Splines, equivariant cohomology, and applied mathematics

Julianna Tymoczko
Smith College

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SRiM @ Slaufer Math, AWM MERP

Outline of this talk:

1. (Torus)-Equivariant cohomology via GKM theory
2. The framework of splines - algebraic and classical
3. Questions motivated by applied math

① (Torus-) Equivariant Cohomology via GKM

<u>Set-up</u>: a suitable geometric object with a well-behaved torus action

<u>Examples</u>: $T \circlearrowright X$ where X is

- a symplectic manifold with a Hamiltonian T-action
- a smooth, complete, complex toric variety
- a partial flag variety G/P with maximal torus

How much of that structure is necessary?*

T acts on X with

- isolated fixed points
- the boundary of each 1-orbit consists of two distinct fixed points
- an additional condition called $\boxed{\text{equivariant formality}}$

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* "necessary": people get this to work without

The inclusion $\imath : X^T \hookrightarrow X$ induces a map

$$\imath^* : H_T^*(X) \longrightarrow H_T^*(X^T)$$

often called the localization map.

For some $T \circlearrowright X$ this map is $\boxed{\text{injective}}$

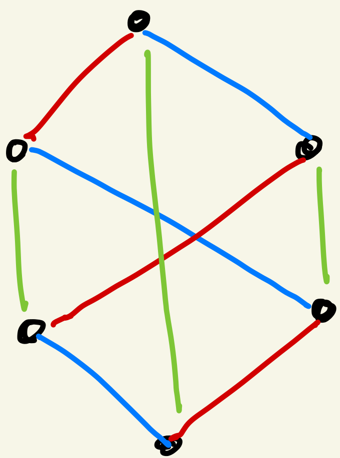
For even more special $T \circlearrowright X$ we can $\boxed{\text{identify the image of } H_T^*(X)}$

GKM guarantees the image $\boxed{\text{comes from a particular graph}}$

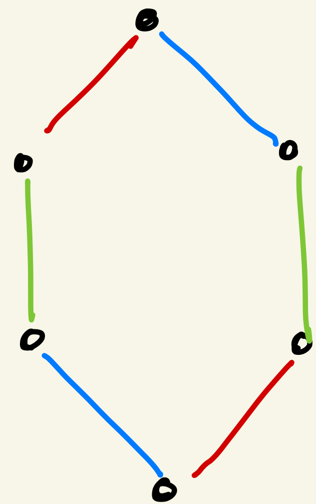
The GKM graph (or moment graph)

- Vertices are T-fixed points
- Edges are 1-dim T-orbits

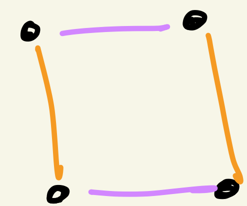
Flag variety



Hess variety



$\mathbb{P}^1 \times \mathbb{P}^1$



The GKM graph

- Edges are labeled with weight of T-orbit

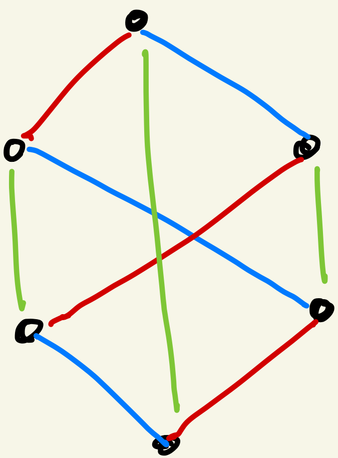
— $= t_1 - t_2$

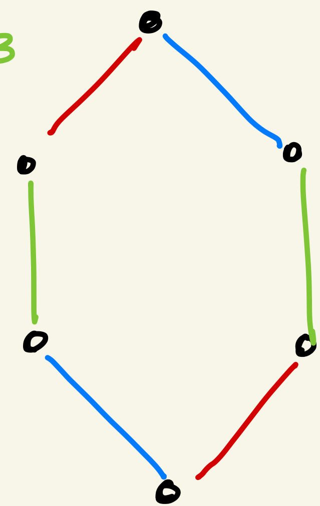
— $= t_2 - t_3$

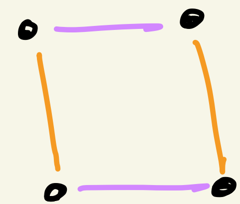
— $= t_1 - t_3$

— $= \alpha$

— $= \beta$







For a single point, we have

$$H_T^*(pt) = \mathbb{C}[t_1, \ldots, t_n]$$

can be changed

The image of the map

$$\iota^* : H_T^*(X) \hookrightarrow H_T^*(X^T)$$

$|V|$ in the graph

sits inside $\mathbb{C}[t_1, \ldots, t_n]^{|X^T|}$

The GKM condition

Consider the elements $\vec{p} \in \mathbb{C}[t_1, \ldots, t_n]^{|X^T|}$ satisfying for each edge uv

$$\boxed{\text{$p(u) - p(v)$ is a multiple of the label $\ell(uv)$ on edge uv}}$$

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$$\boxed{\begin{array}{l}p(u) - p(v) \text{ is a multiple of} \\ \text{the label } \ell(uv) \text{ on edge } uv\end{array}}$$

GKM Theorem:

These $\{\vec{p}\}$ form the image of $H_T^*(X) \hookrightarrow H_T^*(X^T)$

- This identifies $H_T^*(X)$ with a subring and submodule of $H_T^*(pt)^{|X^T|} = \mathbb{C}[t_1, \dots, t_n]^{|X^T|}$ using pointwise mult, add, scaling

- $\boxed{\text{Equivariant formality:}}$ $H_T^*(X)$ is a <u>free</u> module over $H_T^*(pt) = \mathbb{C}[t_1, \dots, t_n]$

② The framework of splines

Given any ring R (commutative, with identity)

and any graph $G = (V, E)$

whose edges are labeled with ideals in R

$$\ell : E \to \{\text{ideals in } R\}$$

the GKM condition defines a subring and R-submodule of $R^{|V|}$

$\boxed{\text{Defn:}}$ The splines on the edge-labeled graph (G, ℓ) with coefficients in R are

$$Spl(G, \ell; R) = \left\{ \text{elements } \vec{p} \in R^{|V|} \text{ with } \boxed{\begin{array}{l} p(u) - p(v) \text{ is a multiple of} \\ \text{the label } \ell(uv) \text{ on edge } uv \end{array}} \text{ for all edges } uv \right\}$$

GKM condition

Example:

Ⓐ If G is the GKM graph,

$\ell(uv)$ is the principal ideal

generated by the T-weight, and

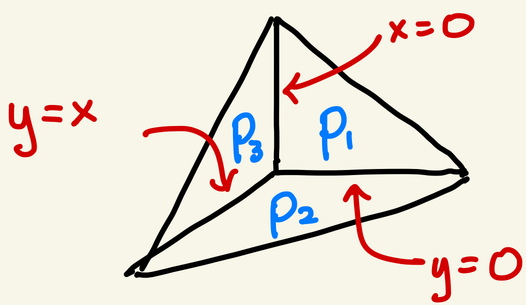
$R = \mathbb{C}[t_1, \dots, t_n] = H_T^*(pt)$ then

$Spl(G, \ell; R)$ is the image

$\iota^*(H_T^*(X))$ inside $H_T^*(X^T) \cong R^{|V|}$

Example:

Ⓑ $\boxed{\text{Classical splines:}}$ piecewise polynomials on a polyhedral decomposition of a space, satisfying a k-differentiability constraint.



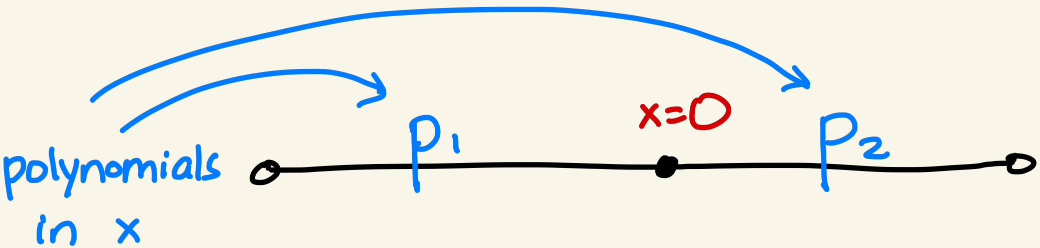
$$p_1\big|_{y=0} = p_2\big|_{y=0}$$

$$p_1'\big|_{y=0} = p_2'\big|_{y=0}$$

$$\vdots$$

$$p_1^{(k)}\big|_{y=0} = p_2^{(k)}\big|_{y=0}$$

k-differentiability is the ideal:



x-axis partitioned by point $x=0$

$$p_1\big|_{x=0} = p_2\big|_{x=0} \iff (p_1 - p_2)\big|_{x=0} = 0$$

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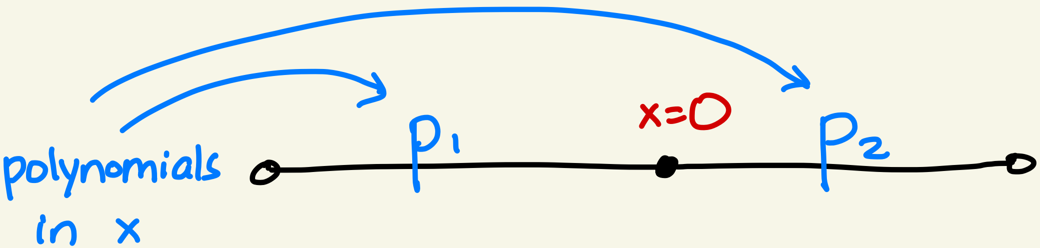

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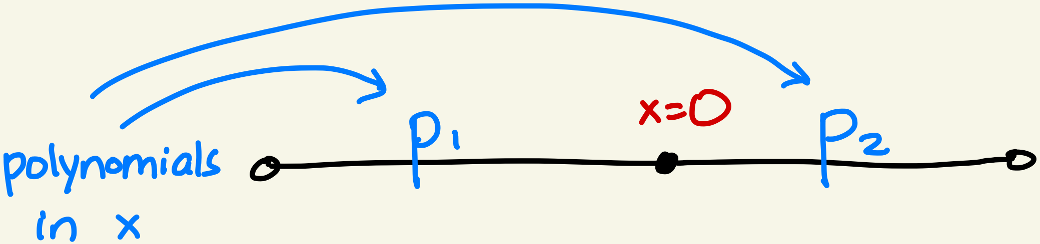

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$$p_1\big|_{x=0} = p_2\big|_{x=0} \iff (p_1 - p_2)\big|_{x=0} = 0 \iff (p_1 - p_2)(0) = 0 \iff x \mid (p_1 - p_2)$$

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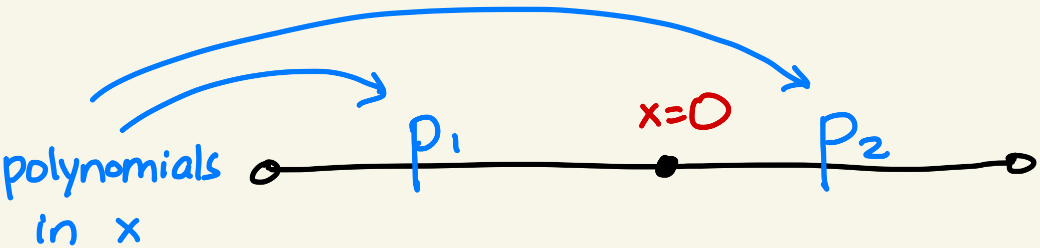

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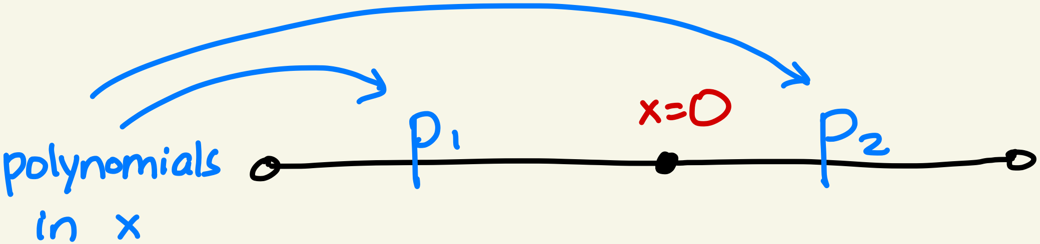

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$$\iff x^2 \mid (p_1 - p_2)$$

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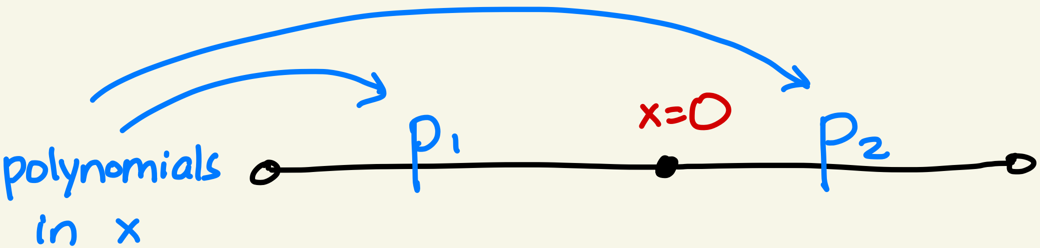

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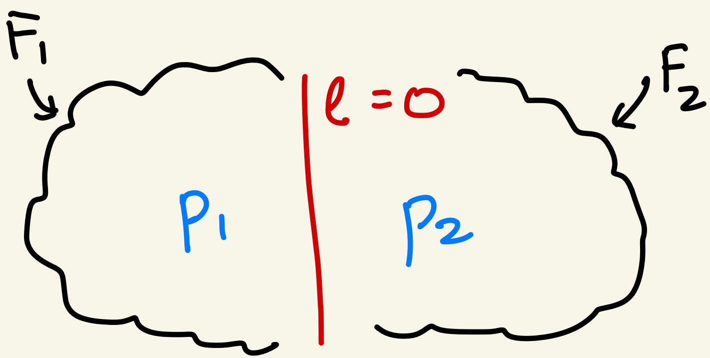
$$p_1^{(k)}\big|_{x=0} = p_2^{(k)}\big|_{x=0} \iff x^{k+1} \mid (p_1 - p_2)$$

More generally, if F_1 and F_2 are facets with $F_1 \cap F_2 = \{ \ell = 0 \}$ for an affine form ℓ then

p_1 & p_2 are piecewise k-differentiable polys



$\Longleftrightarrow$

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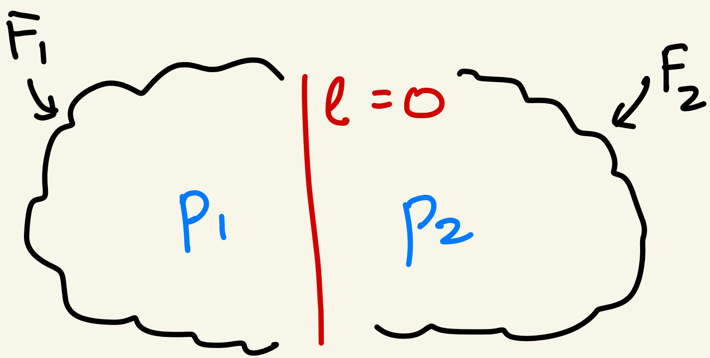
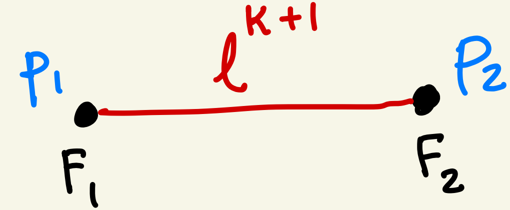


$\iff$

$$\ell^{k+1} \mid (p_1 - p_2)$$

where



in dual graph

Some big differences between these examples:

	Ⓐ toric topology	Ⓑ analysis
Edge label	ℓ	ℓ^{k+1} for any $k > 0$

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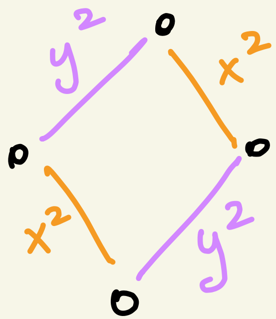
	Ⓐ toric topology	Ⓑ analysis
Edge label	ℓ	ℓ^{k+1} for any $k > 0$
Degree	$d = \infty$	can restrict to polys of $\deg \leq d$
Free over R	YES	SOMETIMES

Example:

$R = \mathbb{C}[x,y]$ Graph is $\mathbb{P}^1 \times \mathbb{P}^1$ and labels are squared



$\boxed{\text{Represent splines as vertex-labeled graphs}}$

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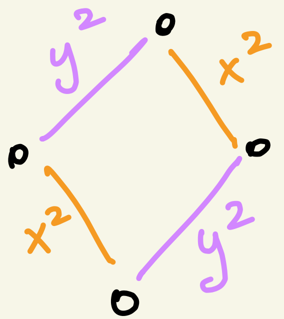
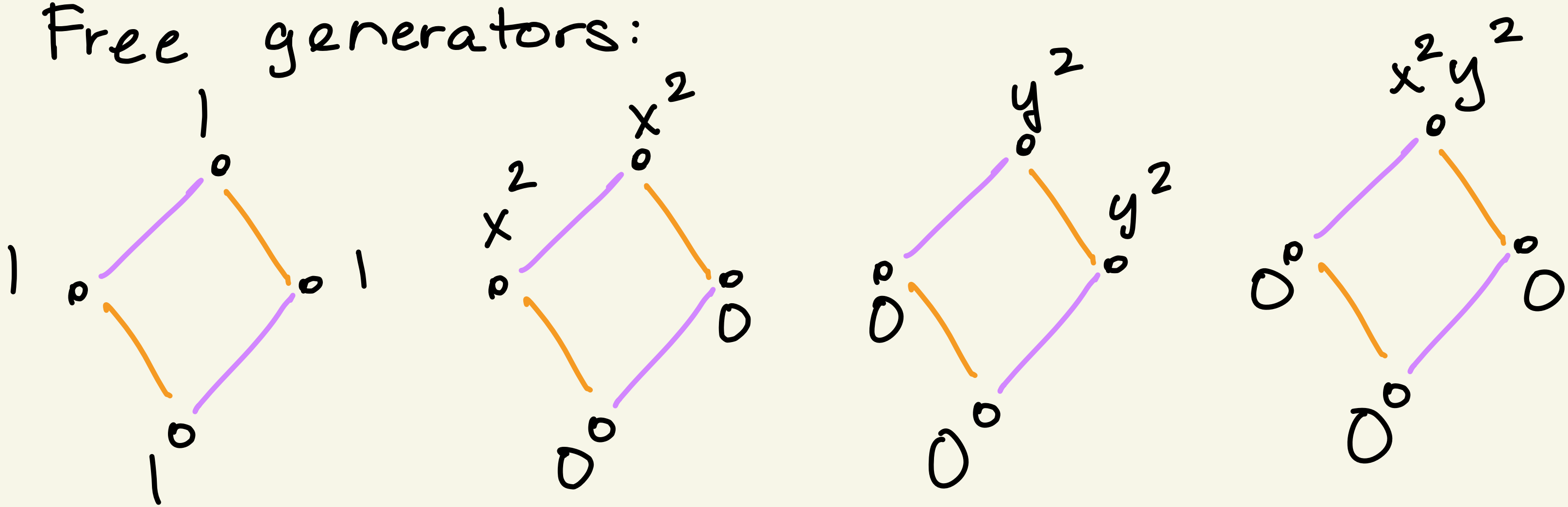


$\boxed{\text{Represent splines as vertex-labeled graphs}}$

Free generators:



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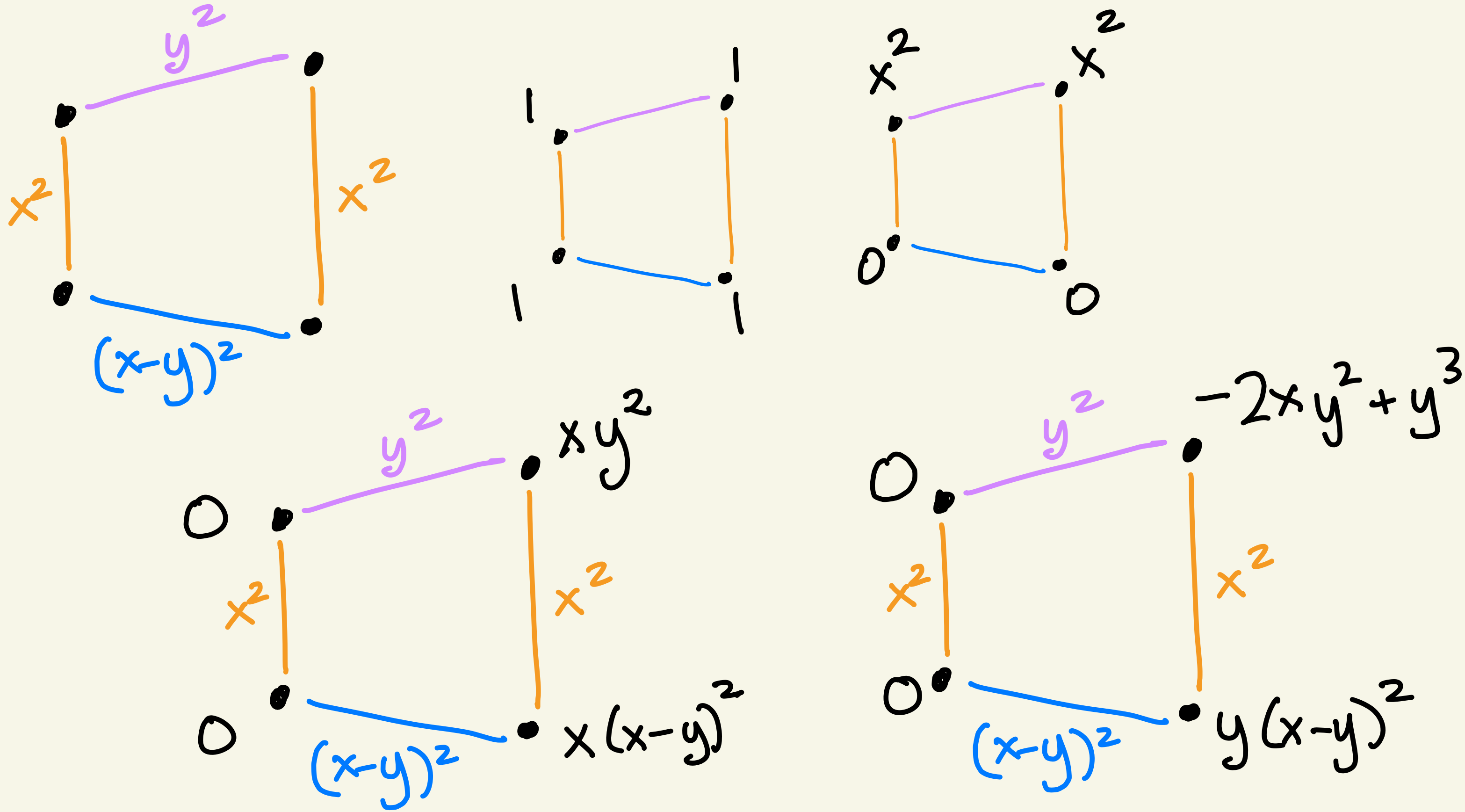

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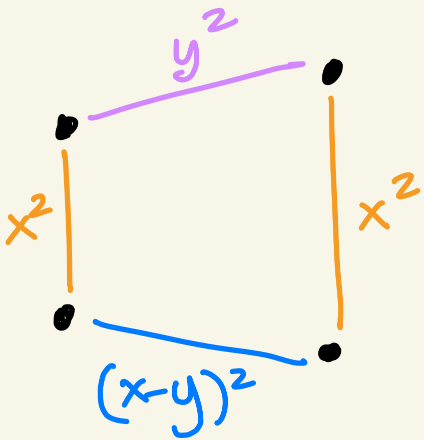
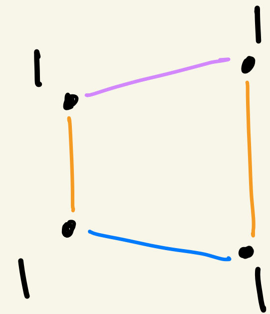
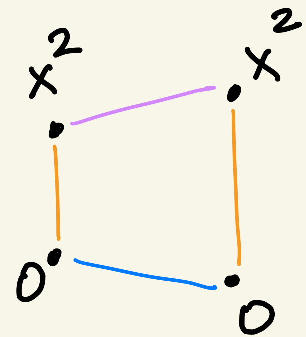
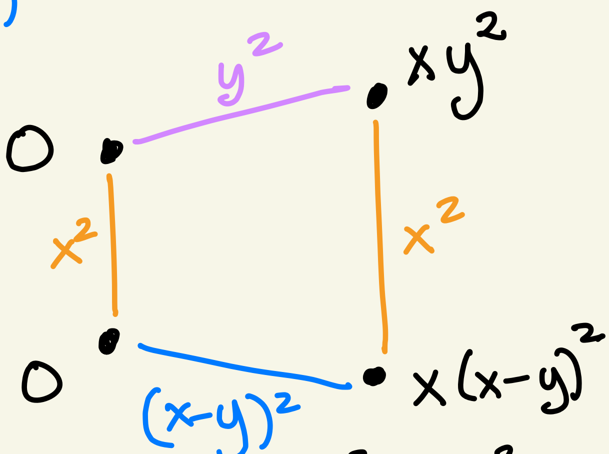
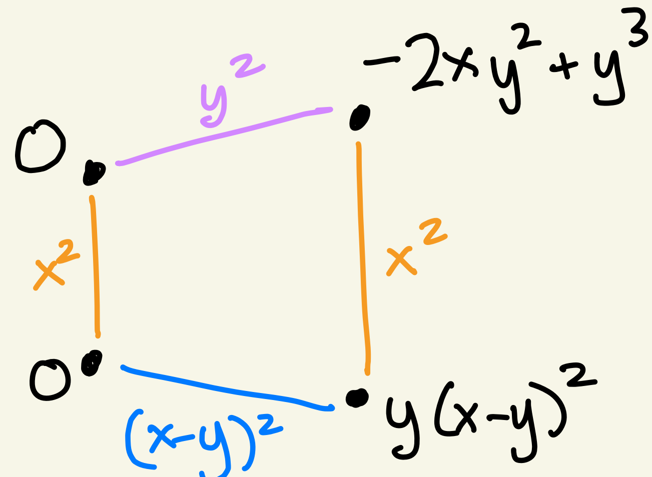








$= x^3 - 2x^2y + xy^2$



$= x^2y - 2xy^2 + y^3$

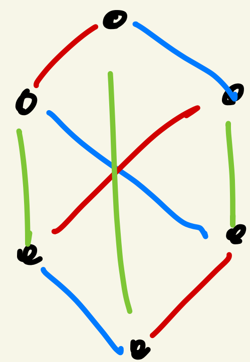
NOT FREE

$$
(2x-y)\left(\begin{tikzcd} 0 \arrow[r, dash] \arrow[d, dash] & xy^2 \arrow[d, dash] \\ 0 \arrow[r, dash] & x(x-y)^2 \end{tikzcd}\right) + x\left(\begin{tikzcd} 0 \arrow[r, dash] \arrow[d, dash] & -2xy^2+y^3 \arrow[d, dash] \\ 0 \arrow[r, dash] & y(x-y)^2 \end{tikzcd}\right)
$$

$$
= 2\left(\begin{tikzcd} 0 \arrow[r, dash, "y^2"] \arrow[d, dash, "x^2"'] & 0 \arrow[d, dash, "x^2"] \\ 0 \arrow[r, dash, "(x-y)^2"'] & x^2(x-y)^2 \end{tikzcd}\right)
$$

Things we can do with splines:

- Subgraphs



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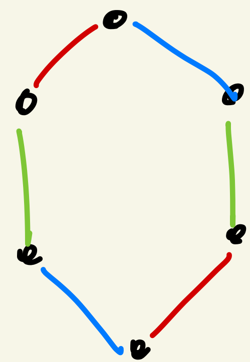

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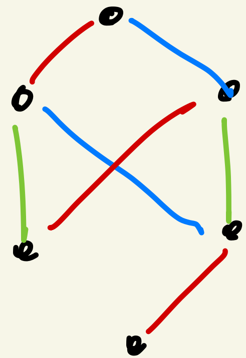
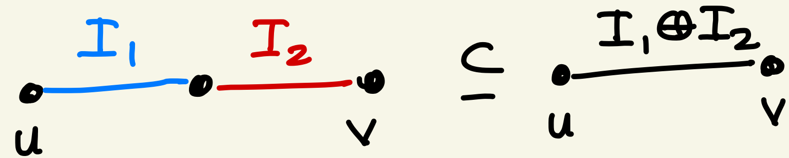


Things we can do with splines:

- Subgraphs

- Path replacement



Things we can do with splines:

- Subgraphs

- Path replacement

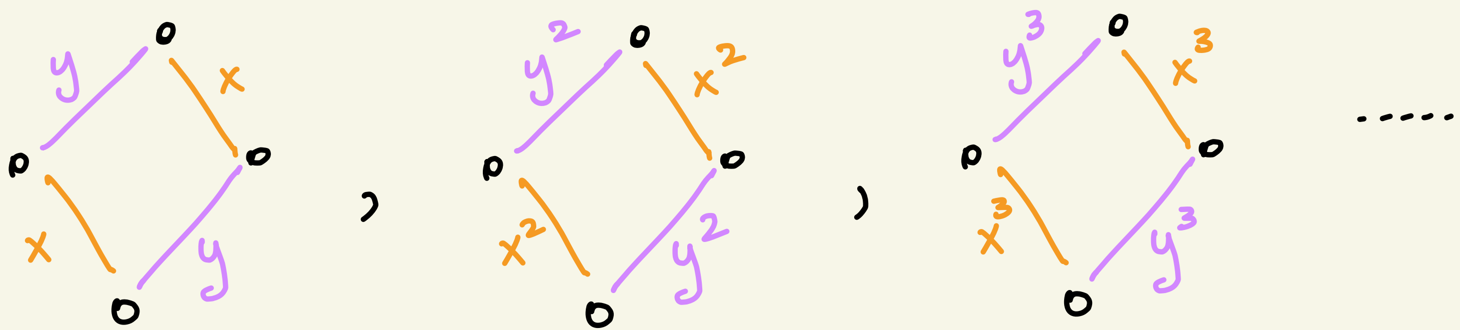
- Change coefficients $\left\{ \begin{array}{l} \mathbb{C}[t_1, \ldots, t_n] \\ \mathbb{R}[t_1, \ldots, t_n] \\ \mathbb{Z}[t_1, \ldots, t_n] \\ \underline{\text{quotient}} \text{ rings} \end{array} \right.$

Things we can do with splines:

- Subgraphs
- Path replacement
- Change coefficients
- and other algebro-combinatorial operations without obvious geometric interps

③ Questions

What does it mean topologically or geometrically to use ℓ^{k+1} ?



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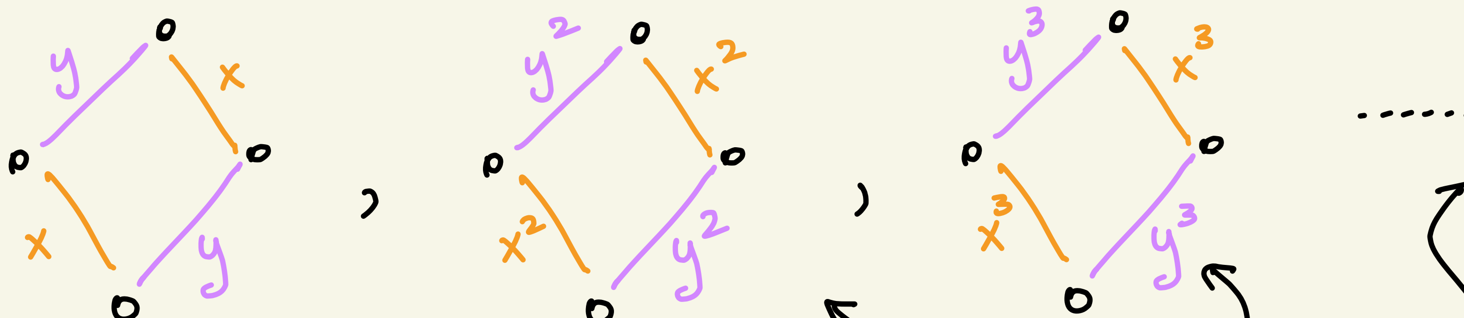


Splines here are $H_T^*(\mathbb{P}^1 \times \mathbb{P}^1)$

What about here?

3 Questions

Can we analyze spline representations?

③ Questions

Can we analyze spline representations?

Any graph automorphism that

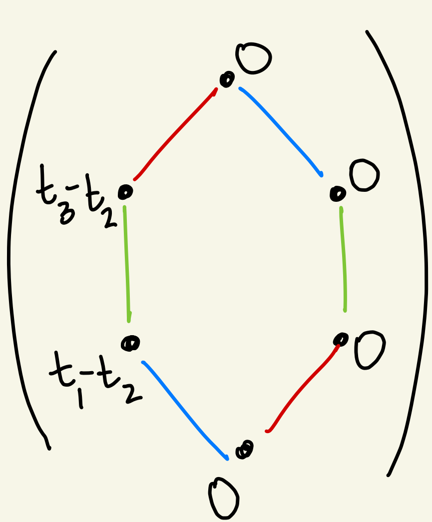
$\boxed{\text{preserves GKM conditions}}$

induces a representation on the module of splines

Example (from the first talk) reg ss Hess vars

$$(w p)(u) = w^{-1}(p(w u))$$

S_1



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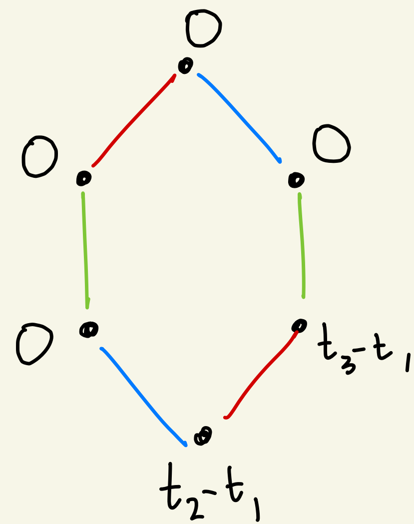
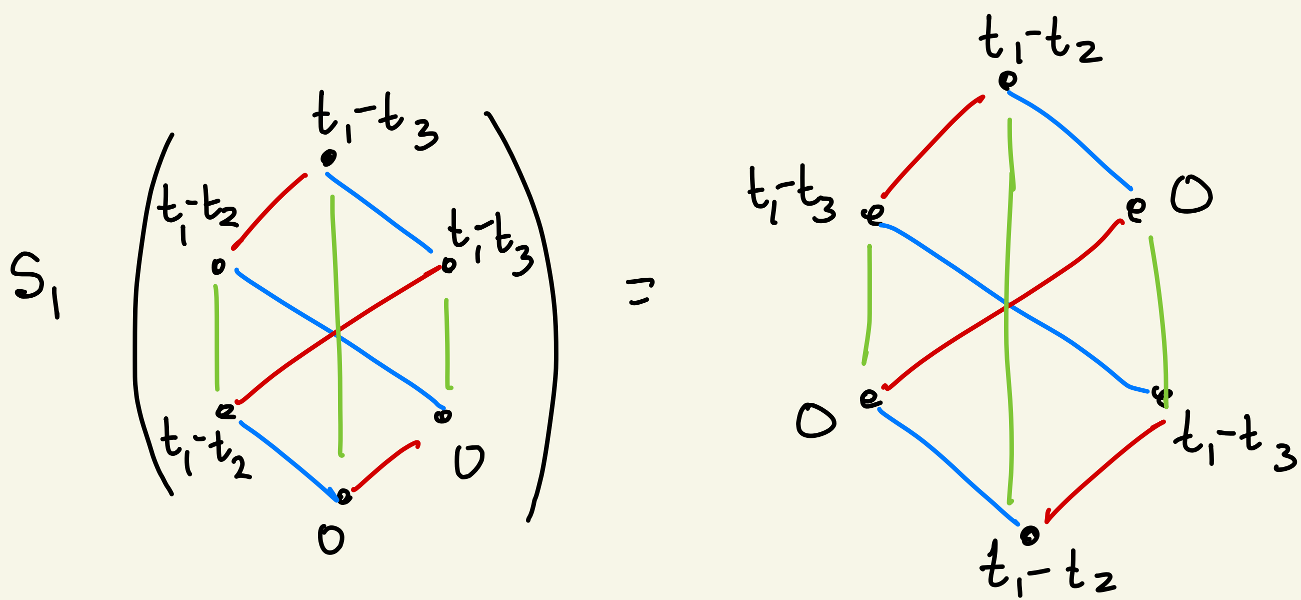


Another example: flag varieties

$$(wp)(u) = p(uw)$$



Spline representations:

Any action that makes combinatorial sense

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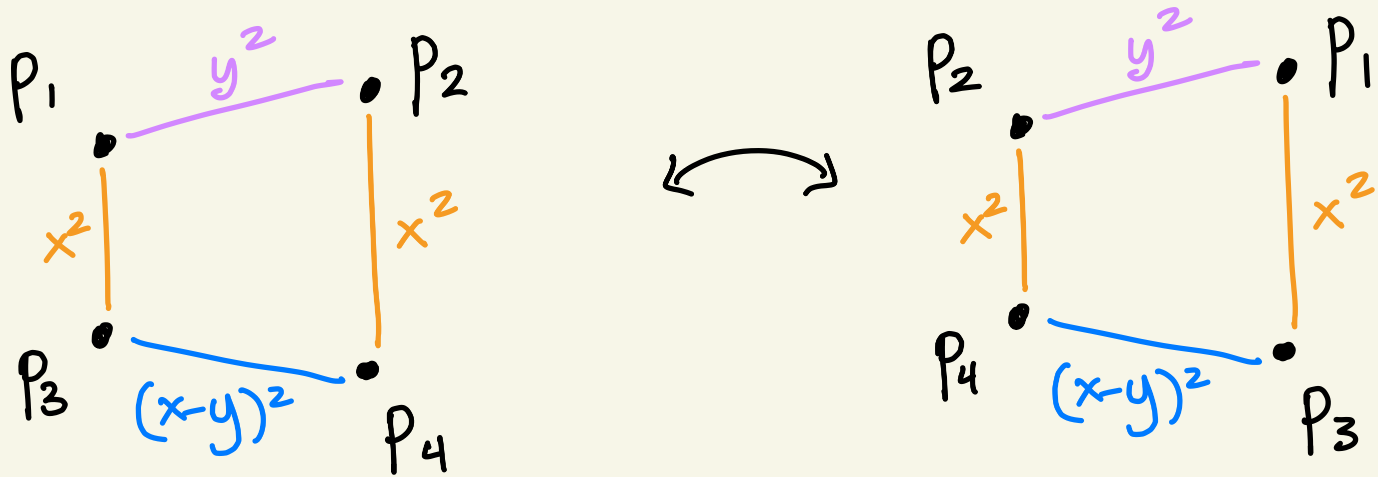
EXCHANGE HORIZONTALLY



Upper bound conjecture:

For splines on graphs dual to planar triangulations, of degree at most 3 and differentiability 1 so labels $\boxed{\ell^2}$

The conjecture is a formula for dimension of spline space, as vector space

Upper bound conjecture:

For splines on graphs dual to planar triangulations, of degree at most 3 and differentiability 1 so labels $\boxed{\ell^2}$

Roughly, it predicts the # of R-module generators

$$\left.\begin{array}{l}\text{Roughly, it}\\ \text{predicts the}\\ \#\text{ of R-module}\\ \text{generators}\end{array}\right\} 1 + \sum_{\text{face } F} \left(\begin{array}{c}\#\text{edges in } F\\ -2\end{array} \right) + \sum_{\substack{\text{"non}\\ \text{singular}\\ \text{faces"}}} 1$$

Upper bound conjecture:

For splines on graphs dual to planar triangulations, of degree at most 3 and differentiability 1 so labels $\boxed{\ell^2}$

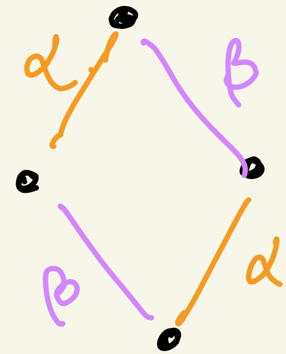
constant spline

non-constant generators of degree ≤ 3

$$\#\text{ of } R\text{-module generators} = \underbrace{1}_{\text{constant spline}} + \sum_{\text{face } F} \binom{\#\text{edges in } F}{-2} + \sum_{\text{"non singular faces"}} 1$$

??

"Singular faces" are the 4-cycle faces with symmetry



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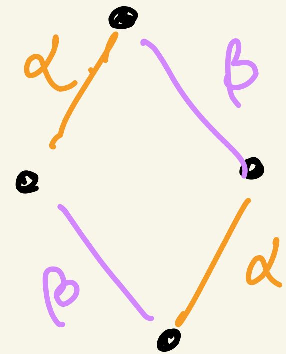
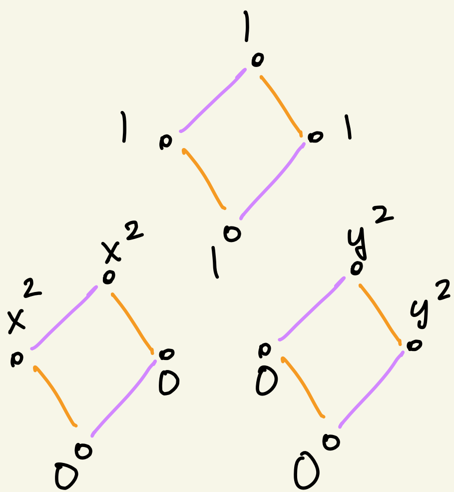
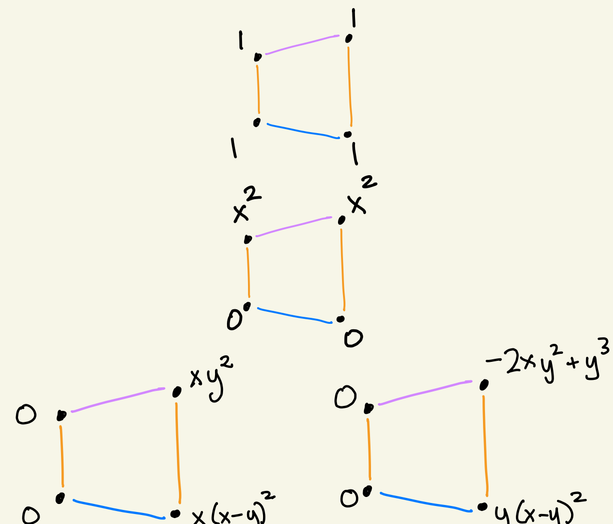




VERSUS



THANK YOU!