Classification of locally standard torus actions

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Joint work with Yael Karshon

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August 22nd 2024
Workshop on Toric Topology
Fields Institute (Toronto)

Locally standard torus action

Assumptions

- $T(= T^d) \simeq \mathbb{T}^d = (S^1)^d$: a d-dim torus.
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- $H := T_x \subset T$: subtorus (i.e., connected);
- $H \curvearrowright T_x M \simeq V(\alpha_1) \oplus \cdots \oplus V(\alpha_k) \oplus \mathbb{R}^{\dim M - 2k}$, where $\alpha_1, \ldots, \alpha_k$ are basis of $\mathfrak{h}^*_{\mathbb{Z}} \simeq \mathbb{Z}^k$ for $k = \dim H \leq d$.

Note: $k = 0 \Leftrightarrow Tx \simeq T$, a free orbit.

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∃ locally standard T-chart

for $\forall x \in M$, $\exists$ open subsets $U_M \subset M$ and $\Omega \subset \mathbb{C}^n \times \mathbb{T}^l \times \mathbb{R}^m$ s.t.

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\begin{tikzcd}
U_M \arrow[r, "\simeq", "diffeo"'] & \Omega \\
T \arrow[u, bend left] \arrow[r, "\rho_{\alpha_1,\ldots,\alpha_d}", "isom"'] & \mathbb{T}^d \arrow[u, bend left]
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where $\rho_{\alpha_1,\ldots,\alpha_d} : t \mapsto (t^{\alpha_1}, \ldots, t^{\alpha_d})$ for basis $\alpha_1, \ldots, \alpha_d$ of $\mathfrak{t}_{\mathbb{Z}}^*$.

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Note: This is equivalent to the definition in [Wiemeler].

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GOAL of this talk

Classify locally standard T-mfd's up to equivariant diffeomorphism.

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Q \supset U_Q \arrow[r, "\phi", "\simeq"'] & \mathcal{O} \arrow[r, hookrightarrow] & \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m
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- each depth k point ($k = \#$of 0 coordinates $\in \mathbb{R}^n_{\geq 0}$ of $\phi(p)$) is in the intersection of exactly k facets, $1 \leq k \leq n$;

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- each depth k point ($k = \#$of 0 coordinates $\in \mathbb{R}^n_{\geq 0}$ of $\phi(p)$) is in the intersection of exactly k facets, $1 \leq k \leq n$;
- using [Schwarz],

$$\theta(z_1, \ldots, z_n; b_1, \ldots, b_l; y_1, \ldots, y_m) = (|z_1|^2, \ldots, |z_n|^2; y_1, \ldots, y_m).$$

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2. $\hat{\lambda} : \mathring{\partial}^1 Q \to \hat{\mathfrak{t}}_{\mathbb{Z}} := (\mathfrak{t}_{\mathbb{Z}})_{primitive}/\{\pm 1\}$: a unimodular labeling, where $\mathring{\partial}^1 Q$ is the set of depth 1 points.

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\begin{tikzcd}
T \arrow[rr, "\rho_{\alpha_1,\ldots,\alpha_d}"] \arrow[d, phantom, "\circlearrowright"] & & \mathbb{T}^d \arrow[d, phantom, "\circlearrowright"] \\
M_{d-1} \cap U_M \arrow[rr, hook, "\widetilde{\phi}"] \arrow[d, two heads, "\pi"'] & & \mathbb{C}^n \times \mathbb{T}^l \times \mathbb{R}^m \arrow[d, two heads, "\theta"'] \\
\mathring{\partial}^1 Q \cap U_Q \arrow[rr, hook, "\phi"] & & \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m \\
\Psi & & \Psi \\
x \arrow[r, mapsto, "\phi"] & (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n; y_1, \ldots, y_m) &
\end{tikzcd}
$$

$\Rightarrow \hat{\lambda}(x) = \pm \eta_j \in \hat{\mathfrak{t}}_{\mathbb{Z}}$

where $\eta_1, \ldots, \eta_d \in \mathfrak{t}_{\mathbb{Z}}$ are the dual basis of $\alpha_1, \ldots, \alpha_d \in \mathfrak{t}^*_{\mathbb{Z}}$, and M_{d-1} is the set of $(d-1)$-dim. orbits.

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\begin{aligned}
&0 \longrightarrow \mathfrak{t}_{\mathbb{Z}} \simeq \mathbb{Z}^d \longrightarrow \mathcal{R}^d \longrightarrow \mathcal{T}^d \longrightarrow 0 \\
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where $\mathcal{T}^d$ (resp. $\mathcal{R}^d$) is the sheaf of germs of smooth maps of $\mathring{Q}$ into T^d (resp. $\mathbb{R}^d$).

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Since $Q \simeq \mathring{Q}$ (homotopy equiv),

$$\begin{array}{ccc} H^1(\mathring{Q}; \mathcal{T}^d) & \xrightarrow{\simeq} & H^2(Q; \mathfrak{t}_{\mathbb{Z}}) \\ \Cup & & \Cup \\ [M_{free} \to \mathring{Q}] & \longmapsto & c \end{array}$$

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Main result (rough statement)

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- [Wiemeler] (2022): if $\exists$ section $M/T \to M$, i.e., if $c = 0$.

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$\overset{iff}{\Leftrightarrow}$

$\exists t : Q_1 \xrightarrow{C^\infty} T(\curvearrowright M_2)$ *s.t.*

$$\widehat{\psi}(x) = t(\pi_1(x)) \cdot \widetilde{\psi}(x).$$

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Let $\mathfrak{P}$ be the category consisting of

- objects: principal T-bdl P over $(Q, \hat{\lambda})$, denoted by $\Pi : P \to Q$;
- morphisms: eq. diffeo's $f : P_1 \to P_2$ s.t.

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Lemma

The quotient functor $\Pi : \mathfrak{P} \to \mathfrak{Q}$ is essentially surj. and full.

Cutting functor Cut : $\mathfrak{P} \to \mathfrak{M}$

For $(P, Q, \hat{\lambda}) \in \mathfrak{P}$, the cut space

$$\mathrm{Cut}(P, Q, \hat{\lambda}) = P/\sim$$

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where $T_{(x)} \subset T$ (subtorus) is generated by $\eta_{i_1}, \ldots, \eta_{i_k} \in \hat{\mathfrak{t}}_{\mathbb{Z}}$ s.t.

$$\begin{array}{ccc}
Q & & \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m \\
\cup & & \cup \\
U & \xrightarrow{\phi} & \mathcal{O} \\
\Psi & & \Psi \\
x & \longmapsto & (x_1, \ldots, \underbrace{0}_{i_1}, \ldots, \underbrace{0}_{i_2}, \ldots, \underbrace{0}_{i_k}, \ldots, x_n; y_1, \ldots, y_m)
\end{array}$$

Note: This is determined by $\hat{\lambda}$.

Lemma (Cutting functor)

Cut : $\mathfrak{P} \to \mathfrak{M}$ *is a functor.*

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Step 1 Define topological atlas on $P/\sim$ by identifying the cut space $\Pi^{-1}(U)/\sim$, for $U \subset Q$ open with $U \cong \mathcal{O} \subset \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m$, with $\Omega \subset \mathbb{C}^n \times \mathbb{T}^l \times \mathbb{R}^m$, where $\Pi : P \to Q$;

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- Step 2 To prove this atlas is smooth, we generalize the argument of [Bredon].

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Outline of a proof for ess. surj.

To prove it, we construct the toric radial-squared blowup functor

$$\mathrm{Bl} : \mathfrak{M} \to \mathfrak{P} \quad s.t. \quad \mathrm{Cut} \circ \mathrm{Bl} \simeq 1_{\mathfrak{M}} \ \textit{and} \ \ \mathrm{Bl} \circ \mathrm{Cut} \simeq 1_{\mathfrak{P}},$$

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by a simultaneous toric radial-squared blowup along codim-1 inv. submfd's. Thus, $\forall (T \curvearrowright) M \in \mathfrak{M}$, $\exists\, \mathrm{Bl}(M) = (P, Q_M, \hat{\lambda}_M) \in \mathfrak{P}$ s.t. $\mathrm{Cut}(P, Q_M, \hat{\lambda}_M) \simeq M \in \mathfrak{M}$. $\square$

Functor chasing

Fix categories $\mathfrak{P}, \mathfrak{M}, \mathfrak{Q}$, and functors $\mathrm{Cut} : \mathfrak{P} \to \mathfrak{M}$, $\mathrm{Cl} : \mathfrak{M} \to \mathfrak{Q}$ and $\Pi : \mathfrak{P} \to \mathfrak{Q}$.

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- *If* $\Pi : \mathfrak{P} \to \mathfrak{Q}$ *is full and* $\mathrm{Cut} : \mathfrak{P} \to \mathfrak{M}$ *is ess. surj., then* $\mathrm{Cl} : \mathfrak{M} \to \mathfrak{Q}$ *is full.*

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This shows

Main result

$(Q, \hat{\lambda}, c)$ classifies locally standard $T \curvearrowright M$ up to equivariant diffeomorphism.

Examples over the interval

Consider $M/T^d = [-1, 1]$ (closed interval).

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- $\hat{\lambda}(1) = -w\eta_1 + k\eta_2 \Rightarrow M \simeq L(k; w) \times T^{d-2}$,

where $w, k > 0$ with $gcd(w, k) = 1$, and $L(k; w)$ is the lens space $S^3/\mathbb{Z}_k$ defined by

$$[z_1 : z_2] = [\xi^w z_1, \xi z_2] \in S^3/\mathbb{Z}_k$$

for $\xi \in \mathbb{Z}_k$.

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Thank you!