Seeds with maximal Buchstaber number

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Buchstaber numbers

K: an $(n-1)$ dimensional simplicial complex on $[m] = \{1, 2, \ldots, m\}$.

Buchstaber number

$$s(K) = \max\{r \in \mathbb{N} \mid H^r(\subseteq T^m) \underset{free}{\curvearrowright} \mathcal{Z}_K\}.$$

Theorem (Erokhovets, 2014)

$$1 \leq s(K) \leq m - n$$

The upper bound $Pic(K) = p = m - n$ is called the Picard number of K.

Freely acting subtori

Let H^p be an p-dimensional freely acting subtorus of T^m on $\mathcal{Z}_K$.

The quotient ${}^{\mathcal{Z}_K}\!/_{H^p}$ gives

a quasitoric manifold if K is a polytopal complex,

a topological toric manifold if K is a star-shaped sphere.

$$\{\text{polytopal complex}\} \subsetneq \{\text{star-shaped complex}\} \subsetneq \{\text{PL sphere}\}$$

Freely acting subtori

Let $T^m \curvearrowright \mathcal{Z}_K$.
A subtorus H^p can be written as

$$\{(e^{2\pi i(s_{11}\phi_1+\cdots+s_{1p}\phi_p)}, \ldots, e^{2\pi i(s_{m1}\phi_1+\cdots+s_{mp}\phi_p)}) \in T^m \mid \phi_j \in \mathbb{R}\}.$$

Define

an $m \times p$ matrix $S = [s_{ij}]$,

a $p \times p$ matrix $S_{\sigma^c} = [s_{ij}]_{i\in\sigma^c, j\in\{1,\ldots,p\}}$ for each facet σ of K.

Proposition

H freely acts on $\mathcal{Z}_K$ if and only if $\det(S_{\sigma^c}) = \pm 1$ for any facet σ of K.

$J = (j_1, \dots, j_m)$: a positive integer m-tuple.

J-construction

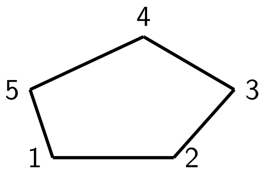
$K(J)$ is the simplicial complex whose minimal non-faces are the set

$$\{\bigcup_{v\in\tau}\{v_1, v_2, \dots, v_{j_v}\} \mid \tau \text{ is a minimal non-face of } K\}.$$

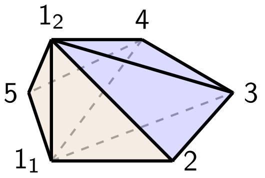
We can consider K as a subcomplex of $K(J)$ by identifying $v_1 = v$.

$MNF(K) = \{\{1,3\},\{1,4\},\{2,4\},\{2,5\},\{3,5\}\}$

$\Longrightarrow MNF(K(2,1,1,1,1)) = \{\{1_1,1_2,3\},\{1_1,1_2,4\},\{2,4\},\{2,5\},\{3,5\}\}$



K



$K(2,1,1,1,1)$

Seeds

Proposition

(1) K is a PL sphere (polytopal, star-shaped) if and only if so is $K(J)$

(2) $Pic(K) = Pic(K(J))$

(3) $s(K) = s(K(J))$ (Ewald, 1986)

Theorem (Choi-Park, 2016)

Any subtorus freely acting on $\mathcal{Z}_{K(J)}$ can be constructed by subtori freely acting on $\mathcal{Z}_K$.

A PL sphere K is called a seed if $K = L(J)$ implies $J = (1, 1, \ldots, 1)$.

Seed inequality

Theorem (Choi-Park)

$m \leq 2^p - 1$ if K is a seed with $s(K) = p = m - n \geq 3$.

Sketch of the proof

(1) Let S be an $m \times p$ matrix representing a freely acting subtorus.

(2) For a seed, the rows of S have to be mutually distinct.

(3) Then the mod 2 reduction of S should not allow repeated rows.

(4) $m \leq |\mathbb{Z}_2^p \setminus \mathbb{O}| = 2^p - 1$

Seed inequality

Theorem

There exists an $(n-1)$ *dimensional seed* K *on* $[m]$ *with* $s(K) = m - n \geq 3$ *if and only if* $2 \leq n$, $m \leq 2^{m-n} - 1$, $m - n \geq 3$.

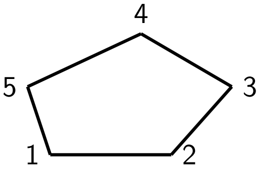
In particular, the inequality is tight.
This is true even for the class of polytopal complexes.
Therefore, there is a quasitoric manifold over a seed with $2 \leq n$, $m \leq 2^{m-n}$.

Seeds

I: a 1-simplex

Suspension

$$\Sigma K := K * \partial I$$



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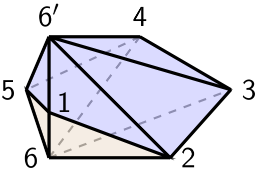


ΣK

Proposition

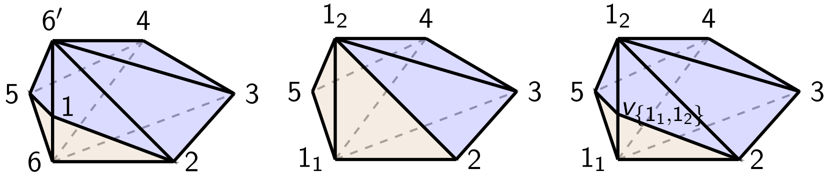
(1) K is a PL sphere (polytopal, star-shaped) if and only if so is ΣK

(2) $Pic(K) + 1 = Pic(\Sigma K)$ and $s(K) + 1 = s(\Sigma K)$

(3) K is a seed if and only if so is ΣK

Stellar subdivision

The stellar subdivision $Ss_\sigma(K)$ at a face σ of K is a subdivision of K with a new vertex v_σ in the relative interior of σ.



ΣK $\quad$ $K(2,1,1,1,1)$ $\quad$ $Ss_{\{1_1,1_2\}}(K(2,1,1,1,1))$

Proposition

(1) K is a PL sphere (star-shaped) if and only if so is $Ss_\sigma(K)$.

(2) If K is polytopal, then so is $Ss_\sigma(K)$.

(3) $s(K)+1 = s(Ss_\sigma(K))$ and $Pic(K)+1 = Pic(Ss_\sigma(K))$.

(4) $Ss_{\{v_1,v_2\}}(K(1,\ldots,1,2,1,\ldots 1))$ is a seed.

Main construction

Choose any subset $\sigma = \{v_1, v_2, \ldots, v_\ell\} \subset [m]$.

Let $J_\sigma = (j_1, \ldots, j_m)$ for $j_v = \begin{cases} 2, & \text{if } v \in \sigma \\ 1, & \text{otherwise.} \end{cases}$

Then σ is a face of $K(J_\sigma)$.

Theorem

$Ss_\sigma(K(J_\sigma))$ is a seed that is not a suspension if K is a seed that is not a suspension and $|\sigma| > 1$.

Main construction

$p \backslash n$	2	3	4	5	6	7	8	9	10	11	12	13	14
3	∂P_5	$(\partial I^3)^*$	$\partial C^4(7)$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
4	∂P_6	$\Sigma\partial P_5$	$\Sigma(\partial I^3)^*$	$\Sigma\partial C^4(7)$	$Ss_{\{1\}}(\partial C^4(7)(2,2,1,\dots,1))$					$Ss_{[7]}(\partial C^4(7)(2,\dots,2))$	$\varnothing$	$\varnothing$	$\varnothing$
5	∂P_7												

Remark

(1) The doubling (that is, $J = (2, 2, \dots, 2)$) of K followed by stellar subdivision from the maximal complex of Picard number p gives a maximal one of Picard number $p + 1$.

(2) For $p = 4$, there are four seeds K with $s(K) = 4$. Our construction gives one of them, and proves the polytopality. Are the other three polytopal?

Complete non-singular fans

Theorem

A seed K with $Pic(K) = 4$ is the underlying complex of a complete non-singular fan if and only if $2 \leq n \leq 8$.

Our construction gives underlying complexes K of complete non-singular fans only up to $n = 7$ from the seeds of Picard number 3.

Thank you for your attention