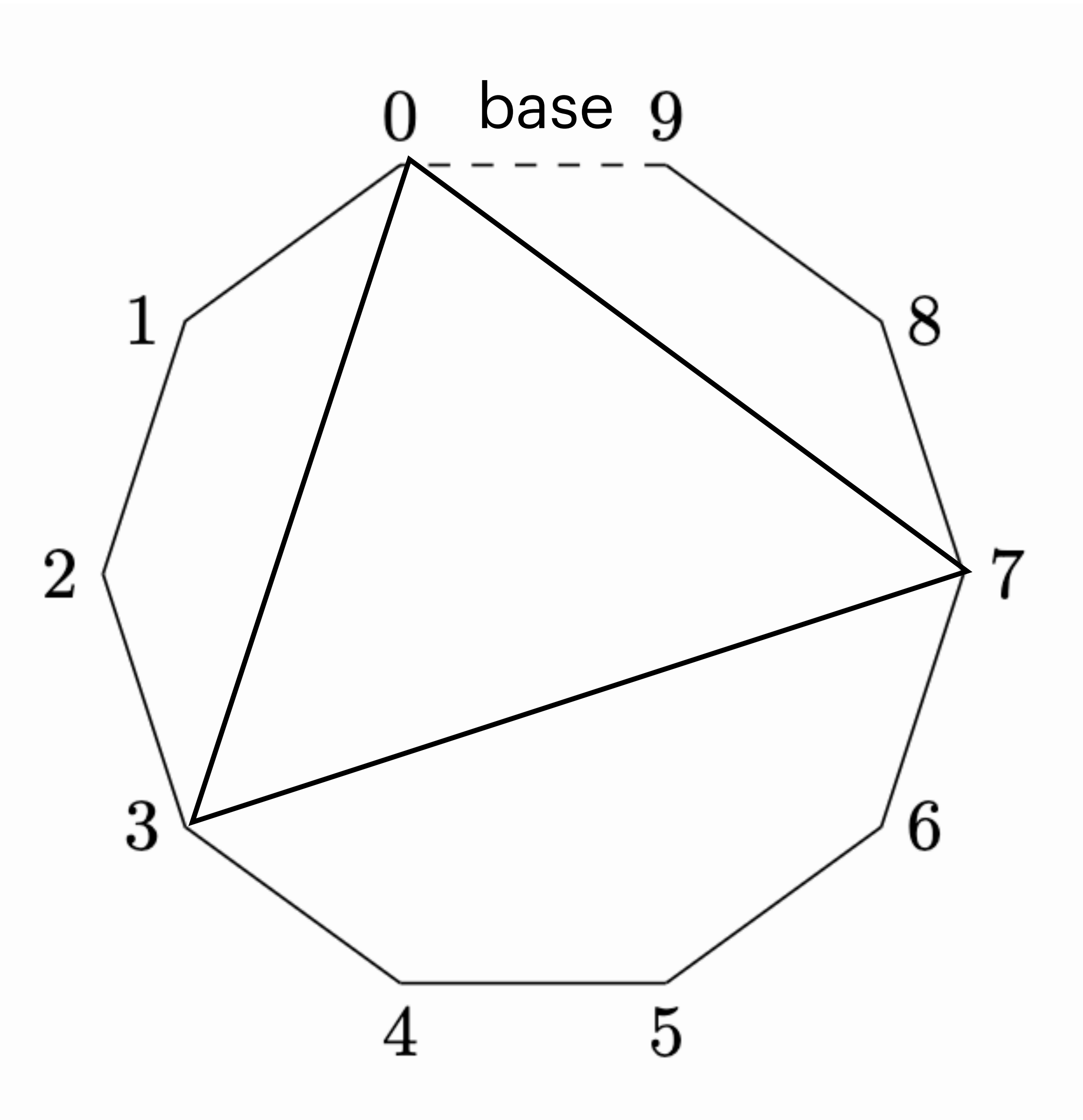

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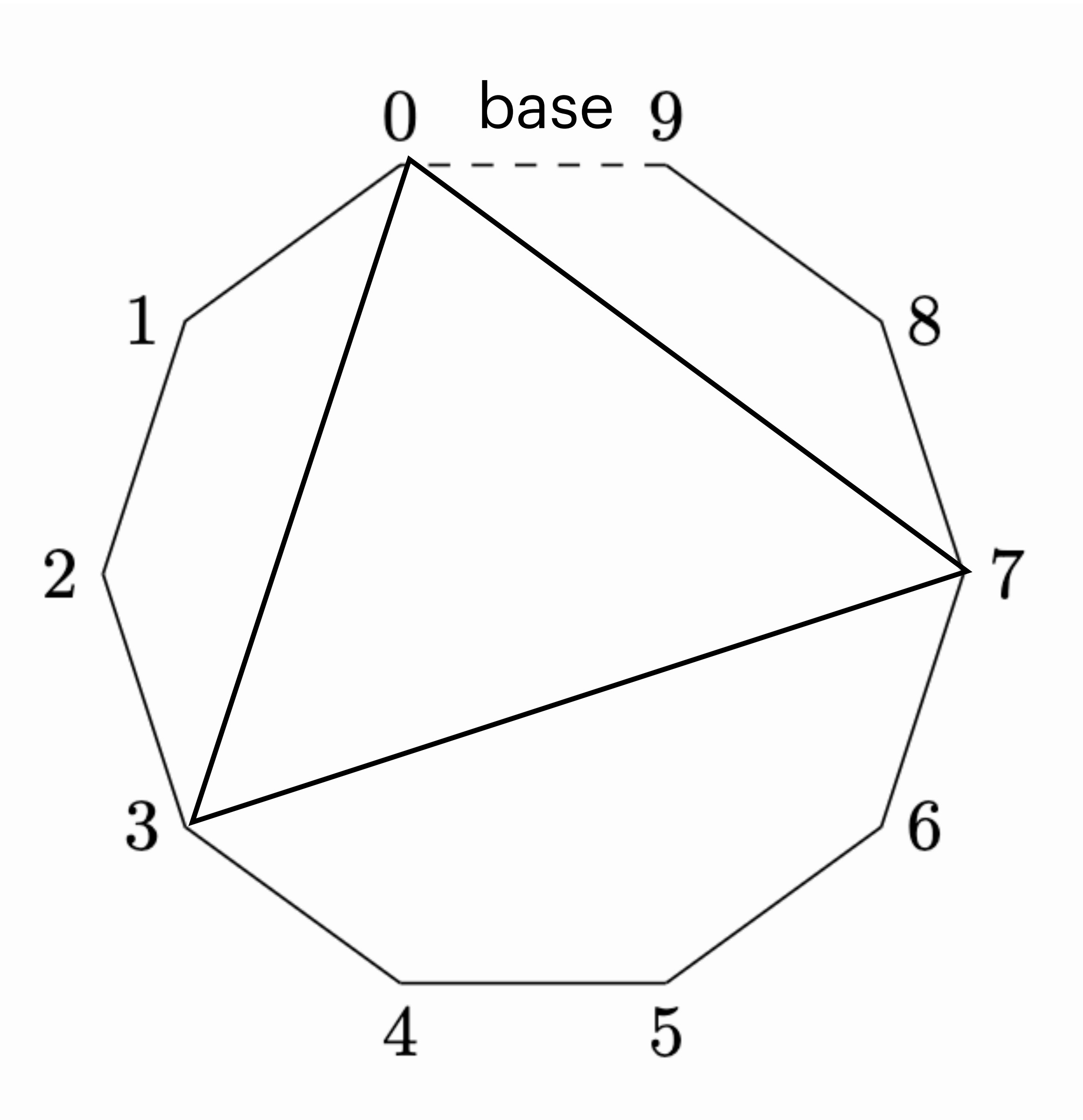

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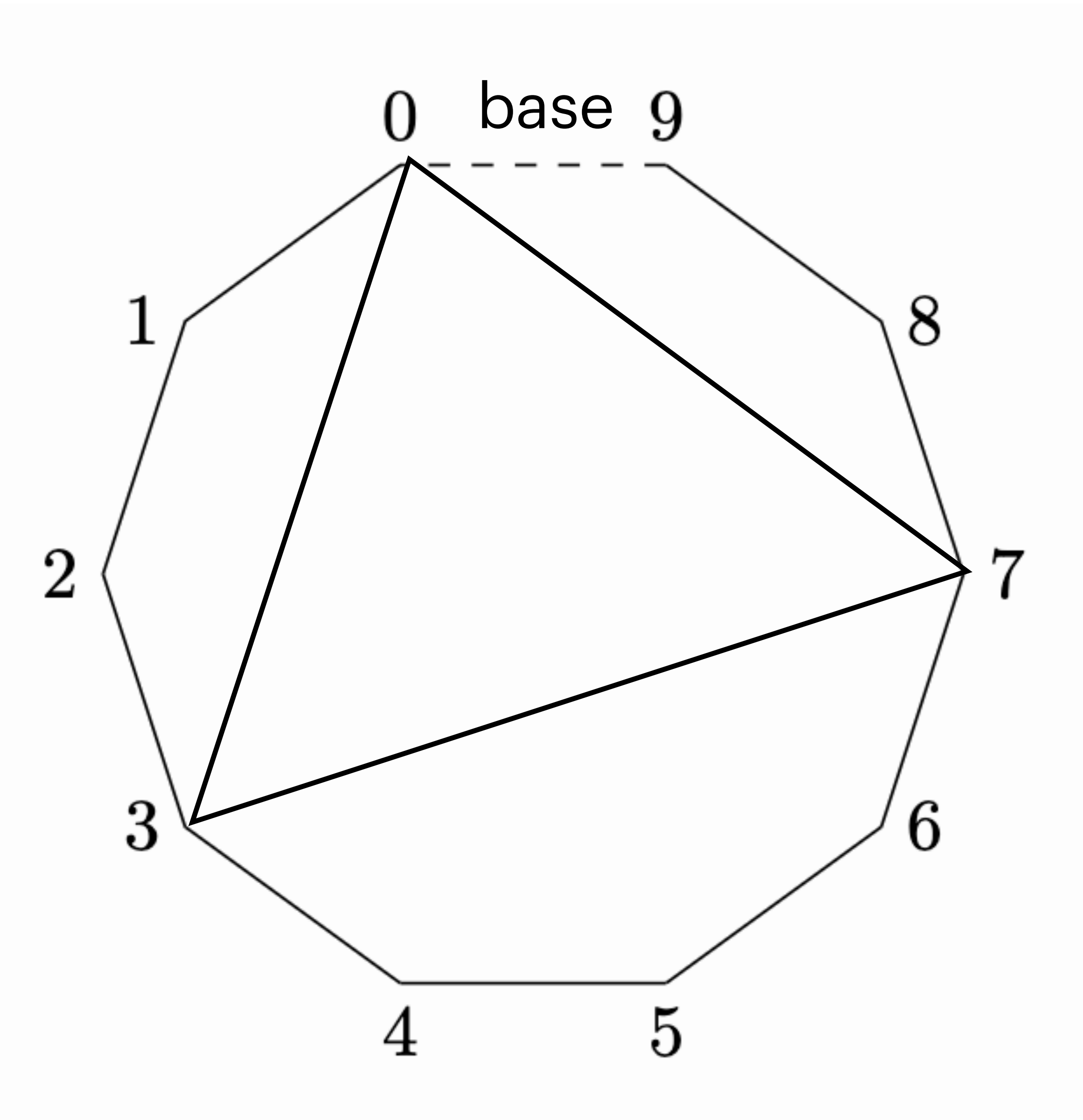

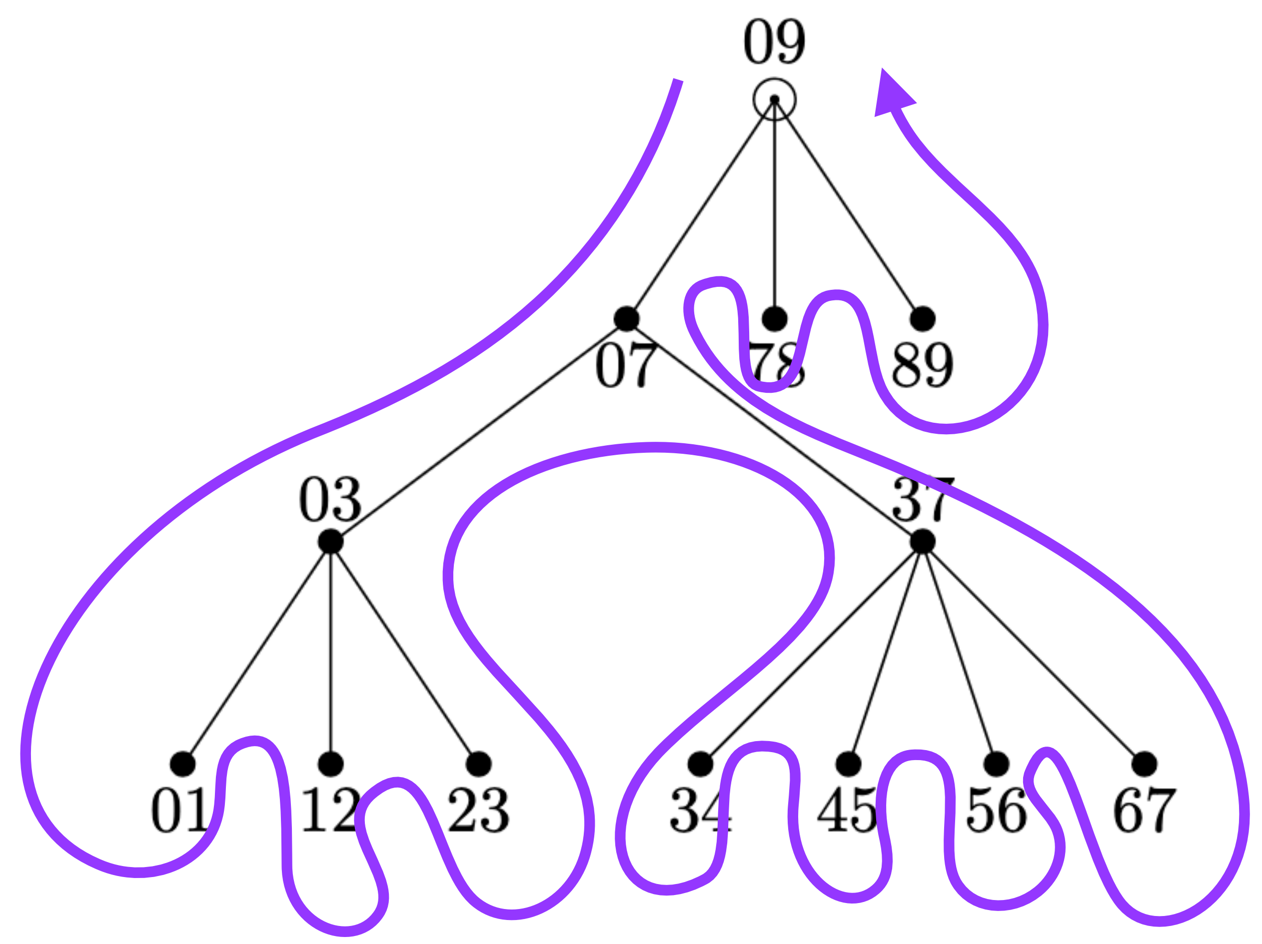

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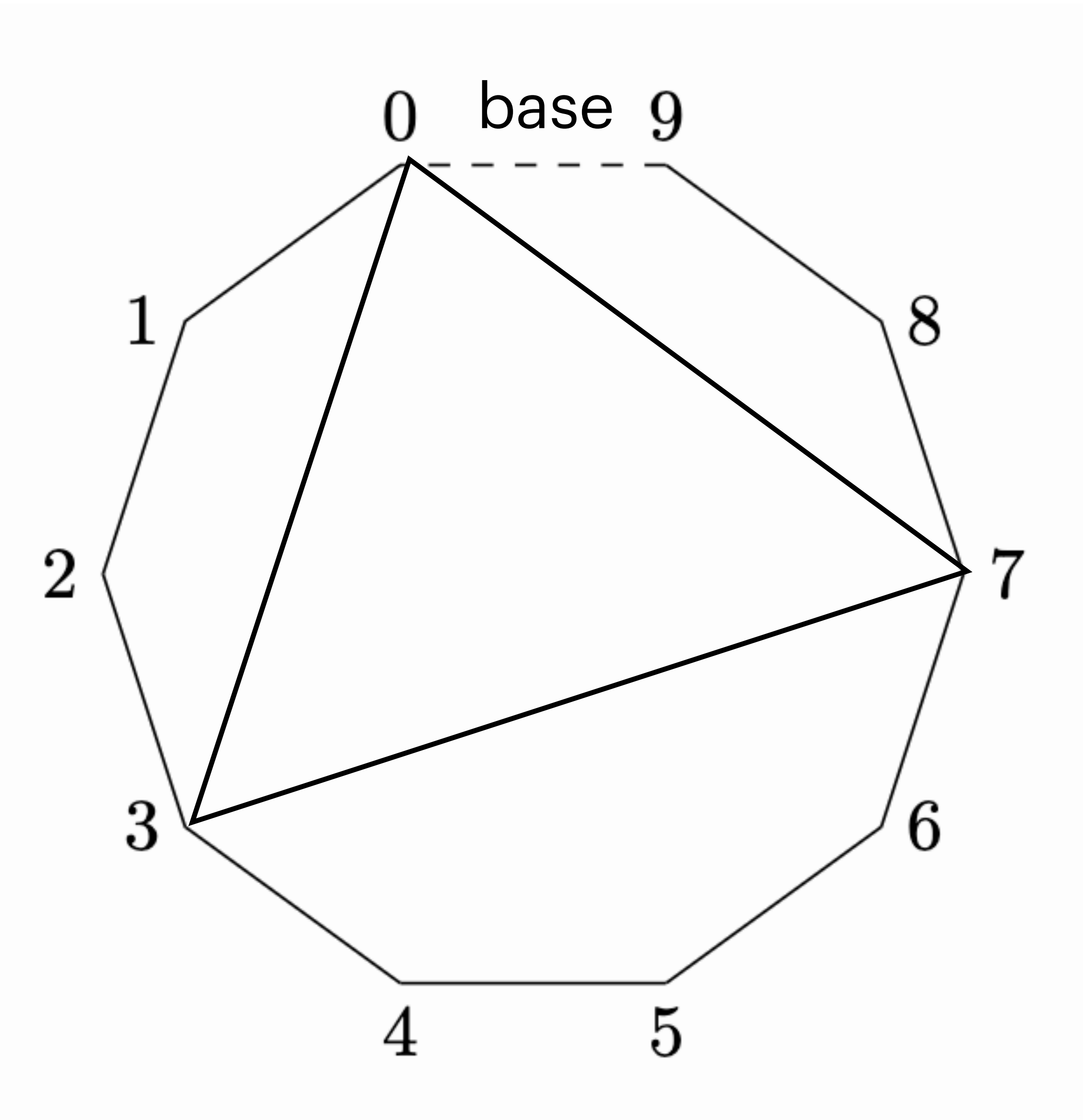

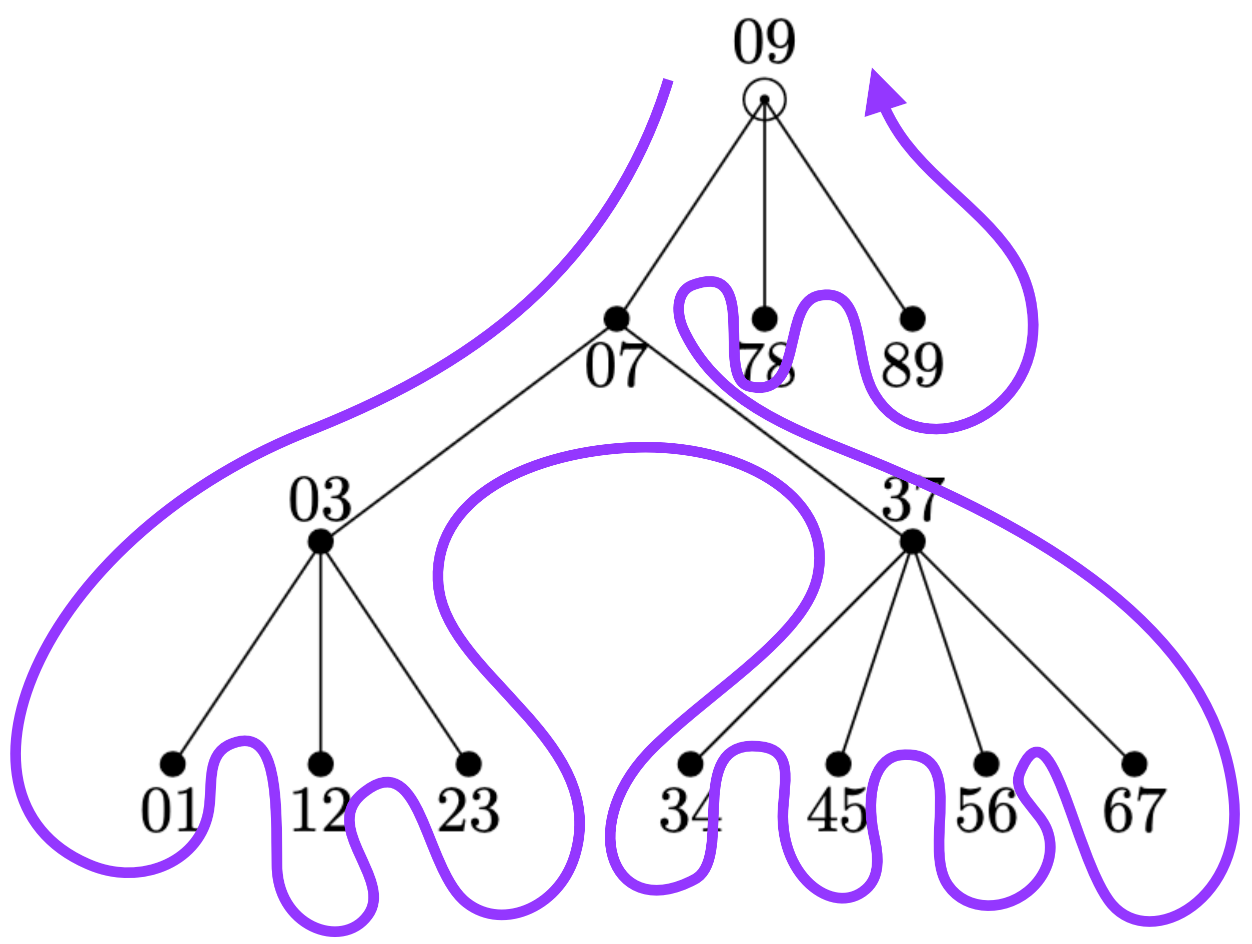

This order will be used when we construct a toric variety from D.

Construction of a toric variety from a polygon dissection

Toric variety

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Example.

1. $\mathbb{C}^n$ is a smooth toric variety.

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n)$$

2. $\mathbb{C}P^n$ is a projective smooth toric variety.

$$(t_1, \ldots, t_n) \cdot [z_0; z_1; \ldots; z_n] = [z_0; t_1 z_1; \ldots; t_n z_n]$$

3. A generalized Bott manifold is a projective smooth toric variety.

$$\mathscr{B}_n \to \cdots \to \mathscr{B}_j = \mathbb{P}(\mathbb{C} \oplus \bigoplus_{k=1}^{n_j} \xi_{j,k}) \to \mathscr{B}_{j-1} \to \cdots \to \mathscr{B}_1 = \mathbb{C}P^{n_1} \to \mathscr{B}_0 = \{\text{a point}\}$$

Here, $\xi_{j,k}$ is a $\mathbb{C}$-line bundle over $\mathscr{B}_{j-1}$.

Associate P_{n+2} with $\mathbb{C}P^n$.

We associate P_{n+2} with the polytope

$$\Delta = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = n(n+1)/2, x_i \geq 0\,(\forall i)\}.$$

Then the edge vectors of Δ generate the lattice $M = \{(x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}$.

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The dual lattice N of M can be identified with the quotient lattice $\mathbb{Z}^{n+1}/(1,\ldots,1)$ of $\mathbb{Z}^{n+1}$ through the dot product on $\mathbb{Z}^{n+1}$. Let ϖ_i $(i = 0,1,\ldots, n+1)$ be the quotient image of $\sum_{k=1}^{i} \mathbf{e}_k$ in N. Then $\{\varpi_1, \ldots, \varpi_n\}$ is a basis of N and $\varpi_0 = \varpi_{n+1} = \mathbf{0}$ by definition.

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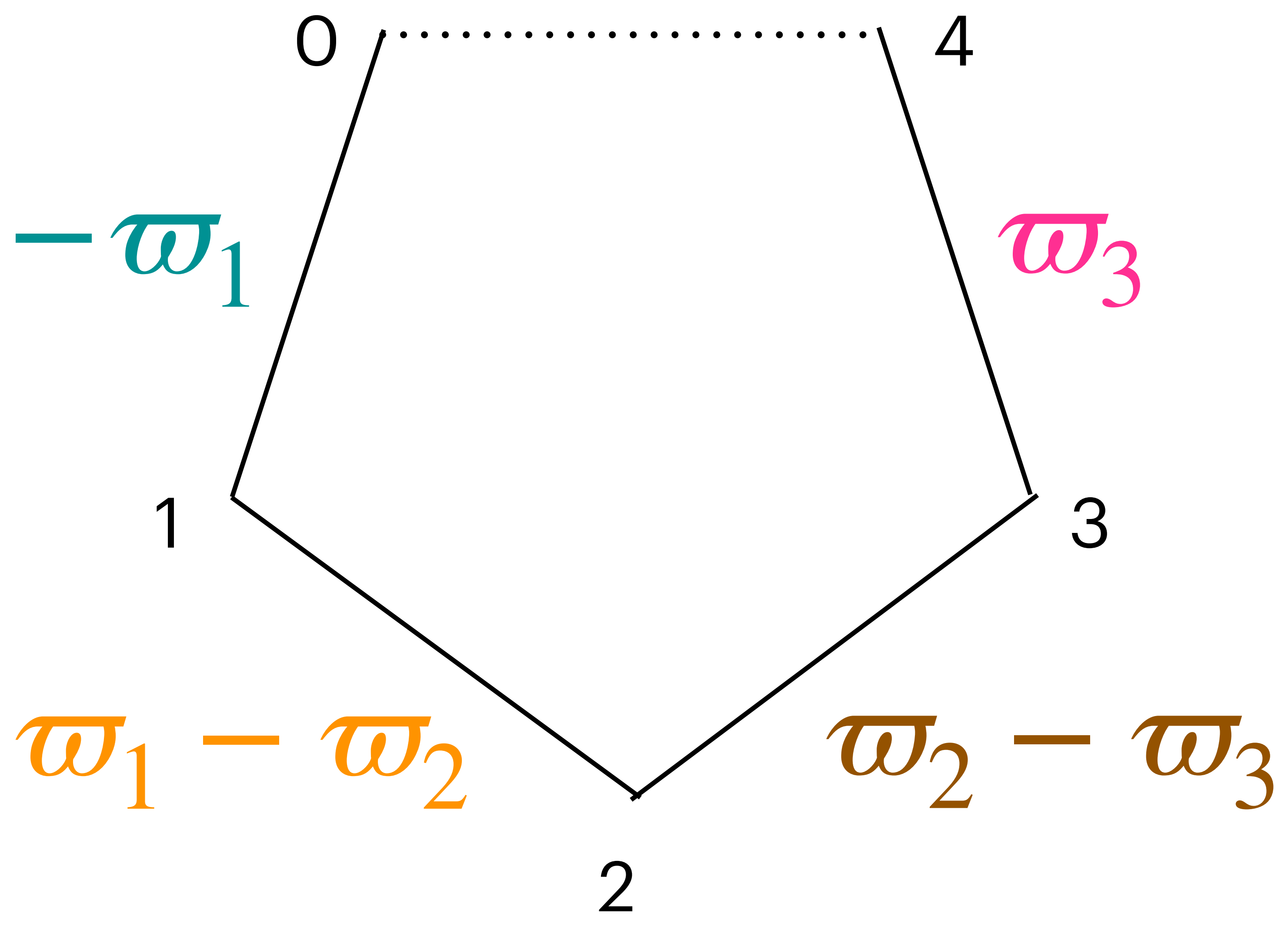
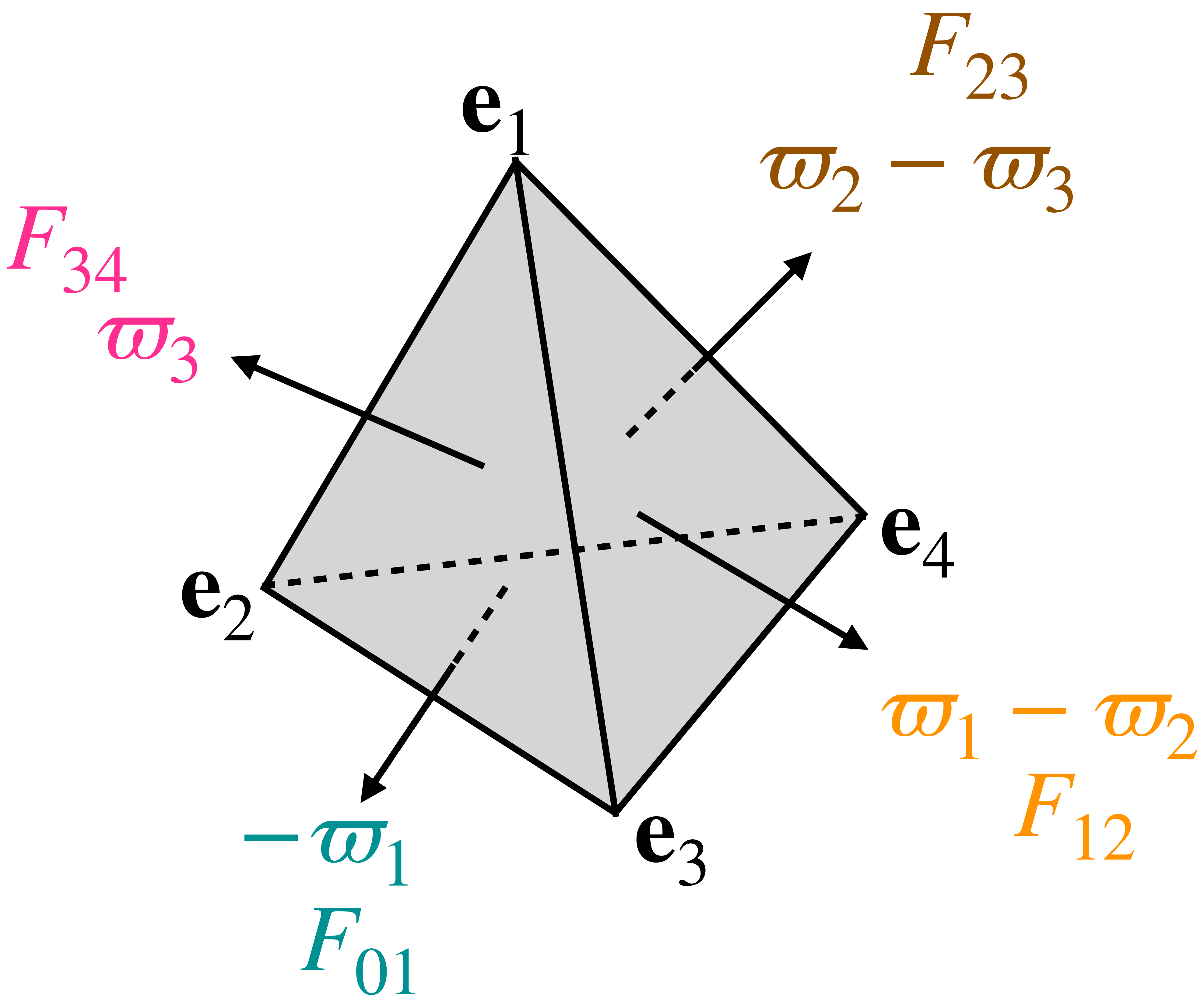
The normal facet vectors of Δ are $\varpi_{i-1} - \varpi_i$ for $i = 1,\ldots,n+1$, which corresponds to the side $(i-1,i)$ of P_{n+2}.

Example





For $i = 1, \ldots, n+1$, we denote $F_{i-1,i}$ by the facet whose outward normal vector is $\varpi_{i-1} - \varpi_i$.

Toric variety corresponding to a polygon dissection

Note that a blowing up of a smooth projective toric variety becomes a smooth projective toric variety.

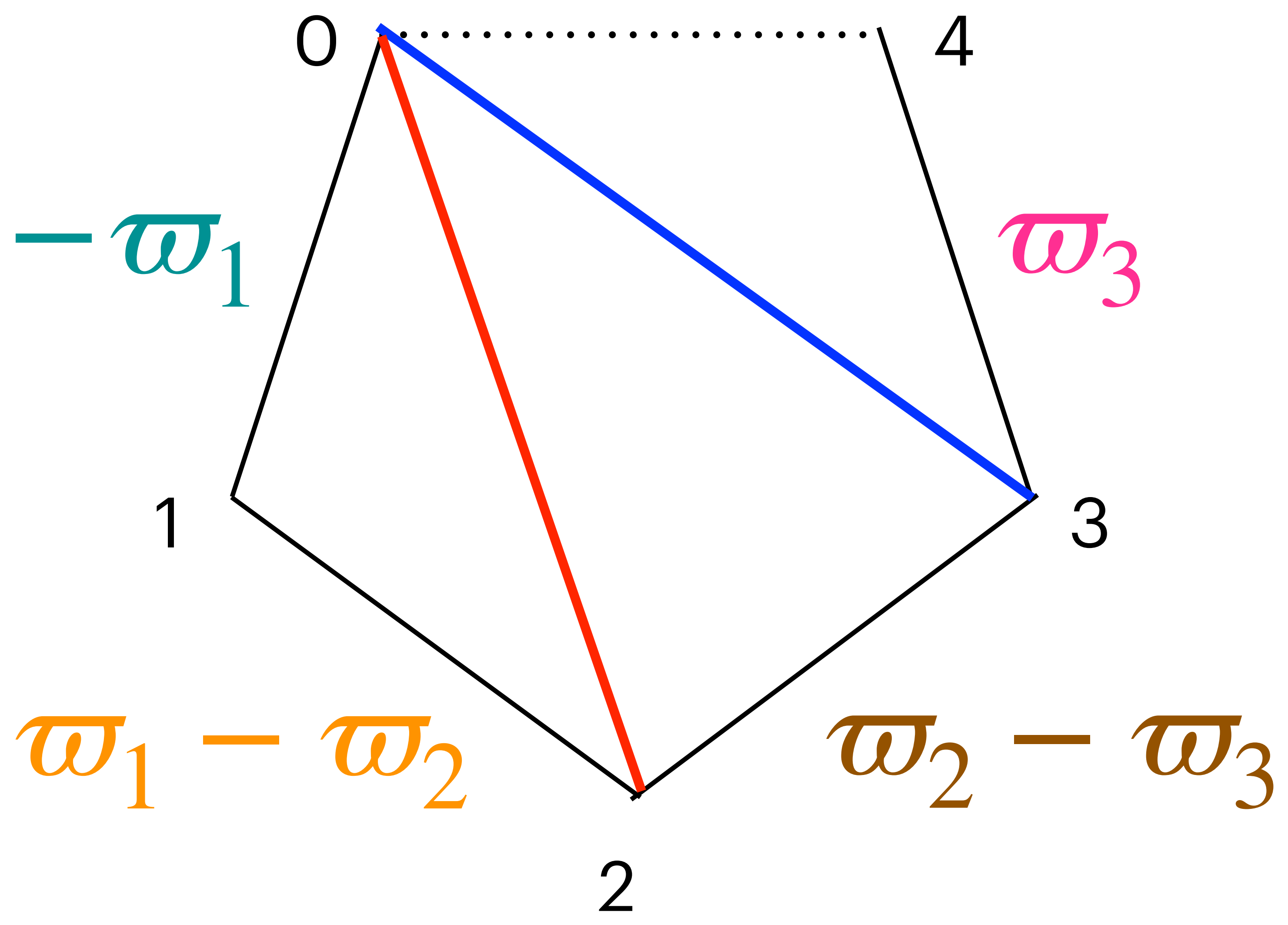
Now we assume $(i_1, j_1) \prec \cdots \prec (i_k, j_k)$.

- We first blow up $\mathbb{C}P^n$ along the subvariety corresponding to the face $F_{i_1,i_1+1} \cap \cdots \cap F_{j_1-1,j_1}$ of Δ. Denote by F_{i_1,j_1} the new facet. Note that $\mathscr{E}_0(i_1j_1) = \{(i_1, i_1+1), \ldots, (j_1-1, j_1)\}$.
- Next, we blow up along the subvariety corresponding to the face $F_{i_2,i_2+1} \cap \cdots \cap F_{j_2-1,j_2}$. Denote by F_{i_2,j_2} the new facet. Note that $\mathscr{E}_0(i_2j_2) = \{(i_2, i_2+1), \ldots, (j_2-1, j_2)\}$.
- Continuing this process until the last diagonal (i_k, j_k), we get a smooth toric variety X_D associated with D.

We call X_D a toric variety of Schröder type. When D is a triangulation, X_D is called a toric variety of Catalan type.

We denote by P_D the polytope obtained from the above process.

Example



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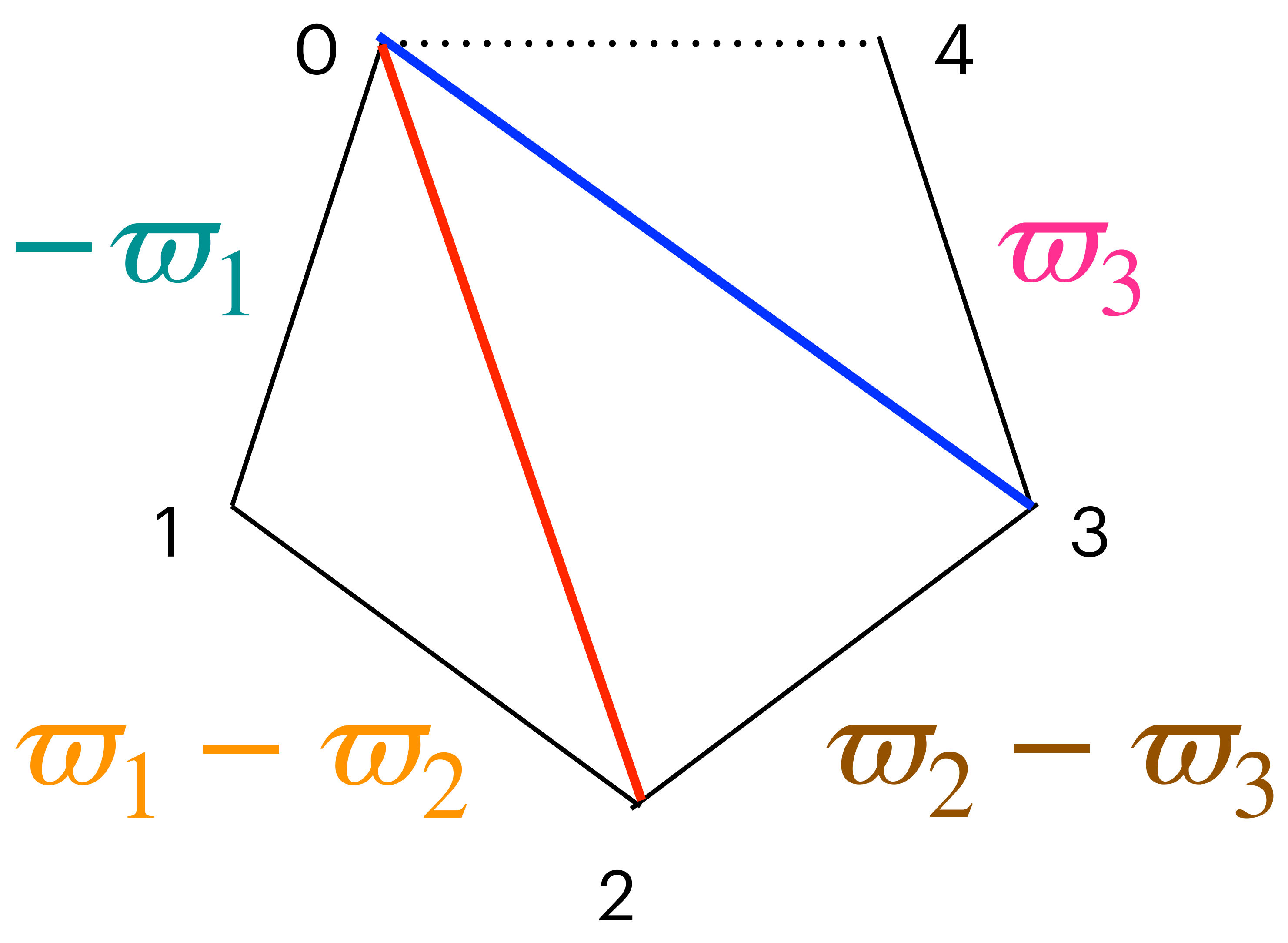
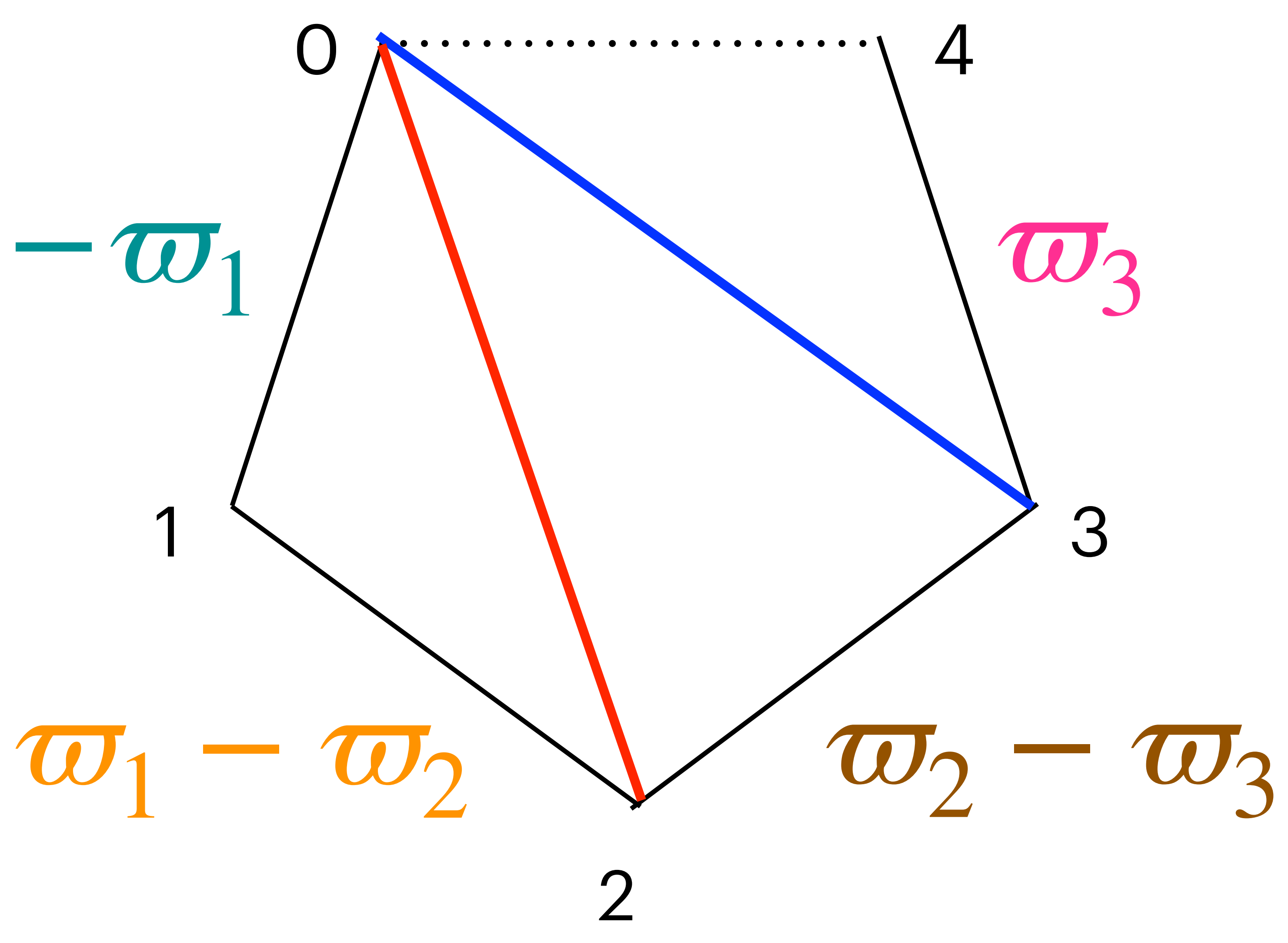


$$(0,3) \prec (0,2)$$

Example



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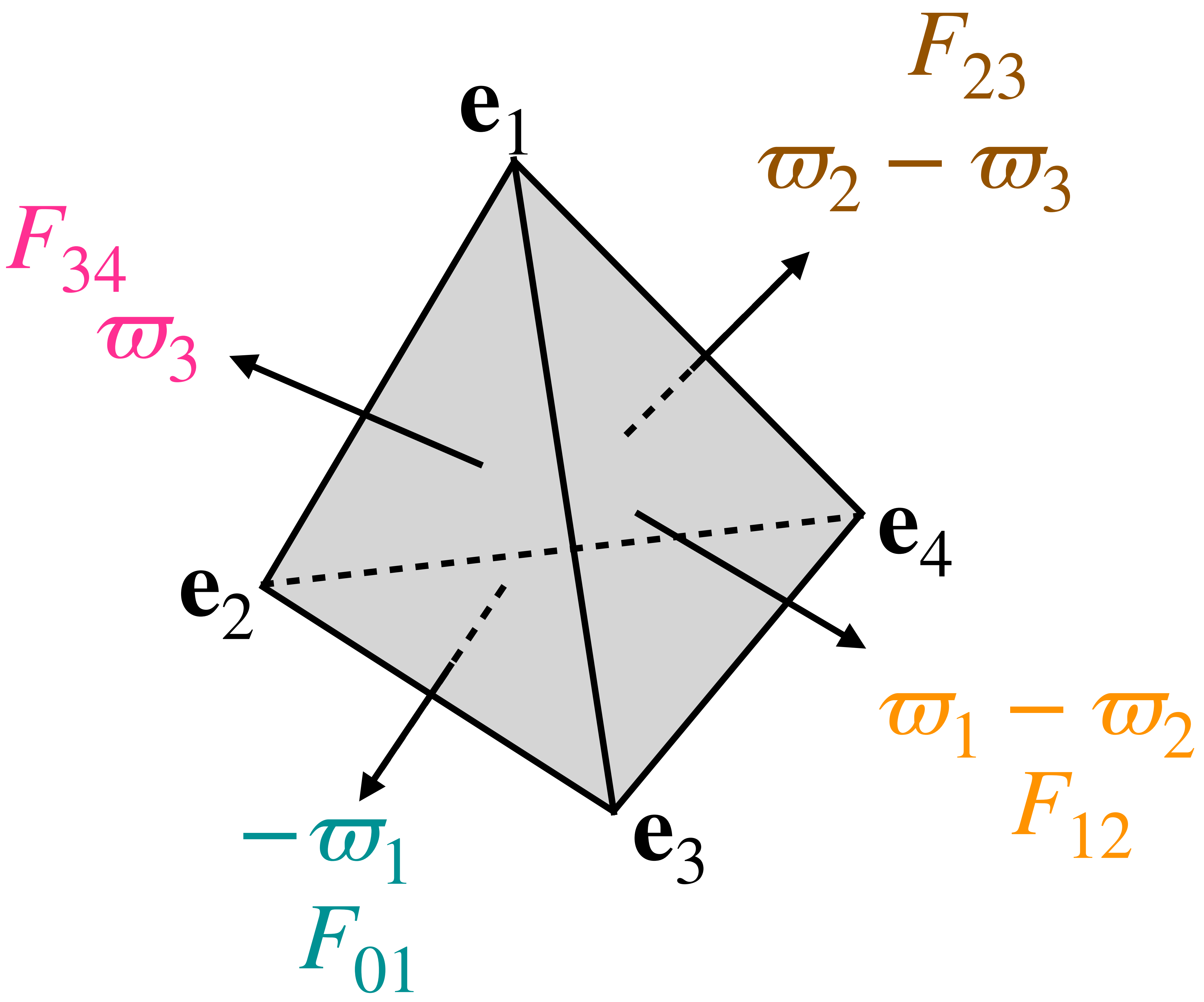

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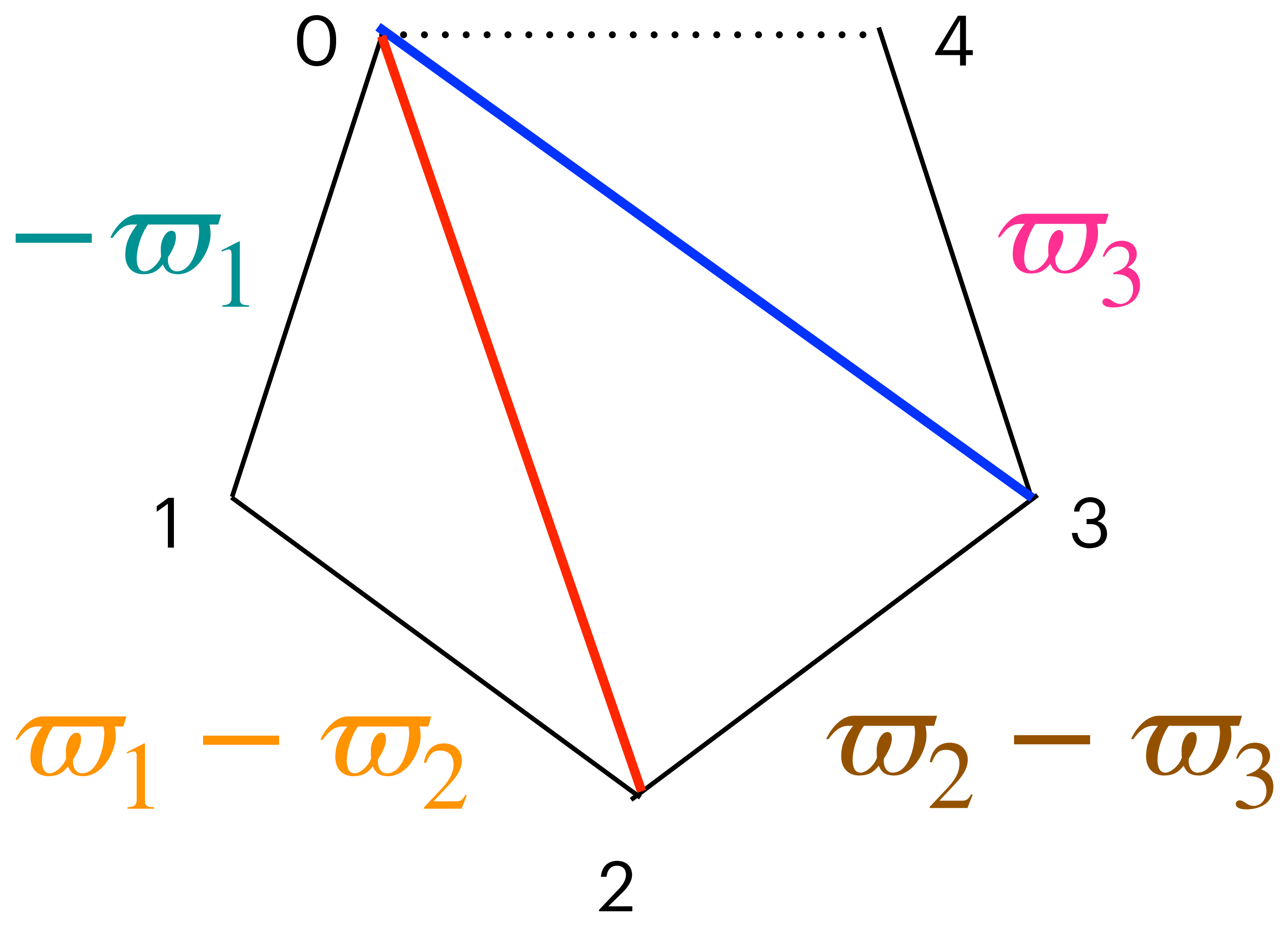

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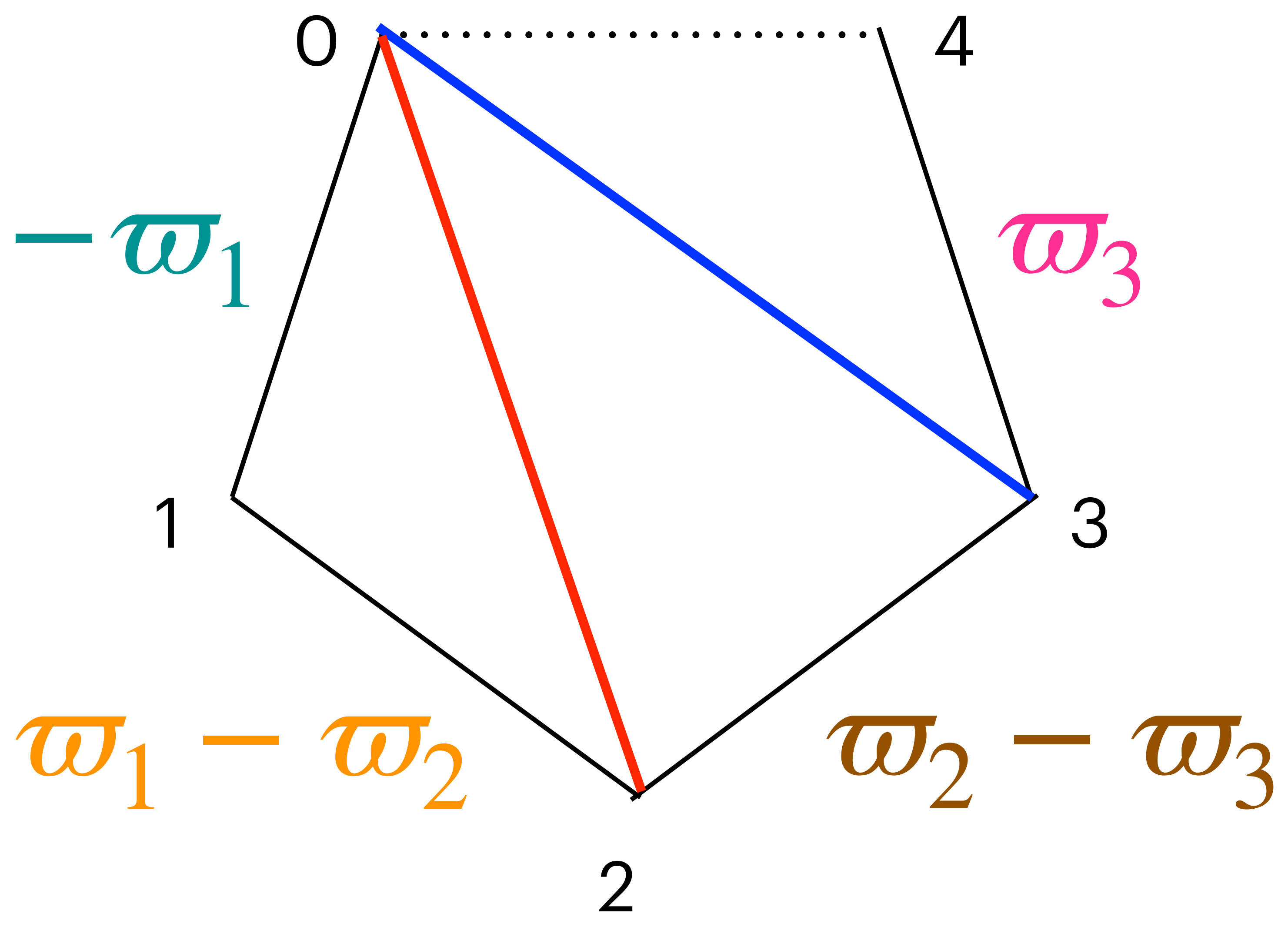

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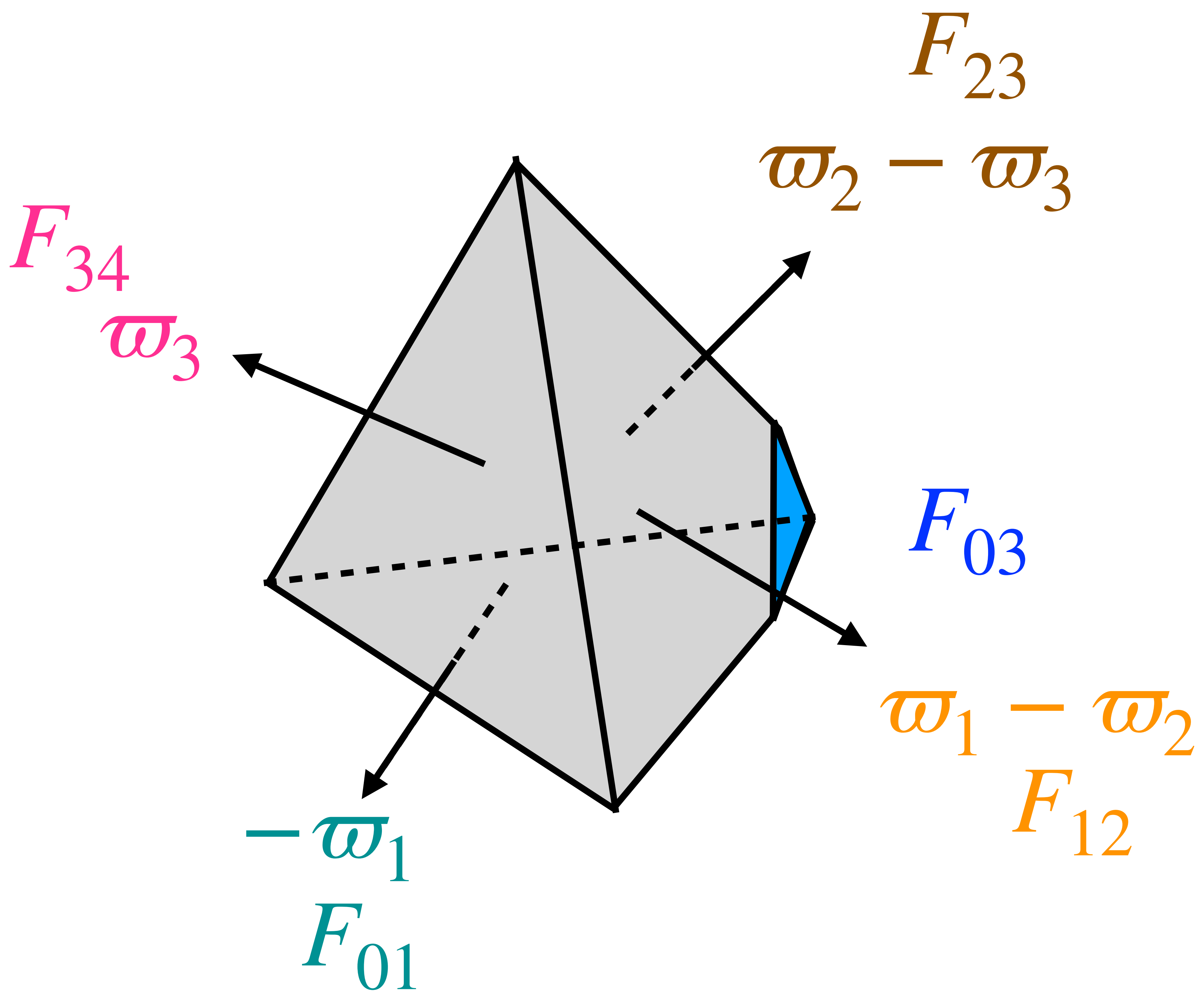

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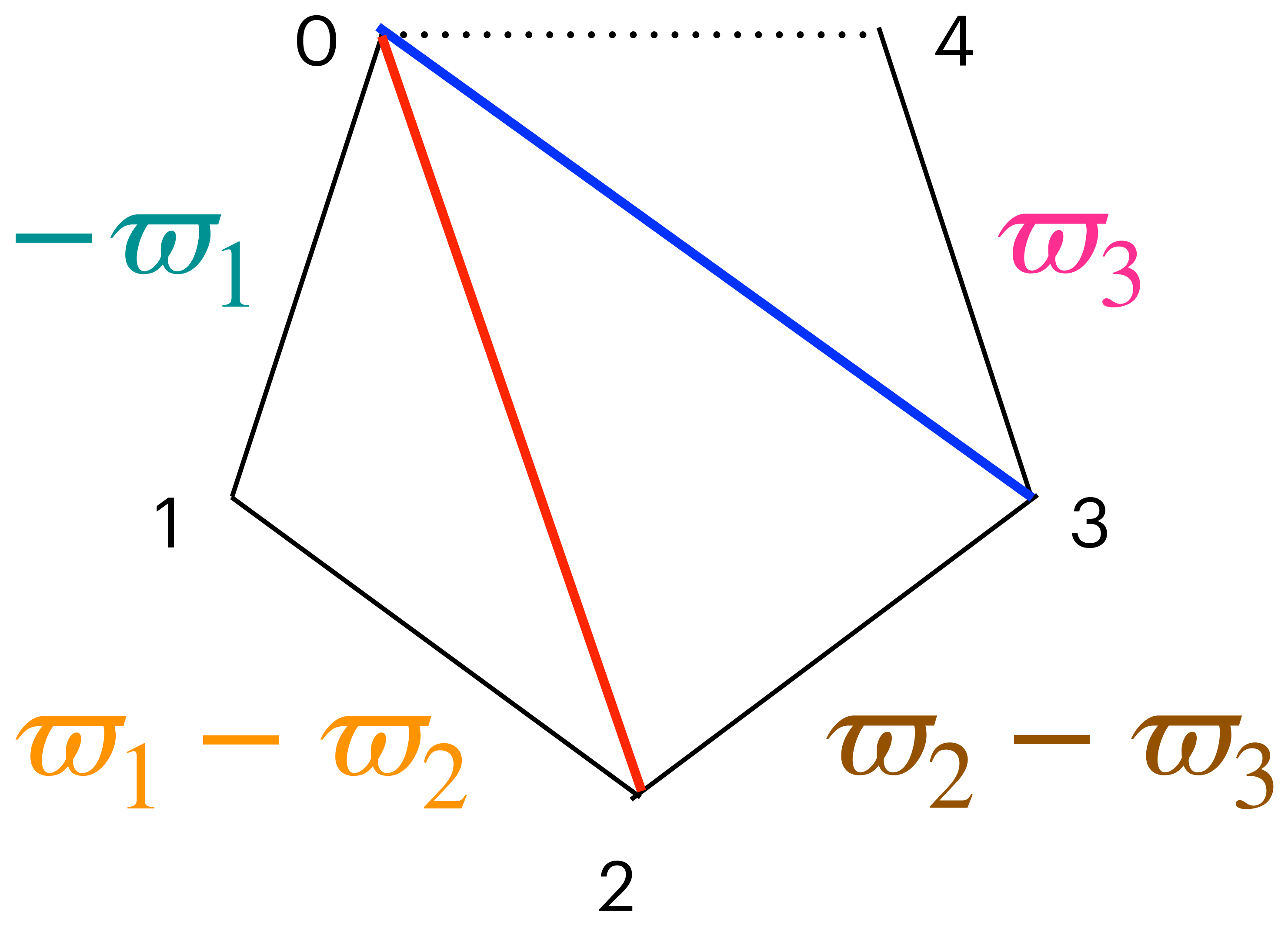

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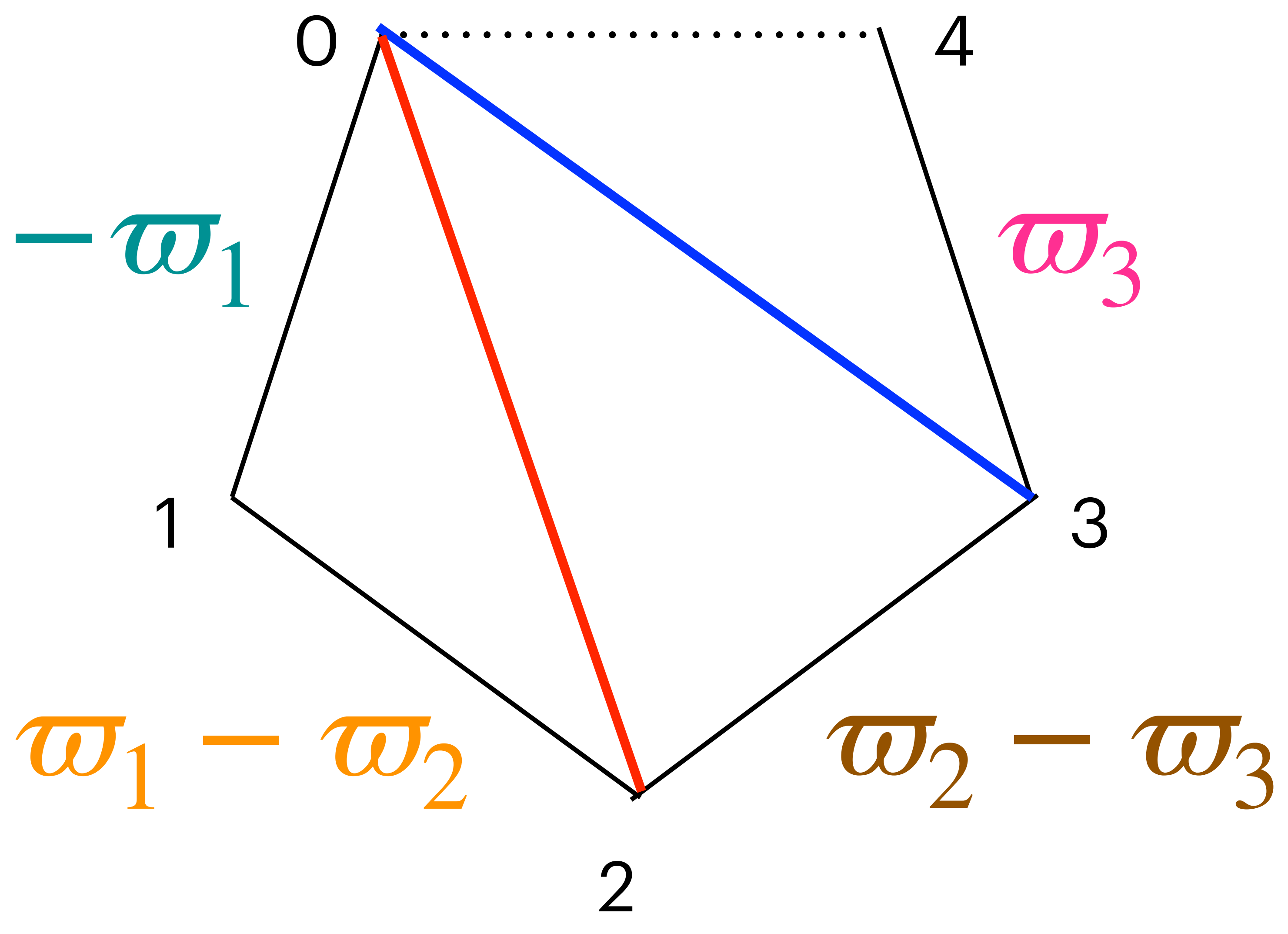

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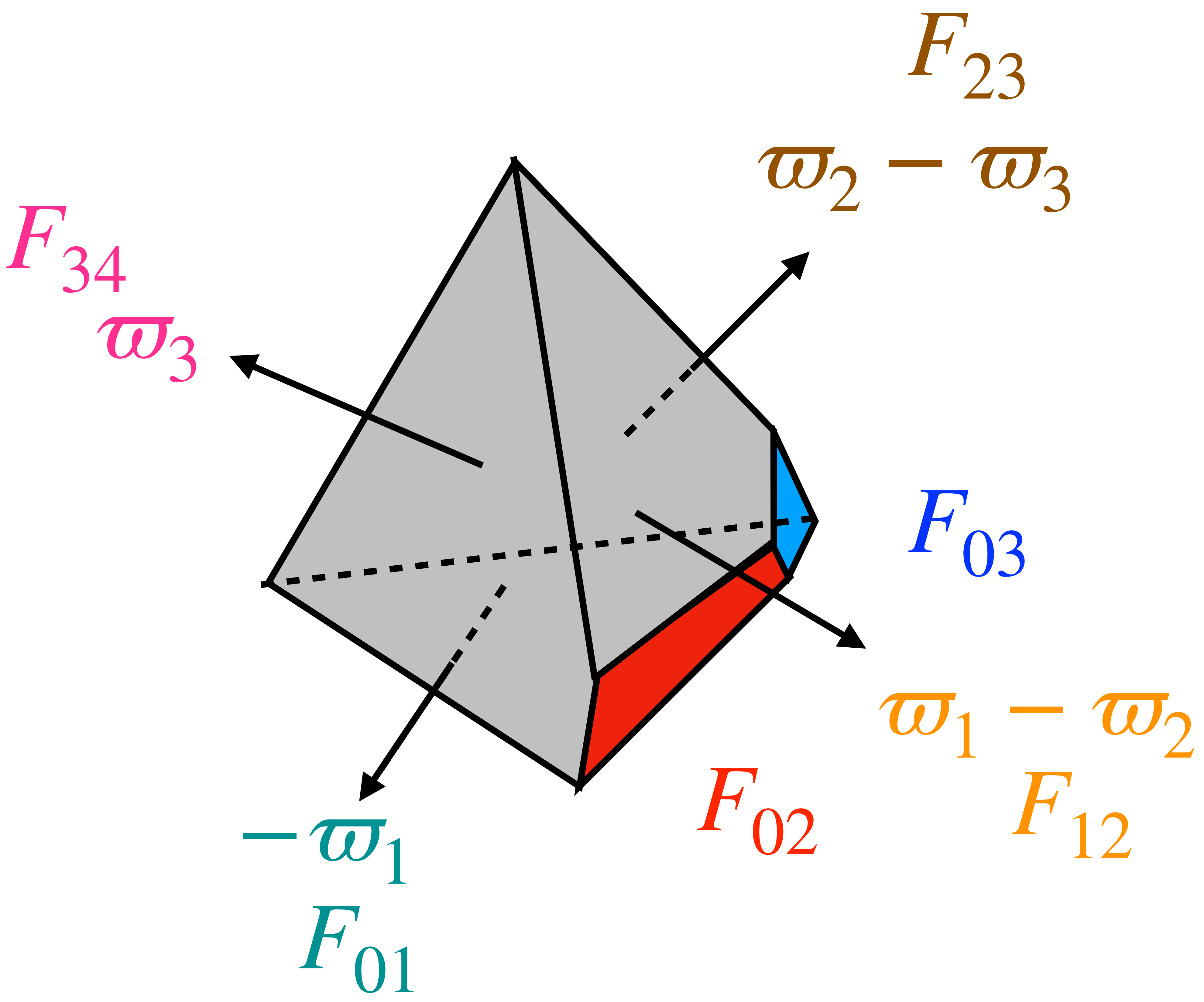

Fano toric variety

A projective smooth variety X is Fano if the anticanonical divisor $-K_X$ is ample.

Example.

1. $\mathbb{C}P^n$ is Fano.

2. $P(\mathbb{C} \oplus \gamma)$ is Fano, where γ is a tautological line bundle over $\mathbb{C}P^n$.

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For a projective fan Σ, a subset R of the primitive ray vectors is called a primitive collection of Σ if

$$\mathrm{Cone}(R) \notin \Sigma \quad \text{but} \quad \mathrm{Cone}(R \backslash \{\mathbf{u}\}) \in \Sigma \quad \text{for every } \mathbf{u} \in R\,.$$

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Note that if Σ_P is the normal fan of a polytope P, then primitive collections of Σ_P correspond to the minimal non-faces of P.

Batyrev's criterion

For a primitive collection $R = \{\mathbf{u}'_1, \ldots, \mathbf{u}'_\ell\}$, we get $\mathbf{u}'_1 + \cdots + \mathbf{u}'_\ell = \mathbf{0}$ or there exists a unique cone σ such that $\mathbf{u}'_1 + \cdots + \mathbf{u}'_\ell$ is in the interior of σ. That is,

$$\mathbf{u}'_1 + \cdots + \mathbf{u}'_\ell = \begin{cases} \mathbf{0}, & \text{or} \\ a_1\mathbf{u}_1 + \cdots + a_r\mathbf{u}_r, & \end{cases}$$

where $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are the primitive generators of σ and $a_1, \ldots, a_r$ are positive integers. The above equation is called a primitive relation, and we define the degree of a primitive collection R as

$$\deg R = \ell - (a_1 + \cdots + a_r).$$

Proposition. (Batyrev 1999)

A projective toric variety X_Σ is Fano when $\deg R > 0$ for every primitive collection R of Σ.

X_D is a Fano generalized Bott manifold

Theorem. (Lee-Masuda-P. 2023, Huh-P. 2022)

The toric variety X_D constructed from a polygon dissection D is a Fano generalized Bott manifold.

(Proof) Let D be a polygon dissection of P_{n+2} with diagonals $(i_1, j_1), \ldots, (i_k, j_k)$.

(1) The polytope P_D corresponding to X_D is combinatorially equivalent to $\prod_{q=0}^{k} \Delta^{|\mathscr{E}_0(\mathsf{P}(i_q,j_q))|-1}$.

(2) The toric variety X_D is Fano.

X_D is a Fano generalized Bott manifold

(Proof of (1))

Let D' be a dissection with diagonals $(i_1, j_1), \ldots, (i_{k-1}, j_{k-1})$. If $P_{D'}$ is combinatorially equivalent to $\prod_{p=0}^{k-1} \Delta^{|\mathscr{E}_0(\mathsf{P}(i_q,j_q))|-1}$, then a proper subset of $\mathscr{E}_0(D')$ corresponds to a face of $P_{D'}$ if and only if it does not contain any of the following sets

$$\mathscr{E}_0(\mathsf{P}(i_0, j_0)), \ldots, \mathscr{E}_0(\mathsf{P}(i_{k-2}, j_{k-2})), \text{ and } \mathscr{E}' = \mathscr{E}_0(\mathsf{P}(i_{k-1}, j_{k-1})) \cup \mathscr{E}_0(\mathsf{P}(i_k, j_k)) - \{(i_k, j_k)\}.$$

Since P_D is obtained from $P_{D'}$ by truncating the face $F_{i_{k-1}, i_{k-1}+1} \cap \cdots \cap F_{j_{k-1}-1, j_{k-1}}$, a subset S of $\mathscr{E}_0(D)$ corresponds to a face of P_D if and only if S does not contain $\mathscr{E}_0(\mathsf{P}(i_q, j_q))$ for all $q = 0, 1, \ldots, k$. Therefore, P_D is combinatorially equivalent to $\prod_{p=0}^{k} \Delta^{|\mathscr{E}_0(\mathsf{P}(i_q,j_q))|-1}$.

X_D is a Fano generalized Bott manifold

(Proof of (2): The toric variety X_D is Fano.)

Recall that the facet vector corresponding to $(i, j) \in \mathscr{E}_0(D)$ is the vector $\varpi_i - \varpi_j$. For simplicity, we denote it by $\mathbf{u}_{ij}$. Set $\mathbf{u}_{i_0,j_0} = \mathbf{0}$.

From (1), the primitive collections of the fan $\Sigma(X_D)$ correspond to the edge sets $\mathscr{E}_0(\mathsf{P}(i_q, j_q))$ for $q = 0,1,\ldots,k$.

Hence the associated primitive relation is

$$\sum_{(i,j)\in\mathscr{E}_0(\mathsf{P}(i_q,j_q))} \mathbf{u}_{ij} = \mathbf{u}_{i_qj_q}.$$

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The primitive relations of X_D recovers the Schröder tree T_D.

Classify up to isomorphism.

Proposition. (Batyrev 1999)

Two smooth Fano toric varieties X_Σ and $X_{\Sigma'}$ are isomorphic as varieties if and only if there is a bijection between the sets of rays of Σ and Σ' inducing a bijection between maximal cones and preserving the primitive relations.

Theorem. (Lee-Masuda-P. 2023, Huh-P., 2022)

The toric varieties X_D and $X_{\widetilde{D}}$ are isomorphic as varieties if and only if the Schröder trees T_D and $T_{\widetilde{D}}$ are isomorphic as unordered rooted trees.

Enumeration

We can enumerate the number of isomorphism classes of toric varieties arising from dissections of P_{n+2} by counting the Schröder trees with n leaves as unordered rooted trees.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	$\cdots$
1		1	1	1	1	1	1	1	1	1	
2			1	2	3	4	5	6	7	8	
3				2	5	10	16	24	33	44	
4					3	12	29	57	99	157	
5						6	28	84	192	382	
6							11	66	231	615	
7								23	157	634	
8									46	373	
9										98	
$\vdots$											
	1	1	2	5	12	33	90	261	766	2312	

Wedderburn-Etherington number

The number of isomorphism classes of n-dimensional toric varieties of Catalan type.

The number of isomorphism classes of n-dimensional toric varieties of Schröder type.

Cohomology ring $H^*(X_D)$

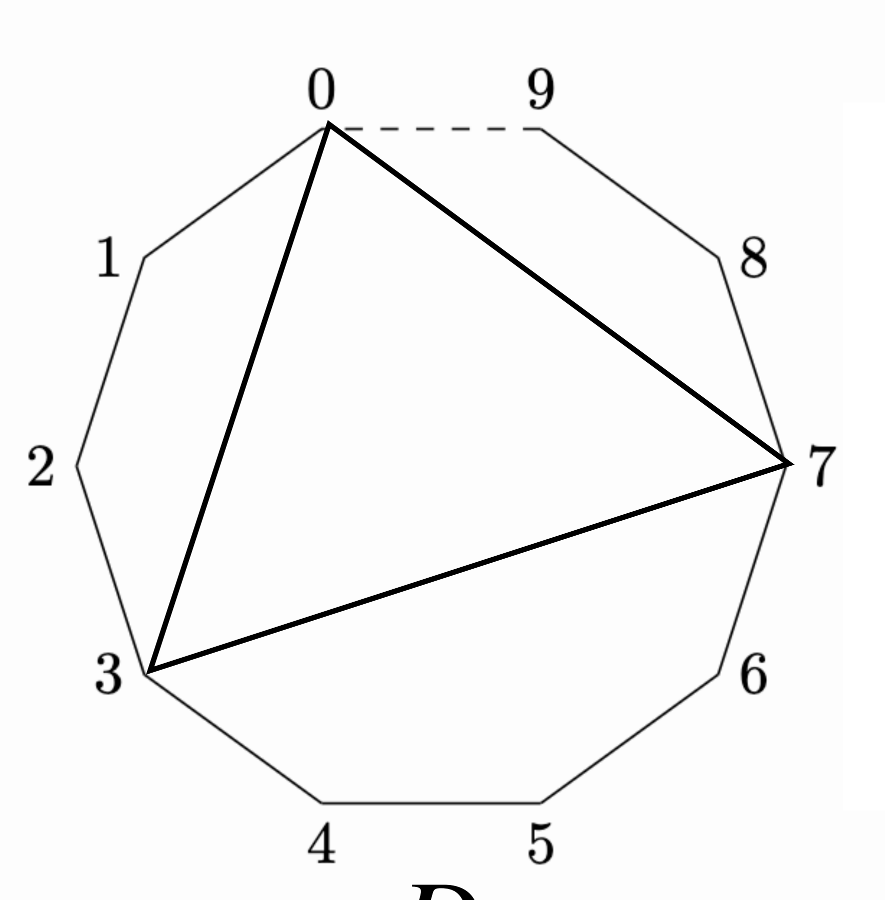
Theorem. (Huh-P., 2022)

Given a k-dissection D of a polygon P_{n+2}, consider the corresponding Schröder tree T_D. For $1 \le i \le k$, let v_i be the ith internal vertex in the preorder listing of T_D. For each i, suppose that v_i has ℓ_i children $w_{i1}, w_{i2}, \dots, w_{i\ell_i}$ from left to right, and $\phi(w_{i\ell_i}) = (a_i, b_i)$. Then the cohomology ring of X_D is

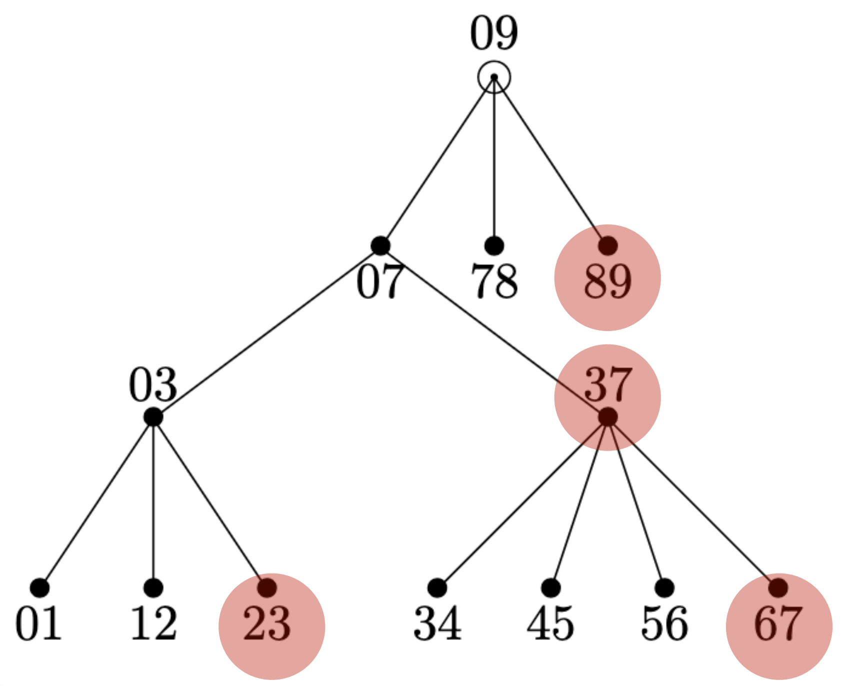
$$H^*(X_D) = \mathbb{Z}[x_{a_1b_1}, x_{a_2b_2}, \dots, x_{a_kb_k}]/\langle p_1, \dots, p_k\rangle,$$

where

$$p_i := x_{a_ib_i} \prod_{j=1}^{\ell_i - 1} \left(- \sum_{u \in S(w_{ij})} x_{\phi(u)} + \sum_{u \in S(v_i)} x_{\phi(u)} \right).$$



D



T_D

The cohomology ring $H^*(X_D)$ is

$$\mathbb{Z}[x_{23}, x_{37}, x_{67}, x_{89}] / \mathcal{I},$$

where

$$\mathcal{I} = \langle x_{23}^3, x_{37}(-x_{23} + x_{37} + x_{67}), x_{67}^4, x_{89}^2(-x_{37} - x_{67} + x_{89})\rangle$$

Cohomological rigidity problem

Theorem. (Huh-P. 2022)

For $k \leq 3$, let D and D' be k-dissection of P_{n+2}. Two toric varieties X_D and $X_{D'}$ are isomorphic as varieties if and only if their integral cohomology rings are isomorphic as graded rings.

Conjecture. (Huh-P. 2022)

Let D and D' be k-dissection of P_{n+2}. Two toric varieties X_D and $X_{D'}$ are isomorphic as varieties if and only if their integral cohomology rings are isomorphic as graded rings.

Torus orbit closures in flag varieties

Flag variety

Flag variety

The flag variety $\mathscr{F}\ell_n$ is the space consisting of all sequences

$$V_\bullet = (\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n),$$

where V_i is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$, $\dim_{\mathbb{C}} V_i = i$, for all $i = 1,\dots,n$.

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If B is the set of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$, then $\mathscr{F}\ell_n = \mathrm{GL}_n(\mathbb{C})/B$.

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Let $\mathfrak{S}_n$ be the set of all permutations on $[n] := \{1,2,\ldots,n\}$. For $w \in \mathfrak{S}_n$, we let $w := \begin{bmatrix} \mathbf{e}_{w(1)} & \mathbf{e}_{w(2)} & \cdots & \mathbf{e}_{w(n)} \end{bmatrix}$.

Then $\mathrm{GL}_n(\mathbb{C}) = \bigsqcup_{w \in \mathfrak{S}_n} \mathrm{B}w\mathrm{B}$ and $\mathscr{F}\ell_n = \bigsqcup_{w \in \mathfrak{S}_n} \mathrm{B}w\mathrm{B}/\mathrm{B}$. (Bruhat decomposition)

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Note that $\mathrm{B}w\mathrm{B}/\mathrm{B} \cong \mathbb{C}^{\ell(w)}$ and $\dim_{\mathbb{C}} \mathscr{F}\ell_n = \ell(w_0) = \dfrac{n(n-1)}{2}$. Here $\ell(w) = \#\{(i,j) \mid i < j \text{ and } w(i) > w(j)\}$.

Torus action on $\mathcal{F}\ell_n$

Let T be the set of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$. Then T acts on $\mathcal{F}\ell_n$ and the T-fixed point set is

$$\{wB = (\{0\} \subsetneq \langle \mathbf{e}_{w(1)} \rangle \subsetneq \langle \mathbf{e}_{w(1)}, \mathbf{e}_{w(2)} \rangle \subsetneq \cdots \subsetneq \langle \mathbf{e}_{w(1)}, \ldots, \mathbf{e}_{w(n)} \rangle) \mid w \in \mathfrak{S}_n\}.$$

Theorem. (Gelfand-Seranova 1987, Lee-Masuda-P. 2021)

There is a moment map $\mu\colon \mathcal{F}\ell_n \to \mathbb{R}^n$ sending $xB \in \mathcal{F}\ell_n$ to

$$-\sum_{j=1}^{n-1} \left\{ \frac{1}{\sum_{\underline{\mathbf{i}} \in I_{j,n}} |p_{\underline{\mathbf{i}}}|^2} \left(\sum_{1 \in \underline{\mathbf{i}} \in I_{j,n}} |p_{\underline{\mathbf{i}}}|^2, \ldots, \sum_{n \in \underline{\mathbf{i}} \in I_{j,n}} |p_{\underline{\mathbf{i}}}|^2 \right) \right\} + (n, n, \ldots, n),$$

where $(p_{\underline{\mathbf{i}}})_{\underline{\mathbf{i}} \in I_{j,n}}$ is the Plücker coordinate of x. In particular, $\mu(wB) = (w^{-1}(1), \ldots, w^{-1}(n))$.

Here we use a different sign convention to that in Tolman's talk, that is, a moment map $\mu\colon (M, \omega, T) \to \mathit{Lie}(T)^*$ satisfies the following: For each $X \in \mathit{Lie}(T)$, $d\mu^X = \iota_{X^\#}\omega$, where $\mu^X(p) = \langle \mu(p), X \rangle$ and $X^\#$ is the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subset T$.

Toric Richardson variety

For each $w \in \mathfrak{S}_n$, we define the Schubert variety $X_w := \overline{BwB/B}$. When $v \leq w$ in Bruhat order, we define the Richardson variety $X_w^v = w_0 X_{w_0 v} \cap X_w$. Then $\dim_{\mathbb{C}} X_w^v = \ell(w) - \ell(v)$.

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It is known that X_w^v is a T-invariant irreducible subvariety of $\mathscr{F}\ell_n$ and

$$(X_w^v)^T = \{z\mathrm{B} \mid v \leq z \leq w\}.$$

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We say that a Richardson variety X_w^v is toric if there is a point $x \in X_w^v$ such that $X_w^v = \overline{T \cdot x}$.

It is known that X_w^v is a T-invariant irreducible subvariety of $\mathscr{F}\ell_n$ and

$$(X_w^v)^T = \{z\mathrm{B} \mid v \leq z \leq w\}.$$

In particular, the moment map image $\mu(X_w^v)$ is the Bruhat interval polytope

$$\mathrm{Q}_{w^{-1}}^{v^{-1}} := \mathrm{ConvHull}\{(z(1), \ldots, z(n)) \mid v^{-1} \leq z \leq w^{-1}\}.$$

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For example, X_{213}^e is toric, but X_{321}^e is not toric because $\dim_{\mathbb{C}} \overline{\mathbb{T} \cdot x} \leq 2$ for any $x \in X_{321}^e$ and $\dim_{\mathbb{C}} X_{321}^e = 3$.

Toric varieties arising from polygon triangulations

Theorem. (Lee-Masuda-P. 2023)
Assume that $v, w \in \mathfrak{S}_n$ satisfy

$$(v = (1, a_2, \ldots, a_n), w = (a_2, \ldots, a_n, 1)) \text{ or } (v = (a_1, \ldots, a_{n-1}, n), w = (n, a_1, \ldots, a_{n-1})).$$

Then the Richardson variety $X_{w^{-1}}^{v^{-1}}$ is a toric variety of Catalan type, and there is a bijective correspondence between the set of isomorphism classes of n-dimensional toric Richardson varieties of Catalan type and the set of unordered full binary trees with $n+1$ leaves.

That is, every toric variety arising from a triangulation of P_{n+2} is a torus orbit closure in $\mathscr{F}\ell_n$.

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Question.

Can we realize a toric variety arising from a polygon dissection as a torus orbit closure in a partial flag variety?

Partial flag variety

The partial flag variety $\mathscr{F}\ell_n^{k_1,\ldots,k_m}$ is the space consisting of all sequences

$$V_\bullet = (\{0\} \subsetneq V_{k_1} \subsetneq V_{k_2} \subsetneq \cdots \subsetneq V_{k_m} = \mathbb{C}^n),$$

where V_{k_i} is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$, $\dim_{\mathbb{C}} V_{k_i} = k_i$, for all $i = 1,\ldots,m$. Then $\mathscr{F}\ell_n^{1,2,\ldots,n} = \mathscr{F}\ell_n$.

There is a natural projection π from $\mathscr{F}\ell_n$ to $\mathscr{F}\ell_n^{k_1,\ldots,k_m}$ which sends $(V_1 \subsetneq \cdots \subsetneq V_n) \mapsto (V_{k_1} \subsetneq \cdots \subsetneq V_{k_m})$.

Theorem. (Gelfand-Serganova 1987)

For $x \in \mathscr{F}\ell_n^{k_1,\ldots,k_m}$, the moment map image of $\overline{T \cdot x}$ is the Minkowski sum of the polytopes $-\sum_{i=1}^{m-1} \Delta_{M_i} + (n,\ldots,n)$,

where Δ_{M_i} is the convex hull of the vectors $\sum_{i \in \underline{\mathbf{i}}} \mathbf{e}_i$ for $\underline{\mathbf{i}} \in I_{k_j,n}$ satisfying $p_{\underline{\mathbf{i}} \neq 0}$.

Note that $L_x = \bigcup_{1 \leq i \leq m-1} \{\underline{\mathbf{i}} \in I_{k_i,n} \mid p_{\underline{\mathbf{i}}}(x) \neq 0\}$ is called the list of x.

Toric varieties arising from polygon dissections

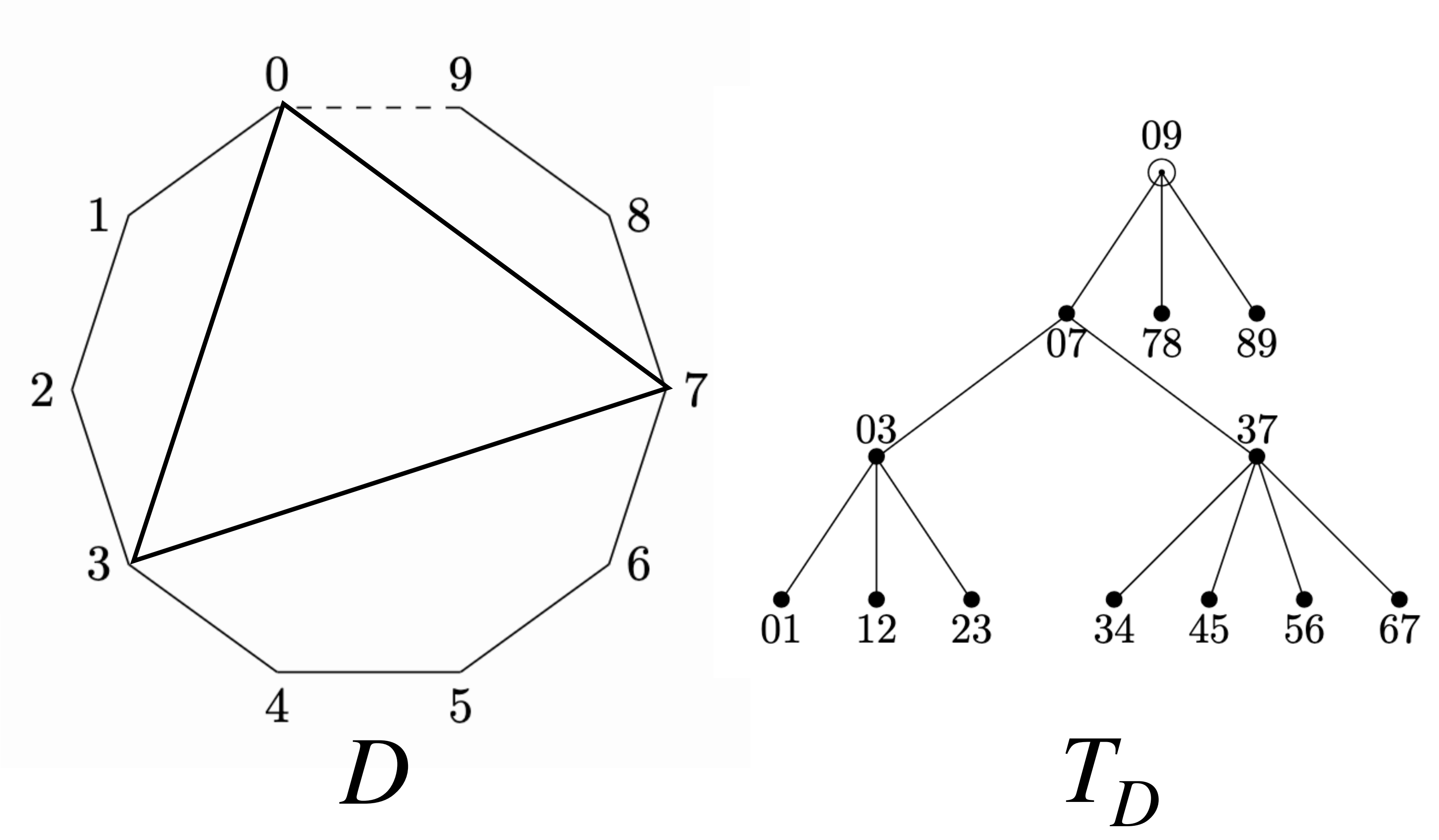
Theorem. (P.)
Let D be a dissection of P_{n+2}. Then the toric variety X_D is a torus orbit closure in $\mathcal{F}\ell_n^{k_1,\ldots,k_m}$, where

$$k_i = \#\{\text{ leaves of depth } \le i-1\} + \#\{\text{ non-leaf vertices of depth } = i-1\}$$

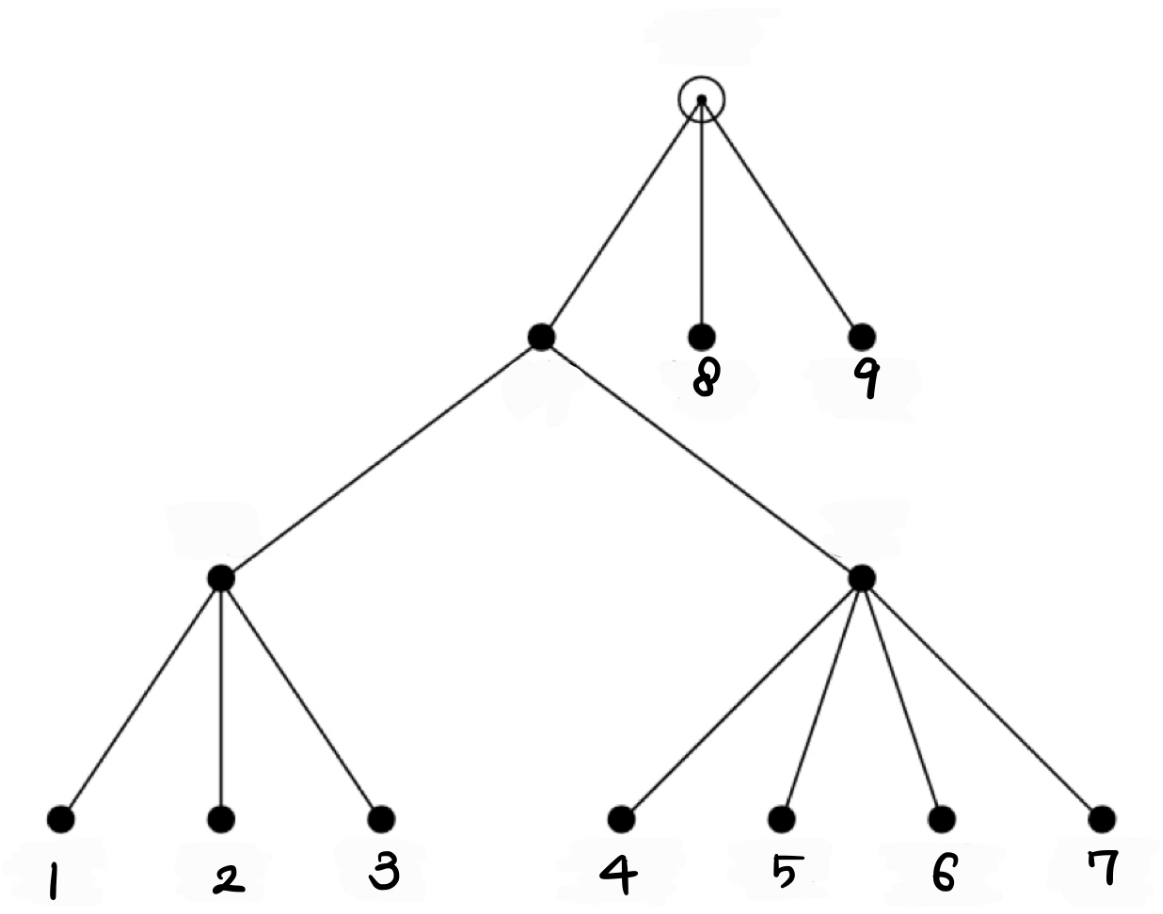
in the Schröder tree T_D. Moreover, it is the image of a toric variety of Catalan type via the natural projection $\pi\colon \mathcal{F}\ell_n \to \mathcal{F}\ell_n^{k_1,\ldots,k_m}$.

There is a point $x \in \mathcal{F}\ell_n^{k_1,\ldots,k_m}$ such that the fan of $\overline{T\cdot x}$ is the same as that of X_D.

$$(k_1, k_2, k_3, k_4) = (1,3,4,9)$$



Toric varieties arising from polygon dissections



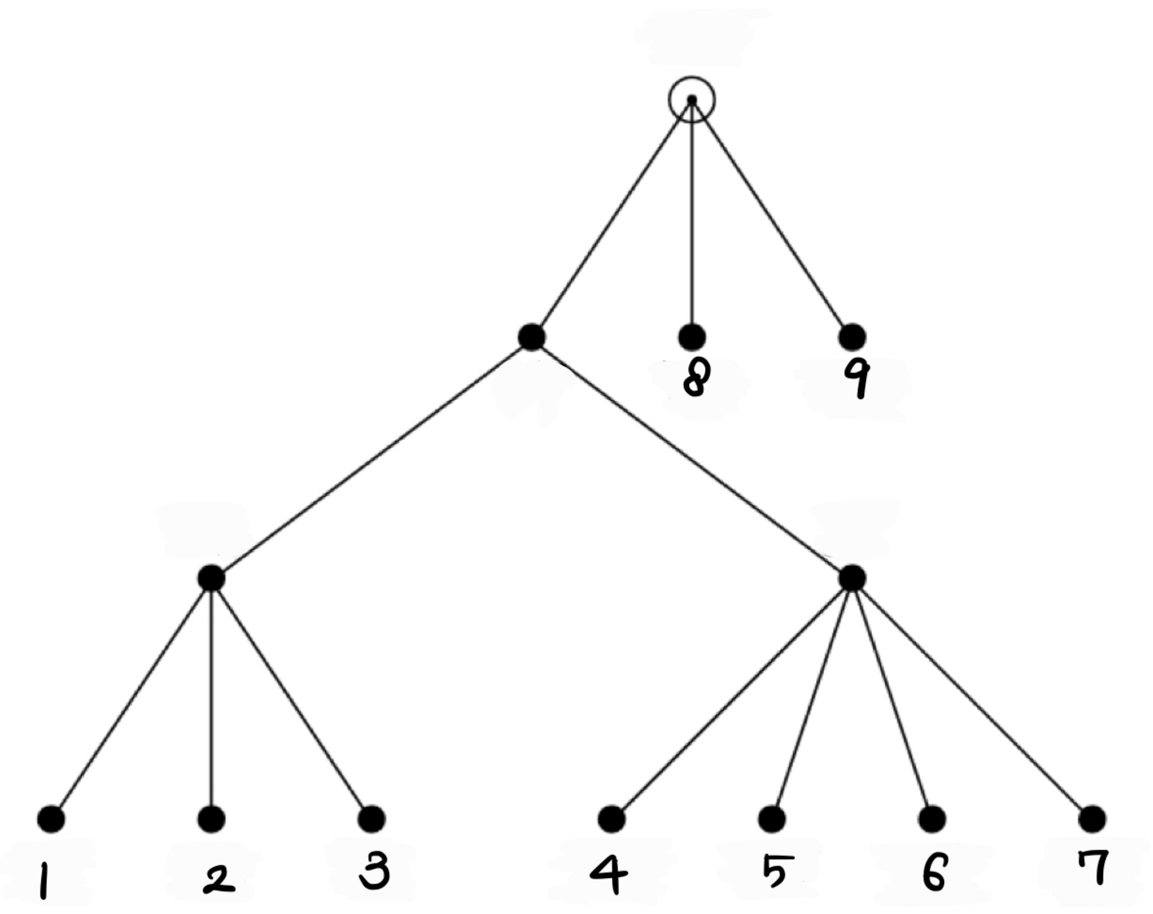
There is a point $x \in \mathscr{F}\ell_9^{1,3,4,9}$ whose list is

$$\{(i) \mid i \in [9]\} \cup \{(i,8,9) \mid i \in [6]\}$$

$$\cup \{(i,j,8,9) \mid i \in \{1,2,3\}, j \in \{4,5,6,7\}\}.$$

canonical triangulation

Toric varieties arising from polygon dissections



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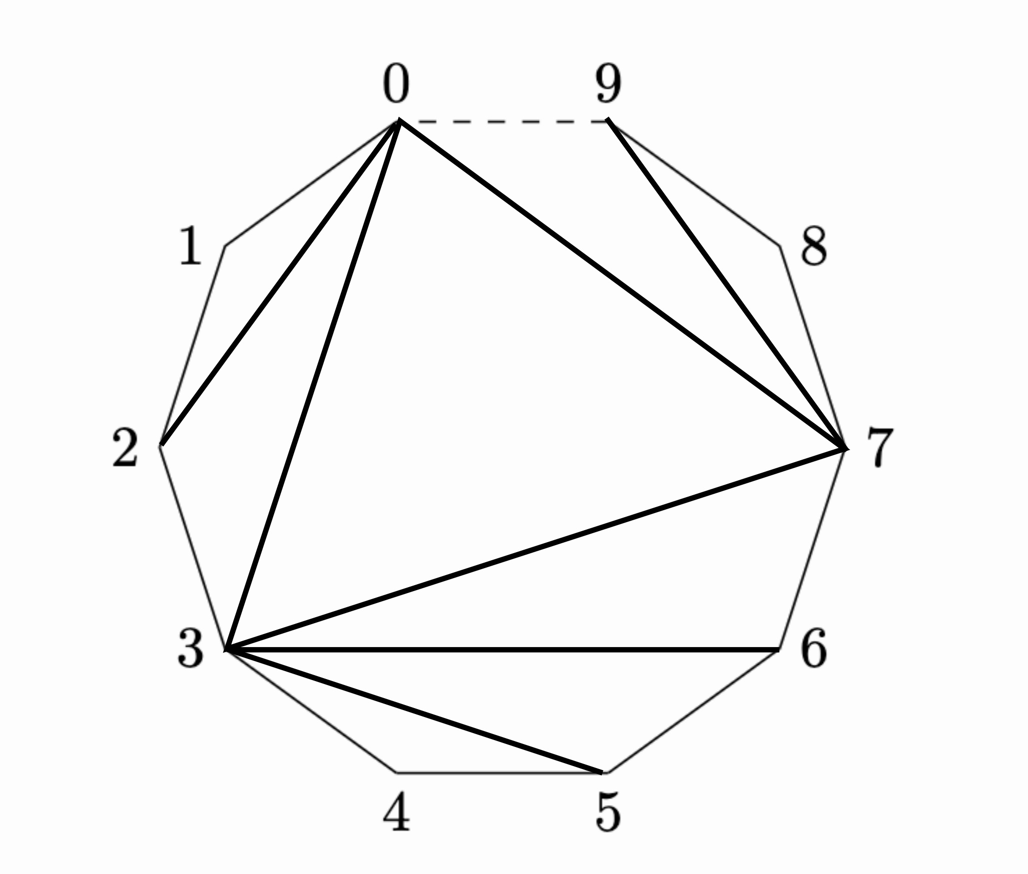
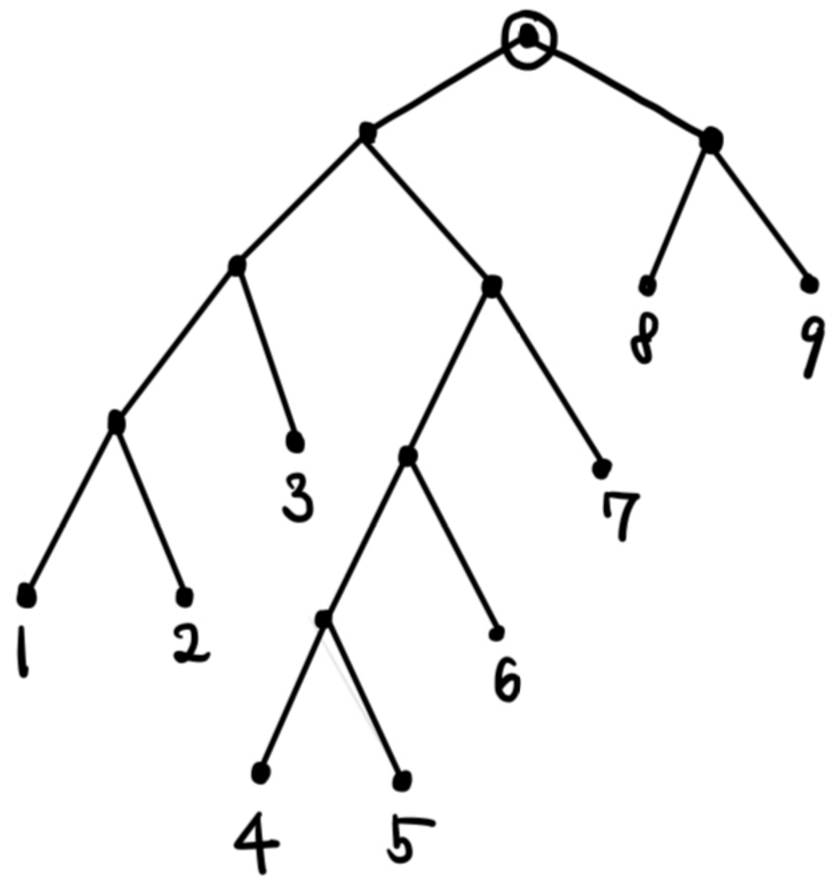


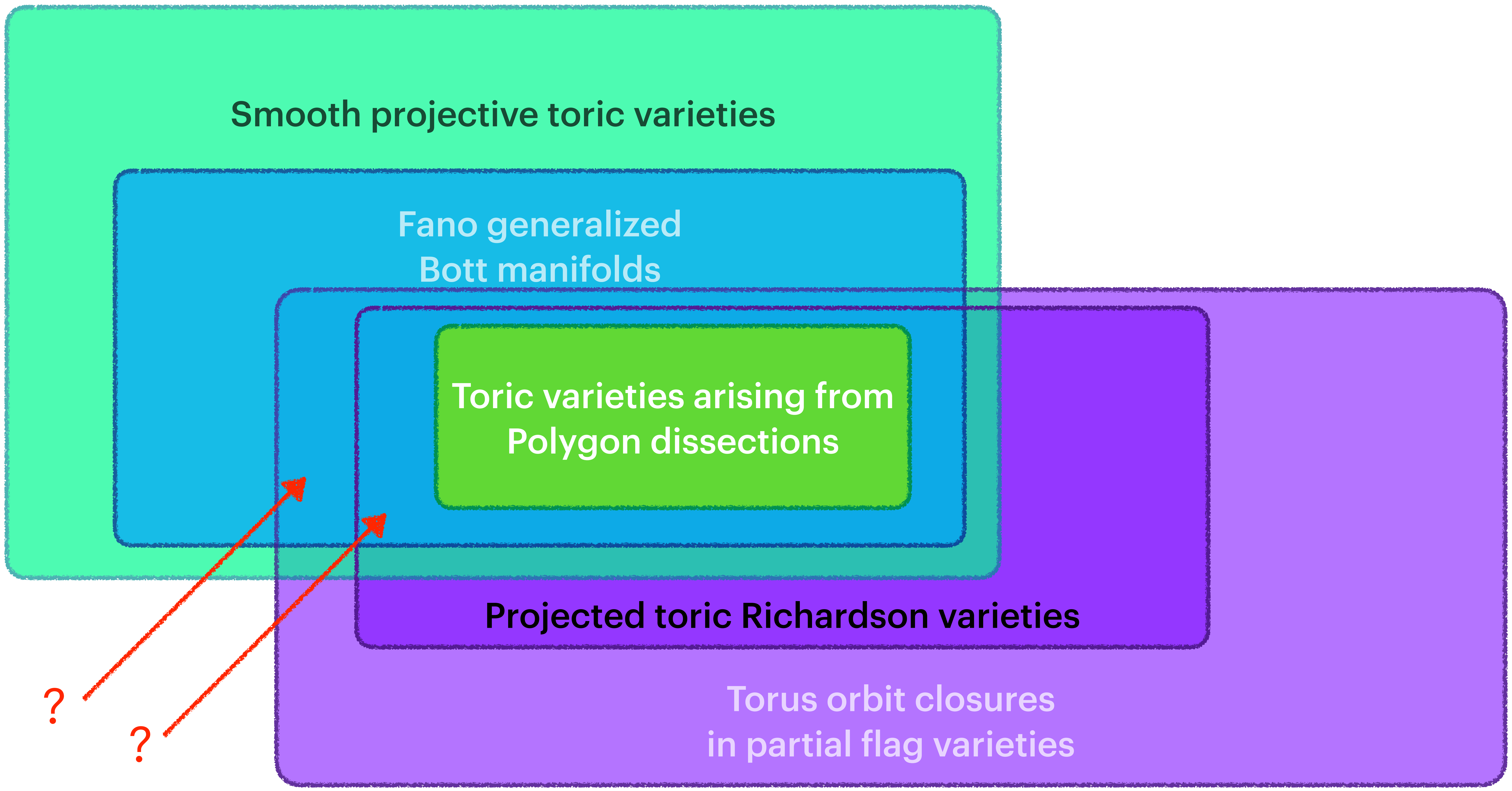
canonical triangulation



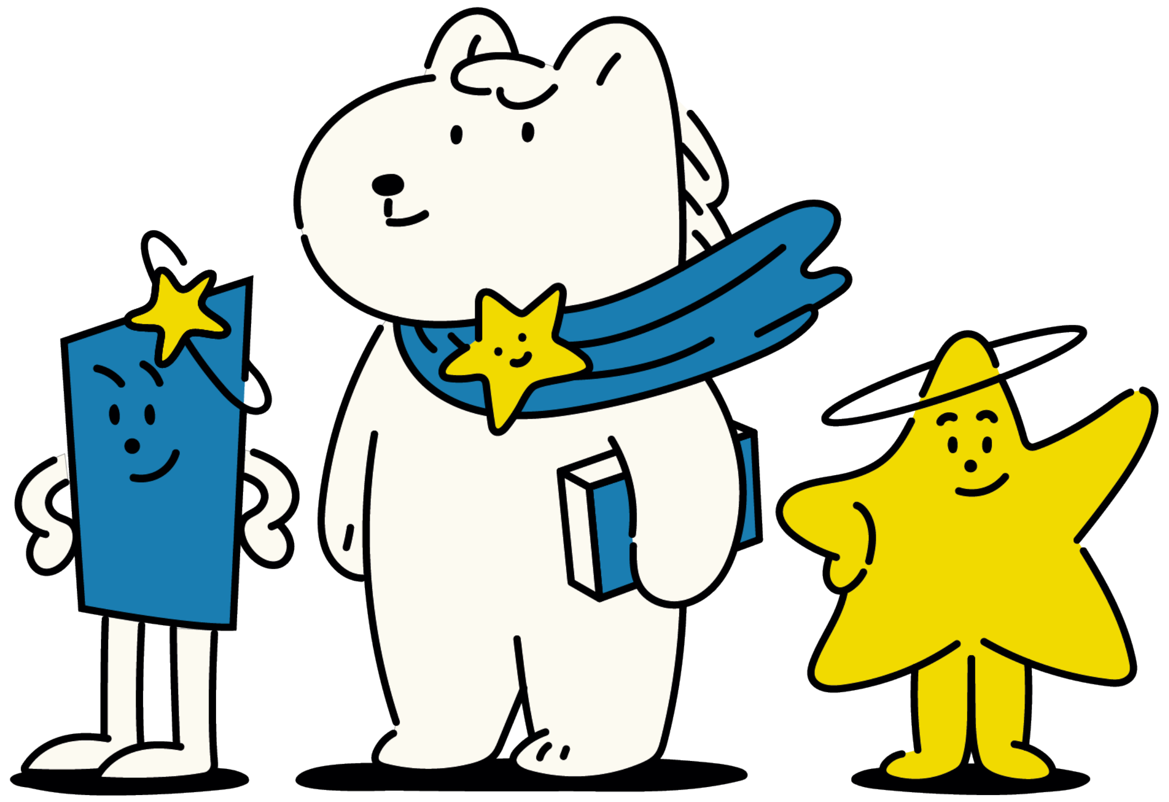
Then X_D is the projection image of the toric Richardson variety $X_{w^{-1}}^{v^{-1}}$ in $\mathscr{F}\ell_9$, where

$$v = 195387624, w = 953876241.$$

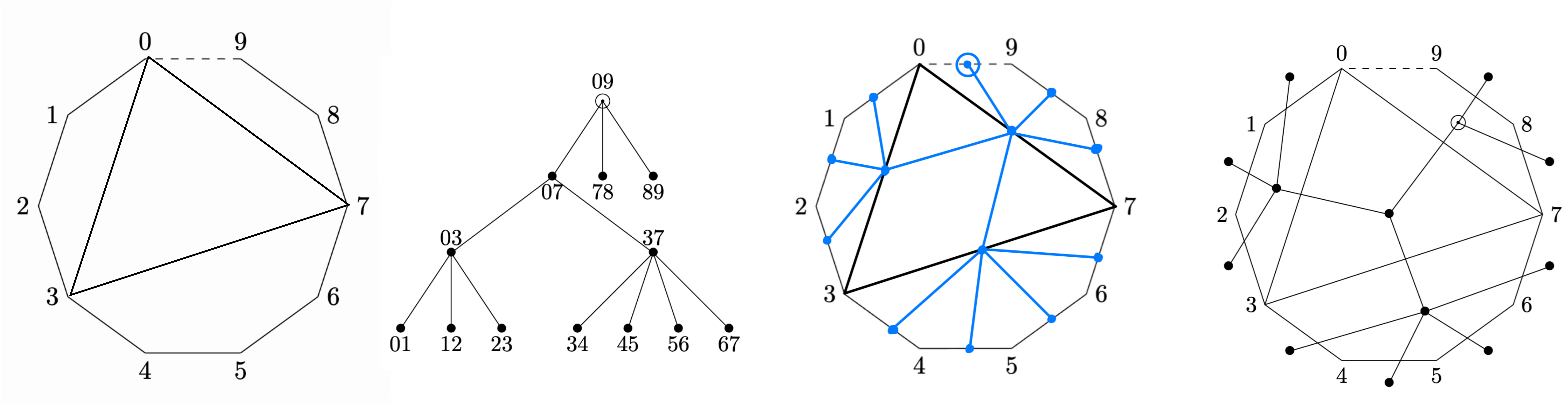
Question


Smooth projective toric varieties
Fano generalized
Bott manifolds
Toric varieties arising from
Polygon dissections
Projected toric Richardson varieties
Torus orbit closures
in partial flag varieties
?
?

Thank you!!



Etherington's bijection



1. The base $(0, n+1)$ corresponds to the root, and the diagonals correspond to the non-leaf vertices, not the root.

2. There is a one-to-one correspondence between the small polygons in a dissection D and the non-leaf vertices of the Schröder tree T_D.

3. Each $\mathscr{E}_0(\mathsf{P}(i_q, j_q))$ corresponds to the set of children of the vertex (i_q, j_q) in T_D.