Vector valued Laurent poly. systems, toric vec. bundles & matroids

Newton polyhedra theory

. Newton $\rightsquigarrow$ Newton polygon   non-Arch. Newton method

(Local) $f(x,y)=0 \rightsquigarrow y(x) \longrightarrow$ leading exp.

Moscow school

. Newton polytope of a Laurent poly.

(Global)

$x = (x_1, \dots, x_n)$ $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$ $\alpha \in \mathbb{Z}^n$

char. of $T = (\mathbb{C}^*)^n$

$$f(x) = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm] = \mathbb{C}[x^\pm] = \mathbb{C}[T]$$

$$\underset{\text{Newt}(f)}{\Delta(f)} = \text{Conv}\{\alpha \mid c_\alpha \neq 0\} \subset \mathbb{R}^n$$

Fix $\mathcal{A} \subset \mathbb{Z}^n$ finite

$$L_{\mathcal{A}} := \{ f(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha \mid c_\alpha \in \mathbb{C}\} \subset \mathbb{C}[T]$$

. $L_{\mathcal{A}} \xleftrightarrow{1-1}$ T-inv. subspaces of $\mathbb{C}[T]$ finite dim.

$$\Delta = \text{Conv}(\mathcal{A}) = \Delta(f) \text{ for } f \in L_{\mathcal{A}} \text{ generic}$$

$$Z(f_1,\dots,f_r) = \{ z \in (\mathbb{C}^*)^n \mid f_1(z) = \dots = f_r(z) = 0 \}$$

$r \leq n$

<u>Problem</u> Fix $\mathcal{A}_1, \dots, \mathcal{A}_r \subset \mathbb{Z}^n$ finite

$\Delta_i =$

Compute discrete geo. & top. inv. of

$Z(f_1, \dots, f_r)$ in term of the Newton polytopes Δ_i

for generic $f_i \in L_{\mathcal{A}_i}$.

NO bodies theory extends this to arbitrary varieties

<u>Thm</u> BKK

(Bernstein–Kushnirenko–Khovanskii)

$r = n$

$\mathcal{A}_1 = \dots = \mathcal{A}_n \qquad \Delta = \Delta_1 = \dots = \Delta_n$.

$\# Z(f_1, \dots, f_n) = n! \, \mathrm{Vol}_n(\Delta)$

Any $\mathcal{A}_1, \dots, \mathcal{A}_n$

$\# Z(f_1, \dots, f_n) = n! \, \mathrm{MVol}_n(\Delta_1, \dots, \Delta_n)$

dim. of space of holo. top forms

Khovanskii: Geo. genus, arith. genus, top. Euler char., Mixed Hodge structure ...

$f \in L_{\mathcal{A}}$ generic $\Delta = \mathrm{Conv}(\mathcal{A})$

<u>Thm</u> geo. genus of $Z(f) = \#(\Delta^\circ \cap \mathbb{Z}^n)$.

rel. interior of Δ

<u>Fancy version</u> Σ fan in $N_{\mathbb{R}} = \mathbb{R}^n$

X_Σ toric var. proj. $\qquad$ $\mathcal{L}_1, \dots, \mathcal{L}_r$ T-equiv. line bundles on X_Σ

$\mathcal{L}_i \longleftrightarrow \Delta_i$ polytope

$s_i \in H^0(X_\Sigma, \mathcal{L}_i)$

$$Z(s_1, \dots, s_r) = \{ z \in X_\Sigma \mid s_1(z) = \dots = s_r(z) = 0 \}$$

Compute discrete geo/top. inv. of $Z(s_1, \dots, s_r)$ for generic s_i.

<u>Rem</u> $A = \Delta \cap \mathbb{Z}^n$ $\quad$ Δ lattice polytope

$$L_A \overset{1-1}{\longleftrightarrow} H^0(X_\Sigma, \mathcal{L}_\Delta)$$

Vec. valued Laurent poly.

$T = (\mathbb{C}^*)^n$ alg. torus

$r \leq n$

$E \cong \mathbb{C}^r$ v.s.

Fix $\mathcal{A} \subset \mathbb{Z}^n$ & $\forall \alpha \in \mathcal{A}$ $E_\alpha \subset E$ nonzero subspace

$\{E_\alpha\}_{\alpha \in \mathcal{A}} \rightarrow$ subspace arrangement

$$L := \bigoplus_{\alpha \in \mathcal{A}} E_\alpha \otimes x^\alpha \subset \overbrace{E \otimes \mathbb{C}[T]}^{\text{vec. valued Laurent poly.}}$$

$L \xleftrightarrow{1-1}$ T-inv. subspaces of $E \otimes \mathbb{C}[T]$.
finite dim.

$Z(f) = \{ z \in (\mathbb{C}^*)^n \mid f(z) = 0 \}$ $\quad f \in L$

Problem

Compute disc. geo./top. inv. of $Z(f)$

in terms of geo./comb. of convex polytopes.

Alternative Set up

matroid

Fix a finite subset $\mathcal{M} = \{e_1, \dots, e_m\} \subset E$

$$\forall \; e \in \mathcal{M} \longmapsto \mathcal{A}_e \subset \mathbb{Z}^n$$

$$L_{\mathcal{A}_e} \subset \mathbb{C}[T]$$

$$L = \left\{ f = \sum_{e_i \in \mathcal{M}} e_i \otimes p_i \;\middle|\; p_i \in L_{\mathcal{A}_{e_i}} \right\}$$

Main result

$$\Delta_i \subset \mathbb{R}^n$$

Def. $L \rightsquigarrow (\Delta_1, \dots, \Delta_r)$ characteristic seq. of polytopes of L

$$\Delta_i = \mathrm{Conv}\left\{ \alpha_1 + \dots + \alpha_i \;\middle|\; \begin{array}{l} \exists\, v_1, \dots, v_i \text{ lin. ind.} \\ v_j \in E_{\alpha_j} \end{array} \right\}$$

Thm (K.-Khovanskii-Spink)

$r = n$

$f \in L$ generic $\qquad L = \bigoplus_{\alpha \in A} E_\alpha \otimes x^\alpha$

$$\# Z(f) = n!\, \mathrm{MVol}(\Delta_1, \Delta_2 - \Delta_1, \dots, \Delta_n - \Delta_{n-1})$$

$\leq$

virtual polytopes

Cor. $r \leq n$

$$[Z(f)] = [\Delta_1][\Delta_2 - \Delta_1] \cdots [\Delta_r - \Delta_{r-1}].$$

$\{e_1, \dots, e_n\}$ st. basis for $E = \mathbb{C}^n$.

<u>Example</u> (BKK) $\mathcal{A}_1, \dots, \mathcal{A}_n$ $\quad \mathcal{A} = \bigcup \mathcal{A}_i$

E_α = Coordinate subspace = $\operatorname{span}\{e_i \mid \alpha \in \mathcal{A}_i\}$.

$\# Z_f = \{f_1 = \dots = f_n = 0\}$.

<u>Example</u> (Hyperplane arrangements)

$\{u_0, \dots, u_N\}$ lin. forms on $\mathbb{C}^{N+1}$

$H_i = \{u_i = 0\} \subset \mathbb{CP}^n$ $\quad \{H_i\}$ hyperplane arrangement

$$v \in \mathbb{CP}^n \longmapsto (u_0(v) : \dots : v_N(v)) \in \mathbb{CP}^N$$

$$X = \mathbb{CP}^n \setminus \bigcup H_i \cong \overset{W}{\text{Lin. subspace in }} \mathbb{CP}^N \cap T^N$$

$$T^N = (\mathbb{C}^*)^{N+1} / \mathbb{C}^*_{\text{diag.}}$$

Define: $\Delta_i = \operatorname{Conv}\left\{ e_{j_1} + \dots + e_{j_i} \;\middle|\; \bigcap_{j \notin \{j_1, \dots, j_i\}} H_j = \emptyset \right\}$

$i = 1, \dots, N-n$

We recover the following:

$P_i = \operatorname{Conv}(\mathcal{A}_i)$

<u>Cor.</u> $A_1, \dots, A_n \subset \mathbb{Z}^{N+1}$ $\quad f_i \in L_{\mathcal{A}_i}$ homog. deg d_i

$i = 1, \dots, n$

$$\#\{x \in \mathbb{CP}^n \setminus \bigcup H_i \mid f_1(u_0(x) \cdots u_N(x)) = \dots = f_n(u_0(x), \dots, u_N(x)) = 0\}$$

$$= n!\, \operatorname{MVol}(\Delta_1, \Delta_2 - \Delta_1, \dots, \Delta_{N-n} - \Delta_{N-n-1}, P_1, \dots, P_n)$$

Alexandrov – Fenchel inequ.

Recall from Kindergarten:

Cauchy - Schwarz inequ.: $\langle v,w\rangle^2 \leq \langle v,v\rangle\langle w,w\rangle$

<u>Thm</u> (AF inequ.) $P_1,\dots,P_{n-2},P,Q$ convex bodies in $\mathbb{R}^n$

$$\mathrm{MVol}(P_1,\dots,P_{n-2},P,Q)^2 \geq \mathrm{MVol}(\dots,P,P)\,.$$
$$\mathrm{MVol}(\dots,Q,Q)$$

- Khovanskii-Teissier: Same inequ. satisfied for intersection numbers of ample (or nef) divisors on an irr. var.
- K.-Khovanskii $\longrightarrow$ Simple proof using Newton-Okounkov bodies

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$$L = \bigoplus_{\alpha\in A} E_\alpha \otimes x^\alpha \longrightarrow \Delta_1,\dots,\Delta_{n-2} \qquad r = n-2$$

(char. polytopes)

As a Corollary of Khovanskii-Teissier inequ. we obtain:

<u>Thm</u> (K.-Khovanskii-Spink) P,Q convex bodies in $\mathbb{R}^n$

$$\mathrm{MVol}(\underbrace{\Delta_1,\Delta_2-\Delta_1,\dots,\Delta_{n-2}-\Delta_{n-3}}_{\text{virtual polytopes!}},P,Q) \text{ satisfies AF inequ.}$$

Non-representable matroids

- $\mathcal{A} \subset \mathbb{Z}^n$ finite $\qquad \mathcal{M}$ matroid

$\alpha \longmapsto E_\alpha \subset \mathcal{M}$

$(E_\alpha)_{\alpha \in \mathcal{A}}$

- $e \in \mathcal{M} \longmapsto \mathcal{A}_e \subset \mathbb{Z}^n$

$$\Delta_i = \left\{ \alpha_1 + \cdots + \alpha_i \;\middle|\; \begin{array}{l} \exists\, v_j \in E_{\alpha_j} \\ v_1, \ldots, v_i \text{ ind.} \end{array} \right\}$$

$(\Delta_1, \ldots, \Delta_r) \qquad r = \operatorname{rank}(\mathcal{M}) \leq n$

<u>Thm</u> (in progress)
(K.-Khovanskii-Spink)

AF inequ. holds for $\Delta_1, \Delta_2 - \Delta_1, \ldots, \Delta_r - \Delta_{r-1})$

- extends Adiprasito-Huh-Katz
& Ardila-Denham-Huh log-concavity results for matroids.

Khovanskii-Pukhlikov

Convex chains & multi-valued supp. functions

$M = \mathbb{Z}^n \qquad M_{\mathbb{R}} = \mathbb{R}^n$ Virtual polytopes $\subset \mathcal{C}(M_{\mathbb{R}})$

$$\mathcal{C}(M_{\mathbb{R}}) = \{\alpha = \sum_i c_i 1_{P_i} \mid c_i \in \mathbb{Z},\ P_i \subset M_{\mathbb{R}} \text{ convex polytope}\}$$

$\alpha : M_{\mathbb{R}} \longrightarrow \mathbb{R} \qquad \mathrm{Vol}(\alpha) = \int \alpha$

integral if all P_i lattice polytopes $\qquad S(\alpha) = \sum_{u \in M = \mathbb{Z}^n} \alpha(u)$

supp. function of $\alpha \rightsquigarrow h^{(x)} = \underbrace{\sum_i c_i h_{P_i}^{(x)}}_{\text{a finite collection of real numbers with multi.}} \in \mathbb{Z}[\mathbb{R}]$

Prop. $\alpha \xleftrightarrow{1-1} h$

Rem Given α, $\{P_i\}$ not unique, $\{h_i\}$ not unique.

- $\alpha = \sum_i c_i 1_{P_i}$ is integral & effective if $c_i \in \mathbb{Z}_{>0}$ $\forall i$.

α eff. int., $h_\alpha = (h_1, \dots, h_n)$ n-valued supp. function

Let $\mathrm{MVol}(\alpha) = \mathrm{MVol}(P_1, \dots, P_n) \qquad h_{P_i} = h_i$.

Prop. $\mathrm{MVol}(\alpha)$ well-def. i.e. ind. of choice of the h_i.

Multi-valued supp. function ass. to $L = \bigoplus_{\alpha} E_\alpha \otimes x^\alpha$.

$\{E_\alpha\}_{\alpha \in \mathcal{A}}$ $\quad \xi \in N \quad$ $E_i^\xi = \bigoplus_{\langle \xi, \alpha \rangle \leq i} E_\alpha$

$F_\bullet^\xi$ Corr. flag

$0 \subset F_1^\xi \subset \cdots\cdots \subset F_k^\xi = E$

$ a_1 \qquad\qquad a_k$

$\xi \overset{h_L}{\longmapsto} (a_1, \cdots, a_k)$

where each a_i is repeated $\dim\left(F_i^\xi / F_{i-1}^\xi\right)$

$\dim E = r = n$

<u>Main thm.</u> (2nd version) $\quad f \in L$ generic

$\subset E \otimes \mathbb{C}[T]$

$$\# Z_f = n! \, \mathrm{MVol}(h_L).$$

$\leq$

Non-degen. Condition

Given

$$L = \bigoplus_{\alpha \in A} E_\alpha \otimes x^\alpha \subset E \otimes \mathbb{C}[T].$$

Q. Necessary condition on $f \in L$ that guarantees $\# Z_f = n!\, \mathrm{MVol}.$?

Truncated system ass. to $f \in L$ & $\xi \in N$:

$$f_1 = \sum_{\langle \xi, \alpha \rangle = a_1} e_\alpha \otimes x^\alpha$$

$$f_2 = \sum_{\langle \xi, \alpha \rangle = a_2} [e_\alpha] \otimes x^\alpha \qquad [e] \in E/F_1$$

$\vdots$

Def. $f \in L$ is ξ-non-degen. if $\forall z \in Z_f \subset T$

$$df_1(z) \oplus \cdots \oplus df_k(z) : T_z T \longrightarrow \bigoplus_{i=1}^{k} F_i/F_{i-1} \quad \text{is surj.}$$

In part., if $r=n$, the truncated system has no roots. $\neq E$

Rem Depends only on $\sigma \in \Sigma_L$ = normal fan of $\Delta_1 + \cdots + \Delta_r$.

Thm. $f \in L$ ξ-non-degen $\forall \xi \in N$

Then $\overline{Z_f} \subset X_{\Sigma_L}$ is smooth & transverse to all the orbits.

Cor. $r=n$ $\quad \# Z_f = n!\, \mathrm{MVol}(\Delta_1, \Delta_2 - \Delta_1, \ldots)$.

Toric Vec. bundles (& their equiv. Chern classes)

$L = \bigoplus E_\alpha \otimes x^\alpha$ $\qquad$ L^* dual space

$\Phi_L : T \longrightarrow Gr(L^*, n)$

analogue of map to Grassmannian ass. to a gl. gen. v.b.

$\Phi_L : X_{\Sigma_L} \longrightarrow Gr(L^*, n)$

Certain t.v. ass. to L

$\mathcal{E}_L = \Phi_L^*(\text{tautological subbundle})$ T-equiv. v.b. (toric v.b.) (classified by Klyachko)

- One can compute the equiv. Chern classes of $\mathcal{E}_L$
- Supp. functions of $\Delta_1, \ldots, \Delta_r$ are equiv. Chern roots of $\mathcal{E}_L$.