# Homotopy Types of Toric Orbifolds from Weyl Polytopes

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# Main Result

A **Weyl group** $W$ acts linearly on a real Euclidean space $V$.

$P$ is the convex hull of the $W$-orbit of a well-chosen point of $V$.

$P$ and $P/W$ are (identified with) polytopes lying in $V$.

There are associated **toric varieties** $X_P$ and $X_{P/W}$.

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**Theorem**

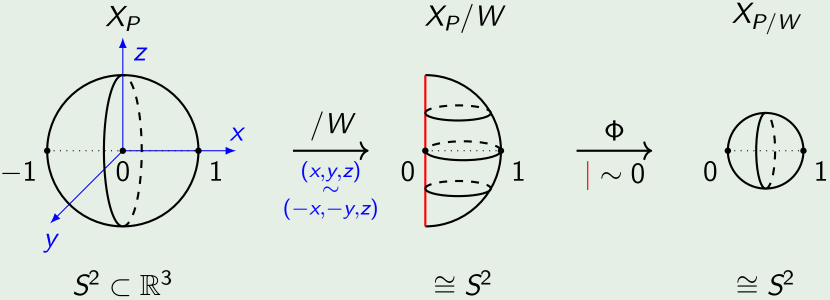
*There is a homotopy equivalence* $\Phi : X_P/W \to X_{P/W}$ *for any Lie type.*

By [Blume'15], $X_P/W \cong X_{P/W}$ as varieties for Lie types $A, B, C$.

By [Horiguchi-Masuda-Shareshian-Song'21], for Lie types $A$, $B$, $C$, $D$, $H^*(X_P/W_K;\mathbb{Q}) \cong H^*(X_{P/W_K};\mathbb{Q})$ for any parabolic subgroup $W_K$. [Song'22] generalized the isomorphism to symmetric polygons.

## Example (of Lie type $A_1$)

In $\mathbb{R}$, $W = \langle s : x \mapsto -x \rangle$, $P = [-1, 1]$, and $P/W \cong [0, 1]$.



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# Toric Varieties from Polytopes

Given a lattice $M$ of rank $n$, its dual lattice $N$ and an $n$-dimensional lattice polytope $P \subset M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, each facet $F$ of $P$ has a normal vector $l_F$ in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, which is integral, primitive and pointing inside the polytope.

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Define the **normal fan** $\Sigma_P$ whose cones are generated by those sets of normal vectors $l_{F_{i_1}}, l_{F_{i_2}}, \cdots, l_{F_{i_k}}$ whose corresponding facets $F_{i_1}, F_{i_2}, \cdots, F_{i_k}$ have non-empty intersection in $P$.

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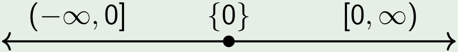
Each cone $\sigma$ in $\Sigma_P$ corresponds to an affine complex toric variety $X_\sigma$, and these affine varieties fit together to form an algebraic variety $X_P$.

## Example

In $\mathbb{R}$, $P = [-1, 1]$.

$\Sigma_P$




$X_\sigma$ $\qquad \mathbb{C} \qquad \mathbb{C}^* \qquad \mathbb{C}$

$$X_P = colim \left( \mathbb{C} \xleftarrow{x \mapsto x^{-1}} \mathbb{C}^* \xrightarrow{x \mapsto x} \mathbb{C} \right) \qquad \cong S^2$$

# Topological Models

$S_N := N_{\mathbb{R}}/N \cong N \otimes_{\mathbb{Z}} S^1$. There is an $S_N$-equivariant homeomorphism

$$X_P \cong (S_N \times P) / \sim,$$

where $(t_1, p_1) \sim (t_2, p_2)$ iff $p_1 = p_2$ lying in the relative interior of some face $F$, and $t_1$, $t_2$ are congruent modulo the subtorus $S_{N(\sigma_{p_1})}$ of $S_N$ for the cone $\sigma_{p_1} \in \Sigma_P$ corresponding to the face $F$.

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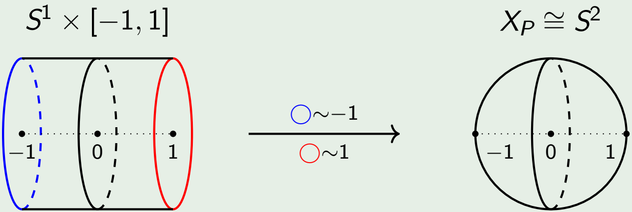

# Group Action

If $G \curvearrowright N$ with $G \curvearrowright P \subset M_{\mathbb{R}}$, then $G \curvearrowright \Sigma_P$ and $G \curvearrowright X_P$.

## Lemma

*For any $g \in G$, the following diagram is commutative:*

$$\begin{array}{ccc} X_P & \xrightarrow{\cong} & (S_N \times P)\,/\sim \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g \times g^{-1}} \\ X_P & \xrightarrow[\cong]{} & (S_N \times P)\,/\sim\,. \end{array}$$

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# Root Systems

In an $n$-dimensional real Euclidean space $(V, \langle \cdot, \cdot \rangle)$, a finite set $R$ of non-zero vectors (**roots**), is a **root system** if

- $R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$, $\forall \alpha \in R$;
- $s_\alpha(\beta) := \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \in R$, $\forall \alpha, \beta \in R$;
- $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

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There is a **simple system** $\Delta \subset R$ such that each root of $R$ can be written as a unique $\mathbb{Z}$-linear combination of simple roots.
Implicitly $R$ is **of rank** $n$, that is, $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

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**Weyl group** $W := \langle s_i := s_{\alpha_i} | \, i \in [n] \rangle$ where $[n] := \{1, 2, \ldots, n\}$.

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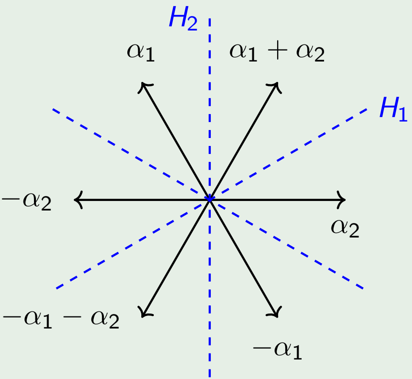
**Weyl group** $W := \langle s_i := s_{\alpha_i} | \, i \in [n]\rangle$ where $[n] := \{1, 2, \ldots, n\}$.

A subset $I \subset [n]$ determines a **parabolic subgroup** $W_I := \langle s_i | \, i \in I\rangle$.

## Example (of Lie type $A_2$)

$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$.
$\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$.
$W \cong S_3 \curvearrowright V$ by permuting coordinates of $\mathbb{R}^3$.




$$R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}.$$

$$s_2(\alpha_1) = \alpha_1 - \frac{2\langle\alpha_1,\alpha_2\rangle}{\langle\alpha_2,\alpha_2\rangle}\alpha_2 = \alpha_1 + \alpha_2.$$

$$W \curvearrowright \mathbb{Z}Span\,(R)\,.$$

# Weyl Polytopes and Quotient Polytopes

The **fundamental chamber** is

$$C := \{x \in V | \ \langle x, \alpha_k \rangle > 0, \ \forall k \in [n]\}.$$

**Theorem (Humphreys'92, Theorem 1.12)**

*$\overline{C}$ is a fundamental domain for the action of $W$ on $V$, i.e., each $x \in V$ lies in $W(x')$ for some unique $x' \in \overline{C}$.*

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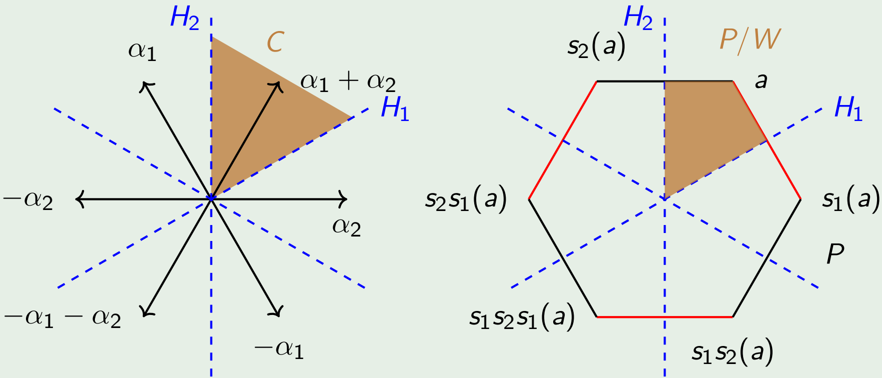
Fix a point $a \in C$, define the **Weyl polytope** $P := Conv(W(a))$.
The quotient $P/W \cong P \cap \overline{C}$ is a polytope.

## Example (of Lie type $A_2$)

$V = (1,1,1)^\perp \subset \mathbb{R}^3$, $\alpha_1 = (1,-1,0)$, $\alpha_2 = (0,1,-1)$.



# Face Structures

Let $F_i := Conv\left(W_{[n]\setminus\{i\}}(a)\right)$, $H_i := (\alpha_i)^\perp$ for $i \in [n]$.

**Lemma (faces of $P$)**

*Define $\Lambda := \bigcup_{I \subset [n]} \{I\} \times \frac{W}{W_{[n]\setminus I}}$, then there is a bijection between $\Lambda$ and the set of all faces of $P$, assigning $(I, s)$ to $s(\cap_{i \in I} F_i)$.*

**Lemma (faces of $P/W$)**

*Define $H_I F_J := (\cap_{i \in I} H_i) \cap \left(\cap_{j \in J} F_j\right)$, then $H_I F_J \cap \overline{C}$ gives out bijectively all faces of $P \cap \overline{C}$ as $I, J$ range among disjoint subsets of $[n]$.*

[Vinberg'91] studied the face structure of $P$, [Burrull-Gui-hu'24] studied that of $P/W$, and [Horiguchi-Masuda-Shareshian-Song'21] studied that of $P/W_K$ for Lie types $A, B, C, D$.

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# Lattice Settings

Let $M$ be the lattice of roots $\mathbb{Z}Span\left(\{\alpha_i\}_{i\in[n]}\right)$, $N$ be the lattice of **coweights** $\mathbb{Z}Span\left(\{\omega_i^\vee\}_{i\in[n]}\right)$.

Choose $a \in C \cap M$, we consider $P = Conv(W(a))$ in $M_\mathbb{R} = V$, and its normal vectors in $N_\mathbb{R} = V$.

We write $\sigma$ for a cone of $\Sigma_P$, and $\sigma_{[n]}$ for a cone of $\Sigma_{P/W}$. For a point $p$ in $P$ (resp. $P \cap \overline{C}$), the cone $\sigma_p$ (resp. $\sigma_{[n],p}$) corresponds to the face of $P$ (resp. $P \cap \overline{C}$) whose relative interior contains $p$.

# The Equivalent Description

**Lemma**

*Identify* $X_P$ *with* $\bigcup_{p \in P} \frac{S_N}{S_{N(\sigma_p)}} \times \{p\}$, *then* $X_P/W$ *is homeomorphic to*

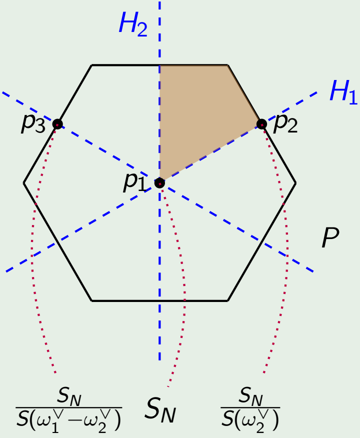
$$\frac{S_N \times (P \cap \overline{C})}{\sim_{ed}}$$

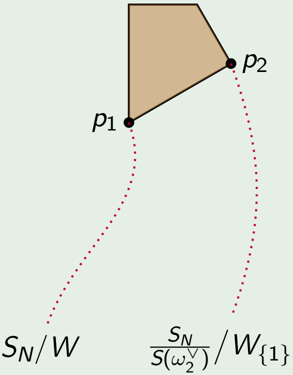
*where* $(t_1, p_1) \sim_{ed} (t_2, p_2)$ *iff* $p_1 = p_2$ *lying in the minimal face* $H_I F_J \cap \overline{C}$ *of* $P \cap \overline{C}$, *and* $t_1$, $t_2$ *represent the same element of* $\frac{S_N}{S_{N(\sigma_{p_1})}} / W_I$.

# Example (of Lie type $A_2$)

$$X_P = \bigcup_{p\in P} \frac{S_N}{S_{N(\sigma p)}} \times \{p\}$$

$$X_P/W = \frac{S_N \times (P \cap \overline{C})}{\sim_{ed}}$$



# The Map $\Phi$

The following diagram is commutative:

$$S_N \times (P \cap \overline{C}) \twoheadrightarrow \frac{S_N \times (P \cap \overline{C})}{\sim} \cong X_{P/W}$$

$$S_N \times (P \cap \overline{C}) \twoheadrightarrow \frac{S_N \times (P \cap \overline{C})}{\sim_{ed}} \cong X_P / W$$

$$X_P / W \overset{\Phi}{\dashrightarrow\!\!\!\!\rightarrow} \frac{S_N \times (P \cap \overline{C})}{\sim}$$

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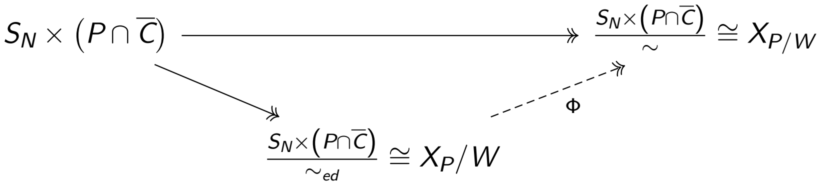
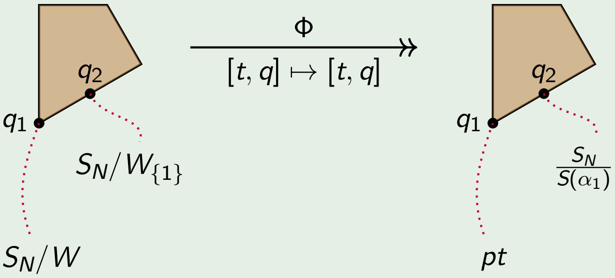



**Theorem (Smale'57)**

*Suppose $X, Y$ are (homeomorphic to) compact simplicial complexes, $f : X \to Y$ is onto. For each $y \in Y$, $f^{-1}(y)$ is locally $k$-connected and $k$-connected.*
*Then the induced homomorphism $f_* : \pi_r(X) \to \pi_r(Y)$ is an isomorphism for $0 \leq r \leq k$, onto for $r = k + 1$.*

# Example (of Lie type $A_2$)

$$X_P/W = \frac{S_N \times (P \cap \overline{C})}{\sim_{ed}} \qquad X_{P/W} = \frac{S_N \times (P \cap \overline{C})}{\sim}$$




$$\Phi^{-1}([t, q_1]) = S_N/W \cong \frac{\Delta^2}{\mathbb{Z}/3} \cong D^2.$$

$$\Phi^{-1}([e, q_2]) = S(\alpha_1)/W_{\{1\}} = \frac{S^1}{\mathbb{Z}/2} \cong D^1.$$

# Fibers of $\Phi$

## Lemma

*Suppose that $[t,p]$ is a point of $X_{P/W}$, and $p$ lies in the relative interior of some face $H_I F_J \cap \overline{C}$. Then the fiber of $\Phi$ is (locally) contractible:*

$$\Phi^{-1}([t,p]) \cong \frac{S_{N(\sigma_{[n],p})}}{S_{N(\sigma_p)}} \Big/ W_I \cong \frac{Span\left(\{\alpha_i\}_{i\in I}\right)}{\mathcal{SP}^\vee_{I,J}} \Big/ W_I,$$

*where* $\mathcal{SP}^\vee_{I,J} := Span\left(\{\alpha_i\}_{i\in I}\right) \bigcap \left(N + Span\left(\{\omega_j^\vee\}_{j\in J}\right)\right)$.

## Proof Sketch (of case $I = [n]$).

$$\Phi^{-1}([t,p]) = S_N/W \cong \frac{\overline{A}}{N/\mathcal{Q}^\vee} = \frac{polytope}{finite\ \ affine\ \ transformations}.$$

□

# Generalization

For $K \subset [n]$, $W_K \curvearrowright P$, and $P/W_K$ is a polytope contained in $V$.

**Theorem**

*There is a homotopy equivalence* $\Phi_K : X_P/W_K \to X_{P/W_K}$.

**Question ([Horiguchi-Masuda-Shareshian-Song'21])**

*Is there an* ***isomorphism*** *between* $X_P/W_K$ *and* $X_{P/W_K}$*?*

Thank You!