# On integral cohomology of weighted Grassmann orbifolds

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# Grassmann manifolds

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and $GL(d, \mathbb{C})$ the set of all non singular complex matrix of order $d$.

For two matrix $A, B \in M_d(n, d, \mathbb{C})$,

$$A \sim B \text{ if and only if } A = BT$$

for some $T \in GL(d, \mathbb{C})$.

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The space $\mathrm{Gr}(d,n)$ is a $d(n-d)$-dimensional smooth manifold.

Denote an element $A \in M_d(n, d; \mathbb{C})$ as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nd} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

where $\mathbf{a}_i \in \mathbb{C}^d$ for $i = 1, \ldots, n$.

# Weighted Grassmann orbifolds (Definition)

For $W := (w_1, w_2, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \in \mathbb{Z}_{\geq 1}$, we define an equivalence relation $\sim_w$ on $M_d(n, d; \mathbb{C})$ by

$$\begin{pmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_n} \end{pmatrix} \sim_w \begin{pmatrix} t^{w_1}\mathbf{a_1} \\ t^{w_2}\mathbf{a_2} \\ \vdots \\ t^{w_n}\mathbf{a_n} \end{pmatrix} T$$

for $T \in GL(d, \mathbb{C})$ and $t \in \mathbb{C}^*$ such that $t^a = \det(T)$.

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for $T \in GL(d, \mathbb{C})$ and $t \in \mathbb{C}^*$ such that $t^a = \det(T)$.

We denote the identification space by

$$\mathrm{WGr}(d, n) := \frac{\mathrm{M}_d(n, d; \mathbb{C})}{\sim_w}.$$

The natural $(\mathbb{C}^*)^n$ action on $\mathbb{C}^n$ induces a $(\mathbb{C}^*)^n$ action on $\mathrm{WGr}(d, n)$.

[1] Hiraku Abe and Tomoo Matsumura. *Equivariant cohomology of weighted Grassmannians and weighted Schubert classes*. Int. Math. Res. Not. IMRN 2015, no. 9, 2499–2524.

# Weighted Grassmann orbifolds

The natural $(\mathbb{C}^*)^n$ action on $\mathbb{C}^n$ induces a $(\mathbb{C}^*)^n$ action on $\mathrm{WGr}(d, n)$.

**Lemma (Brahma-S 2024, Abe-Matsumura)**

*The space $WGr(d, n)$ has an orbifold structure where the charts can be chosen $(\mathbb{C}^*)^n$-invariant.*

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### Remark

*Brahma-S definition is equivalent to the definition of weighted 'Grassmannian' by Abe and Matsumura [1].*

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A **Schubert symbol** $\lambda$ for $d < n$ is a sequence of $d$ integers $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ such that $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_d \leq n$.

# Weighted projective spaces

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For $W = (w_1, w_2, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \in \mathbb{Z}_{\geq 1}$, let

$$c_i := a + \sum_{j=1}^{d} w_{\lambda_j^i} \tag{2}$$

where $\{\lambda^i = (\lambda_1^i, \lambda_2^i, \ldots, \lambda_d^i) : i = 0, \ldots, m = \binom{n}{d} - 1\}$ are Schubert symbols.

# Weighted projective spaces

Define $\sim_w$ on $\mathbb{C}^{m+1} - \{0\}$ by

$$(z_0, z_1, \ldots, z_m) \sim_w (t^{c_0} z_0, t^{c_1} z_1, \ldots, t^{c_m} z_m).$$

The quotient space $\frac{\mathbb{C}^{m+1}-\{0\}}{\sim_w}$ is called the weighted projective space with weights $(c_0, c_1, \ldots, c_m)$ and denoted by $\mathbb{W}P(c_0, c_1, \ldots, c_m)$.
The quotient map $\pi_c \colon \mathbb{C}^{m+1} - \{0\} \to \mathbb{W}P(c_0, c_1, \ldots, c_m)$ is given by

$$\pi_c(z_0, z_1, \ldots, z_m) = [z_0 : z_1 : \cdots : z_m]_{\sim_c}. \tag{3}$$

# Weighted embedding

**Lemma**

*There is an induced $(\mathbb{C}^*)^n$-action on $\mathbb{W}P(c_0, c_1, c_2, \ldots, c_m)$ which commutes with the natural $(\mathbb{C}^*)^m$-action on $\mathbb{W}P(c_0, c_1, c_2, \ldots, c_m)$.*

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**Lemma**

*There is an equivariant 'weighted Plücker embedding'*

$$Pl_w \colon \mathrm{WGr}(d, n) \to \mathbb{W}P(c_0, c_1, c_2, \ldots, c_m).$$

# Invariant subsets of weighted projective spaces

## Proposition

*The map orbit map* $\pi_c \colon \mathbb{C}^{m+1} - \{0\} \to \mathbb{W}P(c_0, c_1, \ldots, c_m)$ *defined in* (3) *induces an equivariant homeomorphism* $f \colon \mathbb{W}P(rc_0, rc_1, \ldots, rc_m) \to \mathbb{W}P(c_0, c_1, \ldots, c_m)$ *for any positive integer* $r$.

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### Remark

*Let* $A = \{(z_1, \ldots z_{i-1}, 0, z_{i+1}, \ldots, z_m,) \in \mathbb{C}^{m+1} - \{0\}\}$ *and* $\gcd\{c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m\} = r$. *Then, the set* $\pi_c(A) = \{[(z_1 : \cdots : z_{i-1} : 0 : z_{i+1} : \cdots : z_m)] \subset \mathbb{W}P(c_0, c_1, \ldots, c_m)\}$ *is homeomorphic to* $\mathbb{W}P(\frac{c_1}{r}, \ldots, \frac{c_{i-1}}{r}, \frac{c_{i+1}}{r}, \ldots, \frac{c_m}{r})$.

# Homology of some fine quotient spaces

Let $S^{2k-1} = \{(z_1, ..., z_k) \in \mathbb{C}^k \mid \sum_{i=1}^{k} |z_i|^2 = 1\}$ and $G \subset SO(2n)$ a finite group acting on $S^{2k-1}$ such that $S^{2k-1} \cap \{z_i = 0\}$ is invariant.

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Let $P_k$ be the set of all non-empty subsets of $\{1, ..., k\}$. Let $\mathcal{L}$ be a subset of $P_k$ and $\{1, ..., k\} \supset \{k_1, ..., k_\ell\} = \sigma \in \mathcal{L}$.

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$$\text{Define } U_\sigma := \{(z_1, ..., z_k) \in S^{2k-1} \mid z_{k_i} \neq 0 \text{ for } i = 1, ..., \ell\},$$

$$\text{and } U(\mathcal{L}) := \bigcup_{\sigma \in \mathcal{L}} U_\sigma.$$

Then $U(\mathcal{L})$ is $G$-invariant for any subset $\mathcal{L}$ of $P_k$.

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**Lemma**

*If $p$ and $|G|$ are coprime, then $H_j(U(\mathcal{L})/G; \mathbb{Z})$ has no $p$-torsion.*

**Theorem (S)**

1. *The cohomology groups* $H^*(WGr(d,n),\mathbb{Z})$ *has no torsion.*

[2] Tetsuro Kawasaki. *Cohomology of twisted projective spaces and lens complexes.* Math. Ann., 206:243–248, 1973.

# Homology of weighted Grassmann orbifolds

### Theorem (S)

1. *The cohomology groups $H^*(WGr(d,n),\mathbb{Z})$ has no torsion.*
2. *Moreover, $H^*(WGr(d,n);\mathbb{Z})$ is concentrated in even degrees.*

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**Theorem (S)**

1. *The cohomology groups $H^*(WGr(d,n),\mathbb{Z})$ has no torsion.*
2. *Moreover, $H^*(WGr(d,n);\mathbb{Z})$ is concentrated in even degrees.*

This was proved for $H^*(\mathrm{WGr}(1,n),\mathbb{Z})$ by Kawasaki [2].

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### Definition ([3])

*Let* $\Gamma$ *be an* $\ell$*-valen graph with orientable edges* $E(\Gamma)$ *and n a positive integer. An orbifold GKM graph is defined by a triple* $(\Gamma, \alpha, \theta)$ *such that the following holds.*

[3] Alastair Darby, Shintaro Kuroki and Jongbaek Song. *Equivariant cohomology of torus orbifolds.* Canad. J. Math.74 (2022), no.2, 299–328.

# Orbifold GKM Graphs

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1. *The map* $\alpha\colon E(\Gamma) \to H^2(BT^n; \mathbb{Q})$ *is satisfying:*

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   - 1.2 *if $e \in E(\Gamma)$ is an oriented edge and $\bar{e}$ its reverse orientation then $r_e\alpha(e) = \pm r_{\bar{e}}\alpha(\bar{e}) \in H^2(BT^n; \mathbb{Z})$ for some positive integers $r_e$ and $r_{\bar{e}}$.*

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2. *The collection* $\theta := \{\theta_{pq} \colon E_p(\Gamma) \to E_q(\Gamma) \mid pq \in E(\Gamma)\}$ *is a connection on* $\Gamma$*, and if* $e, e^{'} \in E(\Gamma)$ *with* $s(e) = s(e^{'})$ *there exists* $c_{e,e'} \in \mathbb{Z} - \{0\}$ *such that* $c_{e,e'}(\alpha(\theta_e(e^{'})) - \alpha(e^{'})) = 0$ *mod* $r_e\alpha(e)$*.*

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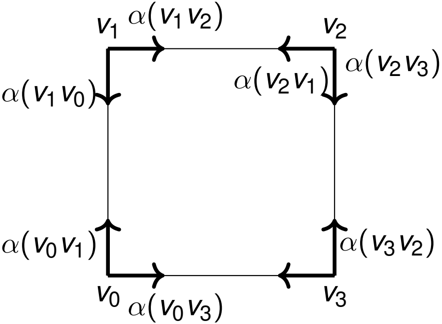
# An orbifold GKM Graph



# An orbifold GKM Graph

The ‘axial function’ $\alpha\colon E(\Gamma) \to H^2(BT^2; \mathbb{Q})$ is defined by

$$\alpha(v_0 v_1) = y_1 - \frac{k_1}{k_0} y_2, \quad \alpha(v_1 v_0) = y_2 - \frac{k_0}{k_1} y_1,$$

$$\alpha(v_1 v_2) = y_1 - \frac{k_2}{k_1} y_2, \quad \alpha(v_2 v_1) = y_2 - \frac{k_1}{k_2} y_1,$$

$$\alpha(v_2 v_3) = y_1 - \frac{k_3}{k_2} y_2, \quad \alpha(v_3 v_2) = y_2 - \frac{k_2}{k_3} y_1,$$

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for some non-zero integers $k_0, \ldots, k_3$ with $k_0^2 \neq k_1 k_3$,

**Definition**

Let $(\Gamma, \alpha, \theta)$ be an orbifold GKM graph. Then the following has a ring structure

$$\{f\colon V(\Gamma) \to H^*(BT^n;\mathbb{Z}) \mid \widetilde{r}_e\alpha(e) \text{ devides } (f(s(e)) - f(t(e)))\},$$

where $\widetilde{r}_e$ is the smallest positive integer satisfying the condition 1.1.2 in the definition of the orbifold GKM graph.

# Equivariant cohomology of GKM graph

**Definition**

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where $\widetilde{r}_e$ is the smallest positive integer satisfying the condition 1.1.2 in the definition of the orbifold GKM graph.

This ring is called the equivariant cohomology ring of $(\Gamma, \alpha, \theta)$, and denoted by denoted by $H^*_{T^n}(\Gamma, \alpha, \theta)$.

**Definition ([4])**

*A $T^n$-orbifold $X$ is said to be a GKM orbifold if the following holds.*

1. $X^{T^n}$ *is finite and discrete.*
2. $X_1 := \{x \in X \mid \dim T^n x \leq 1\}$ *is a finite union of spindles* ($\mathbb{W}P(p,q)$).
3. *If* $\alpha_1, \ldots, \alpha_n$ *are the weight vectors of the irreducible* $T^n$*-representations at* $p \in X^{T^n}$ *for* $T_pX$ *then they are pairwise linearly independent.*

[4] Fernando Galaz-Garcia, Martin Kerin, Marco Radeschi, and Michael Wiemeler. *Torus orbifolds, slice-maximal torus actions, and rational ellipticity*, Int. Math. Res. Not. IMRN (2018), no. 18, 5786–5822.

# GKM property of weighted Grassmann orbifolds

**Proposition**

*Each Grassmann orbifold is a GKM orbifold.*

**Theorem (S)**

*Let $WGr(d, n)$ be a weighted Grassmann orbifold. Then, the equivariant cohomology ring $H^*_{T^n}(X; \mathbb{Z})$ is isomorphic to $H^*_{T^n}(\Gamma, \alpha, \theta)$ as $H^*_{T^n}(BT^n; \mathbb{Z})$-algebra.*

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**ThankYou**