Integral cohomology ring of toric surfaces

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jointly with X. Fu and T. So

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Workshop on Toric Topology

Motivation

Kawasaki '73

$$H^*(\mathbb{C}\mathrm{P}^n_{a_0,\ldots,a_n};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\langle w_1\rangle \oplus \cdots \oplus \mathbb{Z}\langle w_n\rangle ,$$

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$$H^*(\mathbb{C}\mathrm{P}^2_{1,a,b};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\langle w_1 \rangle \oplus \mathbb{Z}\langle w_2 \rangle,$$

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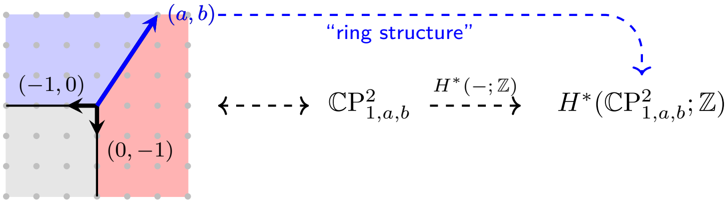
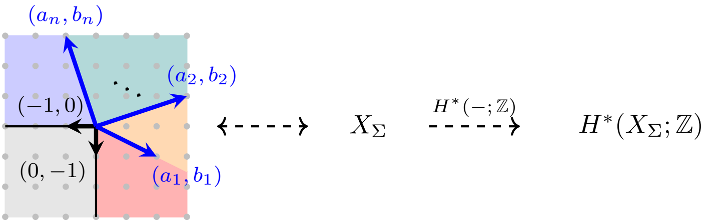
In terms of toric geometry



Questions



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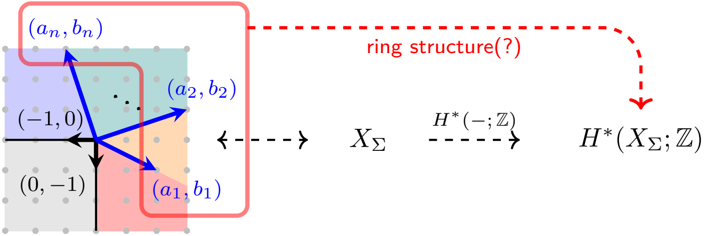

(a_n, b_n)
ring structure(?)
(a_2, b_2)
$(-1, 0)$
X_Σ
$H^*(-;\mathbb{Z})$
$H^*(X_\Sigma;\mathbb{Z})$
$(0, -1)$
(a_1, b_1)

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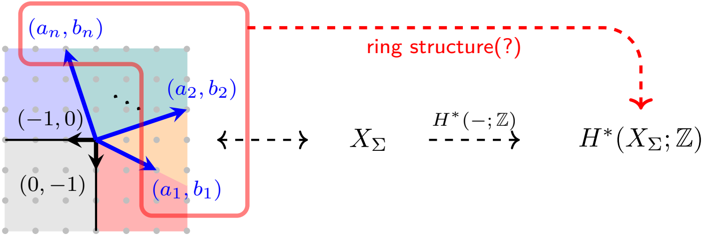

To be more precise,

For Σ as above, $H^k(X_\Sigma;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 4; \\ \mathbb{Z}^n & k = 2; \\ 0 & \text{o.w.} \end{cases}$

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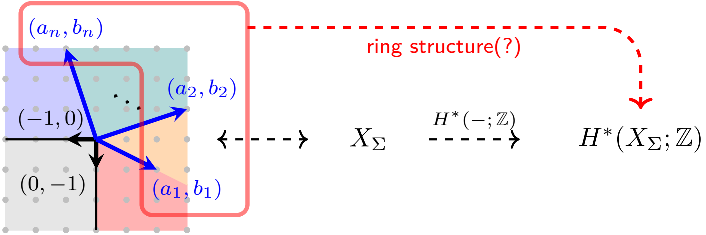

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For Σ as above, $H^k(X_\Sigma;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 4; \\ \mathbb{Z}^n & k = 2; \\ 0 & \text{o.w.} \end{cases}$ Hence, questions are...

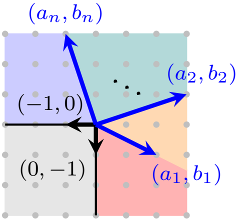
1. Find good "bases" $\{u_1, \dots, u_n\} \subset H^2(X_\Sigma;\mathbb{Z})$ and $v \in H^4(X_\Sigma;\mathbb{Z})$.
2. Find a formula for $M(X_\Sigma) = (c_{ij})_{1 \le i,j \le n}$ with

$$H^2(X_\Sigma;\mathbb{Z}) \otimes H^2(X_\Sigma;\mathbb{Z}) \xrightarrow{\cup} H^4(X_\Sigma;\mathbb{Z}), \quad u_i \cup u_j = c_{ij} \cdot v$$

"in terms of $\{(a_1, b_1), \dots, (a_n, b_n)\}$".

Theorem (Fu–So–S, arXiv:2304.03936)

For a toric surface X_Σ associated with



$\exists$ additive ordered basis $\{u_1, \ldots, u_n\} \subset H^2(X_\Sigma; \mathbb{Z})$ and a generator $v \in H^4(X_\Sigma; \mathbb{Z})$ such that

$$u_i \cup u_j = \boldsymbol{a_i b_j} v, \quad \text{i.e., } M(X) = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_1 b_2 & a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 b_n & a_2 b_n & \cdots & a_n b_n \end{bmatrix}.$$

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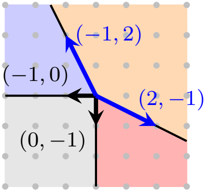
$\Sigma =$ 

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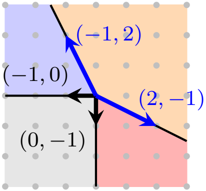
$\Sigma =$ 

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where $\begin{cases} u_1 \cup u_1 = -2 \cdot v; \\ u_1 \cup u_2 = 4 \cdot v; \\ u_2 \cup u_2 = -2 \cdot v, \end{cases}$ i.e., $M(X_\Sigma) = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$.

Preliminaries for the proof: $H^*(X_\Sigma;\mathbb{Z})$

(1) [Danilov '78, Jurkiewicz '80]
For a smooth toric variety X_Σ,

- $H^*_T(X_\Sigma;\mathbb{Z}) \cong \mathsf{SR}[\Sigma] := \mathbb{Z}[x_\rho \mid \rho \in \Sigma^{(1)}]/\mathcal{I}$,
  where $\mathcal{I} = \left\langle \prod_{\rho\in\Gamma} x_\rho \;\mid \mathrm{cone}(\rho \mid \rho\in\Gamma) \notin \Sigma \right\rangle$.
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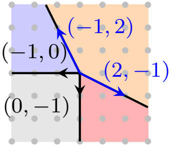
$\Sigma =$ 

$$H^*(X_\Sigma; \mathbb{Q}) = \mathbb{Q}[x_1, x_2, x_3, x_4]/\mathcal{I} + \mathcal{J}$$
$$\mathcal{I} = \langle x_1 x_3, x_2 x_4 \rangle$$
$$\mathcal{J} = \langle 2x_1 - x_2 - x_3, -x_1 + 2x_2 - x_4 \rangle$$

Preliminaries for the proof: $H^*(X_\Sigma; \mathbb{Z})$

(2) [Bahri–Sarkar–S, 17]
For a toric orbifold X_Σ (with '$H^{odd}(X_\Sigma; \mathbb{Z}) = 0$')

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3. Finding 'basis' for each degree of $w\mathsf{SR}[\Sigma]$ or $\overline{w\mathsf{SR}}[\Sigma]$ requires case-by-case computations.

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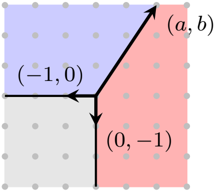
$\Sigma =$ 

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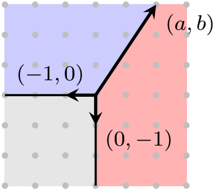
$\Sigma =$ 

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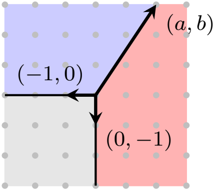
$\Sigma =$ 

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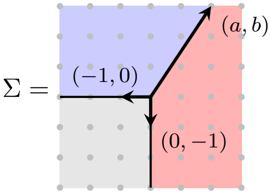

$\longleftrightarrow$ $X_\Sigma = \mathbb{CP}^2_{1,a,b}$

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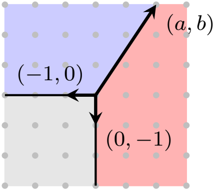
$\Sigma =$ 

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- $w\mathsf{SR}[\Sigma]/\mathcal{J} = \overline{w\mathsf{SR}}[\Sigma] = \mathbb{Z} \oplus \mathbb{Z}\langle[abx_1]\rangle \oplus \mathbb{Z}\langle[x_2x_3]\rangle$
- ring structure: $[abx_1] \cdot [abx_1] = [a^2b^2x_1^2] = \boldsymbol{ab}[x_2x_3]$.

Topological model of a toric variety

Jurkiewicz, '81

Let X be a projective toric variety and P the image of moment map $X \to \mathfrak{t}^*$. Then,

$$X \cong (P \times T^n)/_\sim,$$

where $(x,t) \sim (y,x)$ iff $x = y$ and $t^{-1}s \in T_{F(x)}$.

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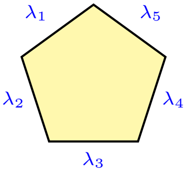

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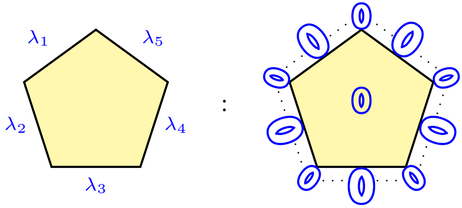

$P \times T^2$

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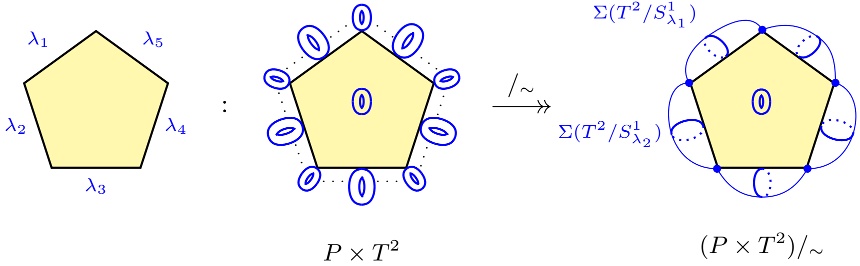

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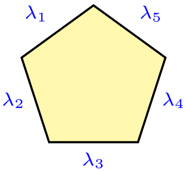

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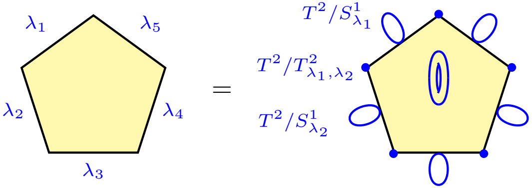

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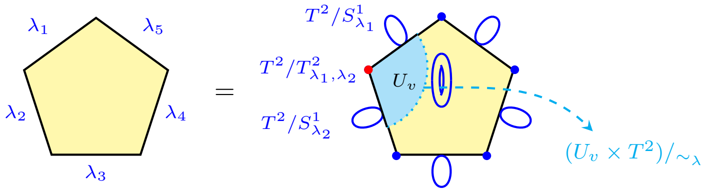

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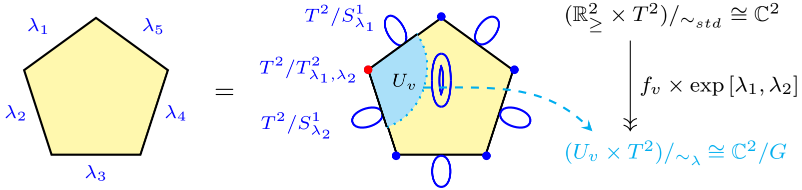

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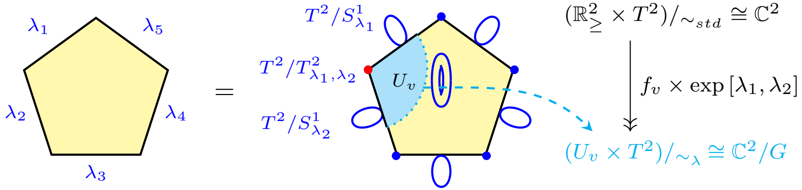

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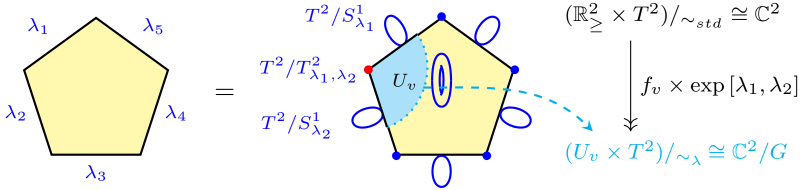

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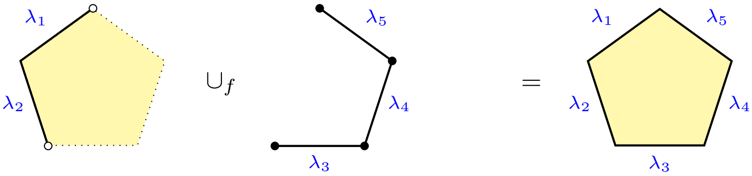

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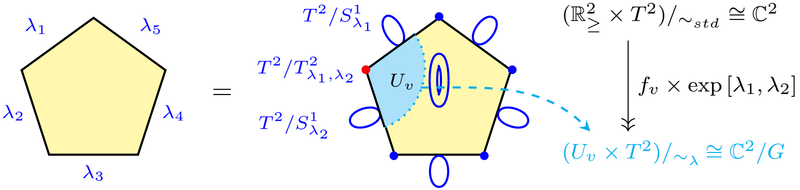

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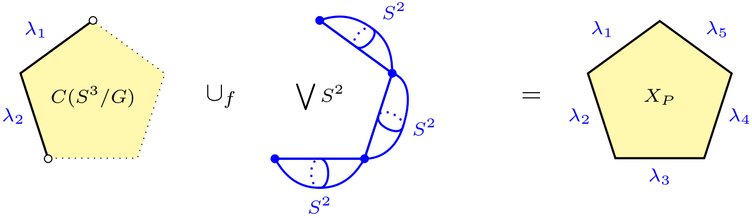
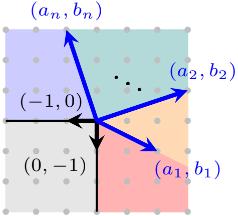
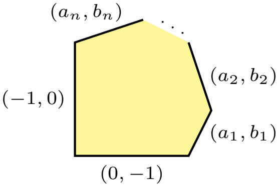


Preliminaries for the proof: cellular basis

(3) For X_Σ corresponding to

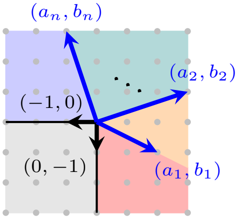


or

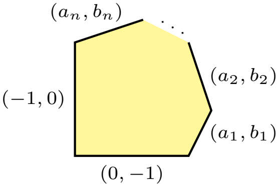


Preliminaries for the proof: cellular basis

(3) For X_Σ corresponding to



or



there is a cofibration

$$S^3 \to \bigvee_{i=1}^{n} S_i^2 \to X_\Sigma$$

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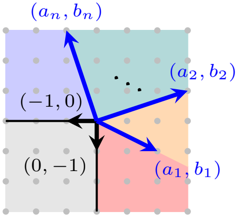

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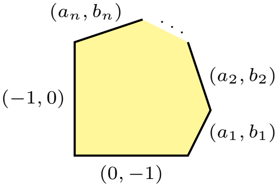

there is a cofibration

$$S^3 \to \bigvee_{i=1}^{n} S_i^2 \to X_\Sigma$$

which gives us:

- $H_2(X_\Sigma; \mathbb{Z}) = \mathbb{Z}\left\langle [S_1^2], \ldots, [S_n^2] \right\rangle$
- $H_4(X_\Sigma; \mathbb{Z}) = \mathbb{Z}\left\langle [D^4] \right\rangle$

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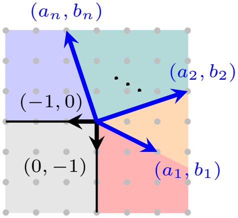

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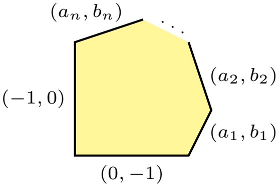

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which gives us:

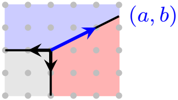
- $H^2(X_\Sigma; \mathbb{Z}) = \mathbb{Z}\langle u_1, \dots, u_n \rangle$, where $\left\langle u_i, [S_j^2] \right\rangle = \delta_{ij}$.
- $H^4(X_\Sigma; \mathbb{Z}) = \mathbb{Z}\langle v \rangle$, where $\left\langle v, [D^4] \right\rangle = 1$.

Summary

So far we have defined two different types of bases:

	Good	Bad
wSR-basis	Easy to see the product structure	Hard to find a basis
Cellular basis	Easy to find a basis	Hard to see the product structure

Example

$\Sigma =$ 

$X_\Sigma = \mathbb{C}\mathrm{P}^2_{1,a,b}$

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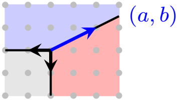
$\Sigma =$  $X_\Sigma = \mathbb{CP}^2_{1,a,b}$

1. (SR-basis)

$$H^*(\mathbb{CP}^2_{1,a,b};\mathbb{Z}) \cong \overline{w\mathsf{SR}}[\Sigma]$$
$$\cong \mathbb{Z} \oplus \mathbb{Z}\langle [abx_1] \rangle \oplus \mathbb{Z}\langle [x_2x_3] \rangle \,/\, \left\langle [abx_1]^2 - ab[x_2x_3] \right\rangle$$

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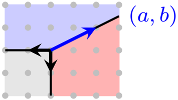
$\Sigma =$  $X_\Sigma = \mathbb{C}\mathrm{P}^2_{1,a,b}$

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2. (Cellular basis) $S^3 \to S^2 \to \mathbb{C}\mathrm{P}^2_{1,a,b}$

$$H^*(\mathbb{C}\mathrm{P}^2_{1,a,b};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\langle u \rangle \oplus \mathbb{Z}\langle v \rangle \quad \text{(as groups)}$$

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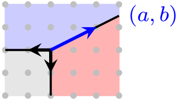
$\Sigma =$ 

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$$\begin{aligned} H^*(\mathbb{C}\mathrm{P}^2_{1,a,b};\mathbb{Z}) &\cong \overline{w\mathsf{SR}}[\Sigma] \\ &\cong \mathbb{Z} \oplus \mathbb{Z}\left\langle [abx_1] \right\rangle \oplus \mathbb{Z}\left\langle [x_2x_3] \right\rangle / \left\langle [abx_1]^2 - ab[x_2x_3] \right\rangle \end{aligned}$$

2. (Cellular basis) $S^3 \to S^2 \to \mathbb{C}\mathrm{P}^2_{1,a,b}$

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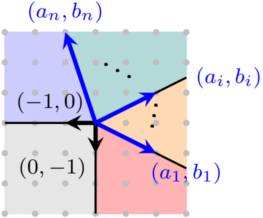
Remark

As $\operatorname{rank} H^2(\mathbb{C}\mathrm{P}^2_{1,a,b};\mathbb{Z}) = \operatorname{rank} H^4(\mathbb{C}\mathrm{P}^2_{1,a,b};\mathbb{Z}) = 1$,

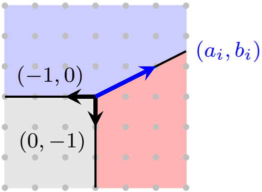
$$[abx_1] \leftrightarrow u \quad \text{and} \quad [x_2x_3] \leftrightarrow v.$$

(with appropriate choices of orientations on S^2 and S^3).

For general cases



"toric morphism"



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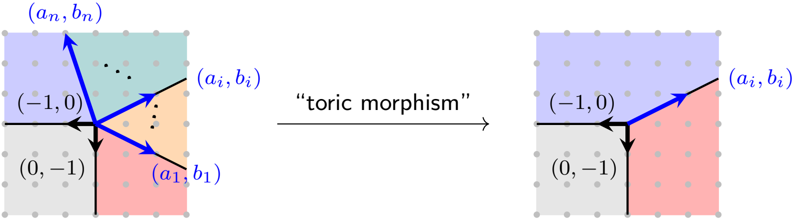

In general

- $\phi\colon \Sigma \to \Sigma' \quad \Rightarrow \quad \phi\colon X_\Sigma \to X_{\Sigma'}$

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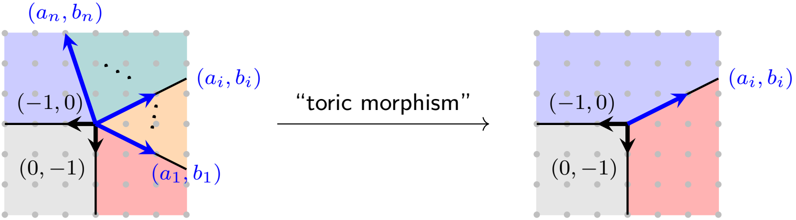

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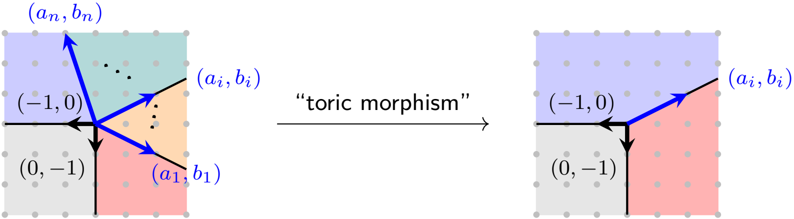


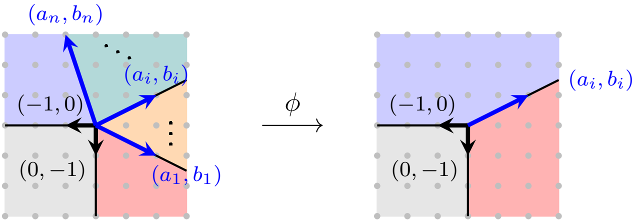
In general

- $\phi\colon \Sigma \to \Sigma' \quad \Rightarrow \quad \phi\colon X_\Sigma \to X_{\Sigma'}$
- Hence, we have: $X_\Sigma \to \mathbb{C}P^2_{1,a_i,b_i}$
- For a toric morphism $\phi\colon \Sigma \to \Sigma'$ of simplicial fans, we have:

$$\begin{array}{ccc} H^*(X_{\Sigma'};\mathbb{Z}) & \xrightarrow{\phi^*} & H^*(X_\Sigma;\mathbb{Z}) \\ \downarrow \cong & & \downarrow \cong \\ \overline{w\mathsf{SR}}[\Sigma'] & \xrightarrow{\overline{w\mathsf{SR}}(\phi^*)} & \overline{w\mathsf{SR}}[\Sigma]. \end{array}$$

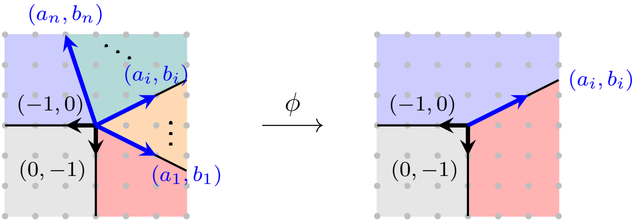
Lemma

For $\phi\colon \Sigma \to \Sigma_i$



Lemma

For $\phi\colon \Sigma \to \Sigma_i$



Then, $\overline{w\mathsf{SR}}(\phi^*)\colon \overline{w\mathsf{SR}}[\Sigma_i] \to \overline{w\mathsf{SR}}[\Sigma]$ is given by

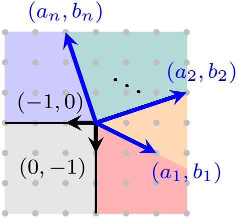
$$[a_i b_i x_1] \mapsto \left[\sum_{k=1}^{i-1} a_k b_i y_k + a_i b_i y_i + \sum_{k=i+1}^{n} a_i b_k y_k\right]$$

$$[x_2 x_3] \mapsto [y_{n+1} y_{n+2}].$$

Revisit the main result

Theorem [Fu–So–S, arXiv:2304.03936]

For a toric surface X_Σ associated with

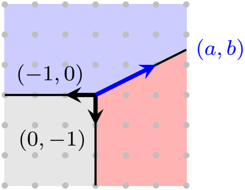


$\exists$ additive basis $\{u_1, \ldots, u_n\} \subset H^2(X;\mathbb{Z})$ and a generator $v \in H^4(X;\mathbb{Z})$ such that

$$u_i \cup u_j = \boldsymbol{a_i b_j} v, \quad \text{i.e., } M(X) = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_1b_2 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1b_n & a_2b_n & \cdots & a_nb_n \end{bmatrix}.$$

Sketch of the proof

Step 1: The case of $\mathbb{CP}^2_{1,a,b}$

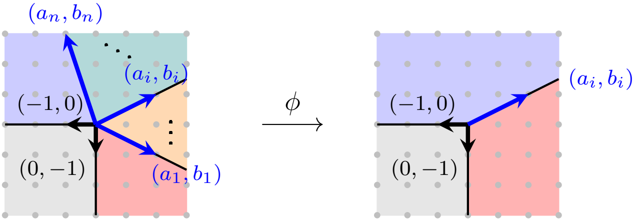
For $\Sigma =$ 

Define $\{u, v\} \subset H^*(X_\Sigma; \mathbb{Z})$ such that

$$H^*(X_\Sigma; \mathbb{Z}) \xrightarrow[\cong]{\Phi} \overline{w\mathsf{SR}}[\Sigma]$$
$$u \mapsto [abx_1]$$
$$v \mapsto [x_2x_3].$$

Then we have: $u^2 = \Phi^{-1}([abx_1]^2) = \Phi^{-1}(ab[x_2x_3]) = abv$.

Step 2: Diagonal entries of $M(X_\Sigma)$



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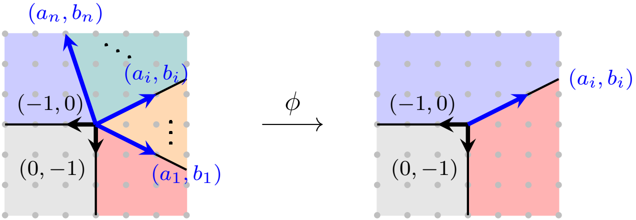


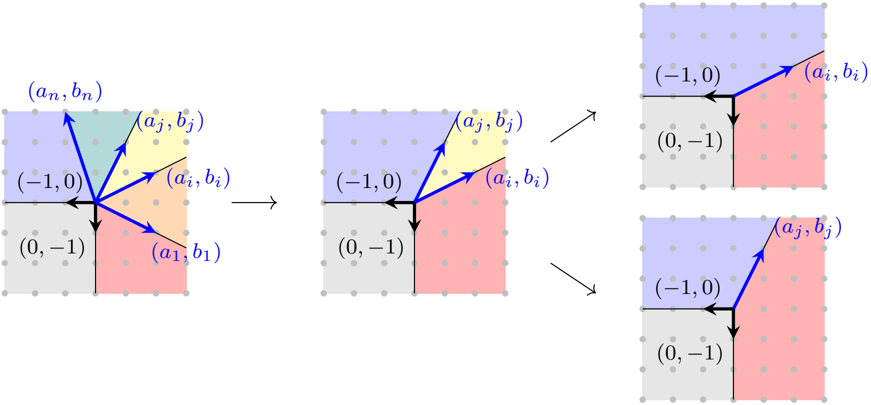
$$\begin{array}{ccccc} S^3 & \longrightarrow & \bigvee_{i=1}^{n} S_i^2 & \longrightarrow & X_\Sigma \\ \Big\| & & \Big\downarrow{\scriptstyle pinch} & & \Big\downarrow{\scriptstyle \phi} \\ S^3 & \longrightarrow & S_i^2 & \longrightarrow & \mathbb{C}\mathrm{P}^2_{1,a_i,b_i} \end{array}$$

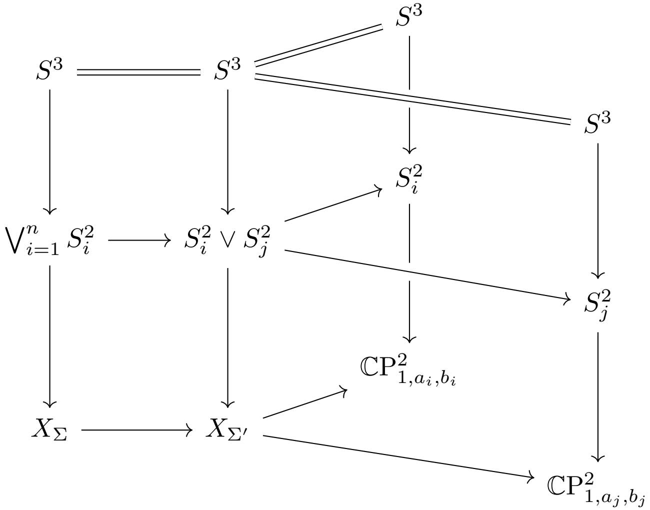
$$
\begin{array}{ccc}
H^*(\mathbb{C}\mathrm{P}^2_{1,a_i,b_i}) & \xrightarrow{\phi^*} & H^*(X_\Sigma) \\
\big\downarrow \cong & & \big\downarrow \cong \\
\overline{w\mathsf{SR}}[\Sigma'] & \xrightarrow{w\mathsf{SR}(\phi^*)} & \overline{w\mathsf{SR}}[\Sigma],
\end{array}
\qquad
\begin{array}{ccc}
u & \xrightarrow{\phi^*} & u_i \\
\big\downarrow & & \\
[a_i b_i x_1] & &
\end{array}
$$

Therefore,

$$
\begin{aligned}
u_i \cup u_i &= \phi^*(u \cup u) \\
&= \phi^*(a_i b_i v) \\
&= a_i b_i \phi^*(v) = a_i b_i v
\end{aligned}
$$

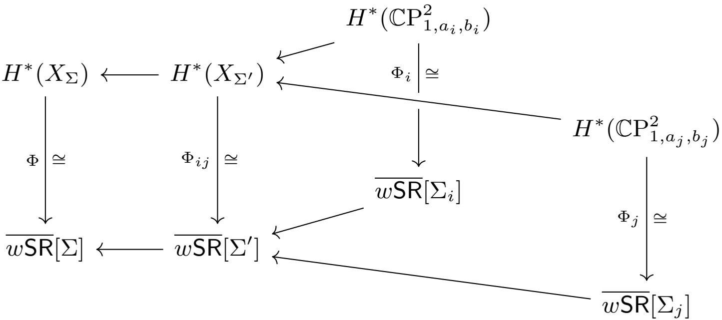
Step 3: Off-diagonal entries of $M(X_\Sigma)$

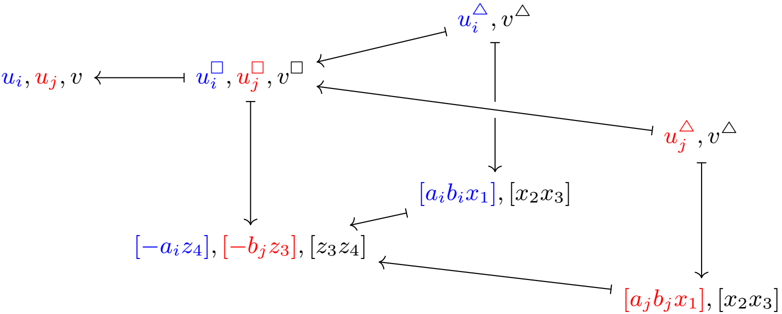


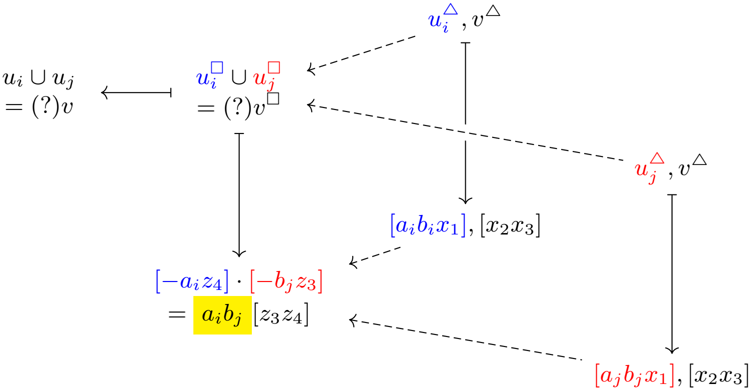


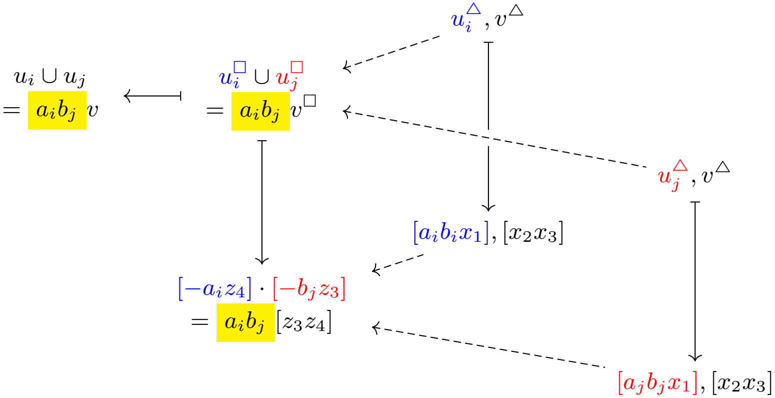
S^3
S^3
S^3
S^3
S_i^2
$\bigvee_{i=1}^n S_i^2$
$S_i^2 \vee S_j^2$
S_j^2
$\mathbb{C}\mathrm{P}^2_{1,a_i,b_i}$
X_Σ
$X_{\Sigma'}$
$\mathbb{C}\mathrm{P}^2_{1,a_j,b_j}$

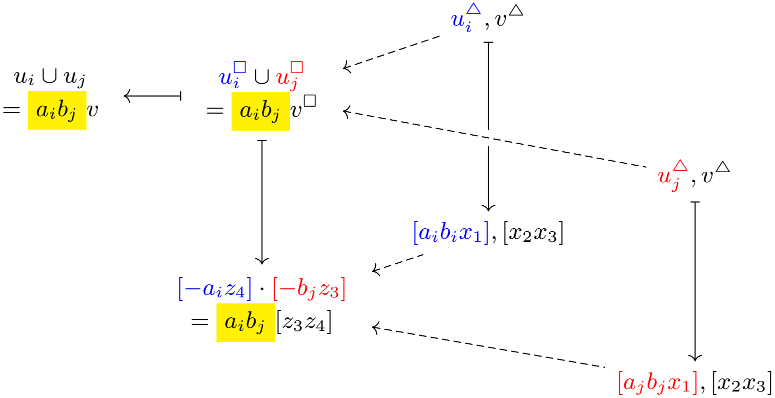
$$
\begin{tikzcd}
 & & H^*(\mathbb{CP}^2_{1,a_i,b_i}) \arrow[dl] \arrow[d, "\Phi_i"', "\cong"] & \\
H^*(X_\Sigma) \arrow[d, "\Phi"', "\cong"] & H^*(X_{\Sigma'}) \arrow[l] \arrow[d, "\Phi_{ij}"', "\cong"] & \overline{w\mathsf{SR}}[\Sigma_i] \arrow[dl] & H^*(\mathbb{CP}^2_{1,a_j,b_j}) \arrow[ull] \arrow[d, "\Phi_j"', "\cong"] \\
\overline{w\mathsf{SR}}[\Sigma] & \overline{w\mathsf{SR}}[\Sigma'] \arrow[l] & & \overline{w\mathsf{SR}}[\Sigma_j] \arrow[ll]
\end{tikzcd}
$$


$H^*(\mathbb{C}P^2_{1,a_i,b_i})$
$H^*(X_\Sigma)$
$H^*(X_{\Sigma'})$
Φ_i $\cong$
$H^*(\mathbb{C}P^2_{1,a_j,b_j})$
Φ $\cong$
Φ_{ij} $\cong$
$\overline{w\mathsf{SR}}[\Sigma_i]$
Φ_j $\cong$
$\overline{w\mathsf{SR}}[\Sigma]$
$\overline{w\mathsf{SR}}[\Sigma']$
$\overline{w\mathsf{SR}}[\Sigma_j]$


$u_i^{\triangle}, v^{\triangle}$
u_i, u_j, v
$u_i^{\square}, u_j^{\square}, v^{\square}$
$u_j^{\triangle}, v^{\triangle}$
$[a_i b_i x_1], [x_2 x_3]$
$[-a_i z_4], [-b_j z_3], [z_3 z_4]$
$[a_j b_j x_1], [x_2 x_3]$


$u_i \cup u_j$
$= (?)v$
$u_i^{\square} \cup u_j^{\square}$
$= (?)v^{\square}$
$u_i^{\triangle}, v^{\triangle}$
$u_j^{\triangle}, v^{\triangle}$
$[a_i b_i x_1], [x_2 x_3]$
$[-a_i z_4] \cdot [-b_j z_3]$
$= a_i b_j [z_3 z_4]$
$[a_j b_j x_1], [x_2 x_3]$


$u_i \cup u_j = a_i b_j v$
$u_i^{\square} \cup u_j^{\square} = a_i b_j v^{\square}$
$u_i^{\triangle}, v^{\triangle}$
$u_j^{\triangle}, v^{\triangle}$
$[a_i b_i x_1], [x_2 x_3]$
$[-a_i z_4] \cdot [-b_j z_3] = a_i b_j [z_3 z_4]$
$[a_j b_j x_1], [x_2 x_3]$



Thank you for your attention.