Geometric families of degenerations from mutations of polytopes

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Fields Institute Workshop on Toric Topology

August 19, 2024

Based on: ArXiv: 2408.01785 (Joint with Laura Escobar and Christopher Manon) and 2408.01788 (Joint with Adrian Cook and Laura Escobar and Christopher Manon)

Toric degenerations

Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope with vertices in $\mathbb{Z}^d$ and let $P \cap \mathbb{Z}^d = \{\alpha_0, \ldots, \alpha_n\}$.

The **toric variety** X_P is the closure of the image of the map $(\mathbb{C}^*)^d \to \mathbb{P}^n$ given by $x \mapsto (x^{\alpha_0}, \ldots, x^{\alpha_n})$.

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Question. Given a variety $X \subseteq \mathbb{P}^n$, is there a polytope P such that the toric variety X_P approximates X?

Definition. A **toric degeneration** of a variety $X \subseteq \mathbb{P}^n$ is a flat family $\pi : \mathfrak{X} \to \mathbb{C}$, where the general fiber is X and the special fiber is a toric variety X_P.

Newton–Okounkov bodies

Let X be a projective variety of dimension d.

Equip $\mathbb{Z}^{d+1}$ with a total order $\succ$. A function $\nu : \mathbb{C}[X] \setminus \{0\} \to \mathbb{Z}^{d+1}$ is a **valuation** if

- $\nu(f+g) \succeq \min\{\nu(f), \nu(g)\}$,
- $\nu(fg) = \nu(f) + \nu(g)$, and
- $\nu(c) = 0$ for all $c \in \mathbb{C}^*$.

Definition. (Okounkov, Lazarsfeld–Mustață, Kaveh–Khovanskii) The **Newton–Okounkov body** for (X, ν) is
$\Delta(X,\nu) := \overline{\text{cone}(\text{im}(\nu))} \cap (\{1\} \times \mathbb{R}^d)$.

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When the tropicalization of X is *well-behaved* [Kaveh–Manon, 2016] construct valuations ν such that $\Delta(X,\nu)$ is a lattice polytope.

Tropical geometry and Newton–Okounkov bodies

Choose a presentation $\mathbb{C}[x_1, \ldots, x_n]/I$ for $\mathbb{C}[X]$, I homogeneous.

$\mathrm{Trop}(I) := \{w \in \mathbb{R}^n \mid \mathrm{in}_w(I) \text{ contains no monomials}\}$.

$\mathrm{Trop}(I)$ is a fan in $\mathbb{R}^n$ with cones $C_w = \overline{\{x \in \mathbb{R}^n \mid \mathrm{in}_x(I) = \mathrm{in}_w(I)\}}$ for $w \in \mathbb{R}^n$.

A cone C of $\mathrm{Trop}(I)$ is **prime** if $\mathrm{in}_w(I)$ is prime for some/all $w \in C^\circ$.

Theorem. (Kaveh-Manon, 2016) Let C be a prime cone of $\mathrm{Trop}(I)$ and $\{u_1, \ldots, u_r\} \subset C$ be maximally linearly independent. There is a valuation ν_C of A such that its Newton-Okounkov body $\Delta(A, \nu_C) \subset \mathbb{R}^d$ is the convex hull of the columns of the matrix with rows $u_1, \ldots, u_r$.

Mutations of Newton-Okounkov bodies

Theorem. (Escobar–H, 2019) Let C_1 and C_2 be two prime cones of $\mathrm{Trop}(I)$ of maximal dimension sharing a codimension-1 face. There exist natural projections $\mathsf{p}_1, \mathsf{p}_2 : \mathbb{R}^d \to \mathbb{R}^{d-1}$ such that

$$\Delta(X, \nu_{C_1}) \xrightarrow{\mathsf{p}_1} \Delta_{C_1 \cap C_2} \xleftarrow{\mathsf{p}_2} \Delta(X, \nu_{C_2})$$

and the fibers are intervals of the same length (up to a global constant).

We obtain two piecewise-linear bijection $\Delta(X, \nu_{C_1}) \to \Delta(X, \nu_{C_2})$.

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Remark 1. Ilten interprets this piecewise-linear bijection as a generalization of the combinatorial mutations of Akhtar-Coates-Galkin-Kasprzyk. (In the context of their study of mirror symmetry for Fano manifolds)

Remark 2. In the case of the Grassmannian of 2-planes in $\mathbb{C}^m$ the second bijection is connected to cluster mutations.

Families of degenerations from mutations of polytopes

$X \rightsquigarrow \mathrm{Trop}(X) \rightsquigarrow$ a collection of Newton–Okounkov polytopes and piecewise-linear bijections between them.

... as suggested above, mutations of polytopes also appear in the theory of cluster algebras/varieties, mirror symmetry, the study of Fano manifolds/varieties....

"Million-dollar question": is there a systematic theory that can unify these??

Families of degenerations from mutations of polytopes

In [Escobar-H-Manon, 2024] we have proposed a theory which generalizes the theory of toric varieties by

- replacing the classical lattice $M \cong \mathbb{Z}^r$ with a collection of lattices which are related by piecewise-linear bijections ("mutations"), and
- replacing the Laurent polynomial ring $\mathbb{K}[x_1^{\pm}, \cdots, x_r^{\pm}]$, together with its usual valuation with a more general $\mathbb{K}$-algebra equipped with a valuation.

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By doing the above, we gain multiple benefits:

- We systematize and generalize the phenomenon in [Escobar–H].
- We exhibit a family $\{\mathfrak{X}_\alpha\}$ of degenerations of a single variety X, where each member $\mathfrak{X}_\alpha$ of the family has as its central fiber the toric variety associated to a polytope which is mutation-related to the others in the family.
- We develop a generalization of the classical theory of polytopes together with a combinatorics-geometry dictionary.

Polyptych lattices

A **polyptych lattice** of rank r is $\mathcal{M} = (\{M_i\}_{i \in I}, \{\mu_{ij}\}_{i,j \in I})$ such that

- $M_i \simeq \mathbb{Z}^r$.
- $\mu_{ij} : M_i \to M_j$ is a piecewise linear bijection.
- $\mu_{ii} = \text{id}$ and $\mu_{jk} \circ \mu_{ij} = \mu_{ik}$.

Example. The trivial polyptych lattice of rank r is $\mathcal{M}_\circ^r := (\{\mathbb{Z}^r\}, \{\text{id}\})$.

Example. Let $\mathcal{M}_2 = (\{M_1, M_2\}, \{\mu_{12}\})$, where $M_1 \simeq M_2 \simeq \mathbb{Z}^2$ and $\mu_{12}(x, y) = (\min\{0, y\} - x, y)$.

An **element** of $\mathcal{M}$ is $\mathrm{m} = (m_i)_{i \in I}$ such that for all $i \in I$, $m_i \in M_i$ and for all $i, j \in I$, $\mu_{ij}(m_i) = m_j$.

Given $\mathcal{S} \subseteq \mathcal{M}$, the *i*-**th chart** of $\mathcal{S}$ is the set $S_i := \{s \mid \exists m \in \mathcal{S}, m_i = s\}$.

Polyptych: "A sculpted or painted object composed of at least two, and usually more than three, panels. Most polyptychs functioned as altarpieces; the panels of some are hinged so that they may be opened and closed." (Oxford Reference)

Greek: "poly" = "many" , "ptukhe" = "fold"

Our collaborator Chris Manon likes to abbreviate this and call them "p-lattices". This has the advantage of linguistic suggestiveness:

- "p" for "polyptych"
- "p" for "piecewise (linear)"
- In particular, "polyptych lattice" abbreviates to "PL" which *also*, very conveniently, (or, confusingly?) suggests "Piecewise Linear" (!)

Suggestion from Lauren Williams (Mirror Symmetry Workshop, King's College London, June 2024):

- "p" for "pretentious" ?

PL halfspaces

A **point** of $\mathcal{M}$ is a collection $p = \{p_i : M_i \to \mathbb{Z} \mid i \in I\}$ such that

- $\forall i, j \in I$, $p_j \circ \mu_{ij} = pi_i$
- $\forall i \in I$ and $\forall m, m' \in M_i$, we have

$$p_i(m) + p_i(m') = \min_{j \in I}\{p_j(\mu_{ij}(m) + \mu_{ij}(m'))\}$$

where the $+$ on the RHS denotes addition in the lattice M_j.

We denote by $\mathrm{Sp}(\mathcal{M})$ the collection of points of $\mathcal{M}$.

Example. $\mathrm{Sp}(\mathcal{M}_2) \cong \{(a, a', b) \in \mathbb{Z}^3 \mid a + a' = \min(0, b)\}$.

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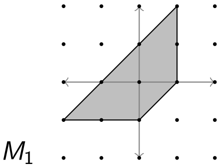
The **PL-halfspace** associated to $p \in \mathrm{Sp}(\mathcal{M})$ and $a \in \mathbb{Z}$ is $\mathcal{H}_{p,a} := \{\mathrm{m} \in \mathcal{M} \mid p(\mathrm{m}) \geq a\}$.

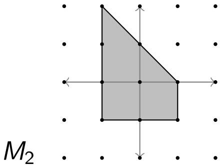
Example. A PL halfspace in $\mathcal{M}_2$:

PL polytopes

An **PL polytope** is a bounded finite intersection of PL halfspaces. A PL polytope is **integral** if for all $i \in I$ its chart in M_i is a lattice polytope.

Example. An integral PL polytope in $\mathcal{M}_2$:





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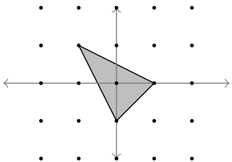
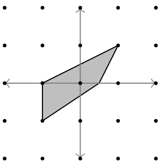
Example. A PL polytope that is **not** lattice:





Compactifications via PL polytopes

1. Toric case.

Let P be a lattice polytope in $\mathbb{R}^n$. The toric variety X_P is a compactification of the torus $\mathrm{Spec}(\mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}])$.

The homogeneous coordinate ring of the toric variety X_P is given by $\mathbb{C}[X_P] = \bigoplus_{k=0}^{\infty} \mathbb{C}[x^m \mid m \in \mathbb{Z}^r \cap kP]$.

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We have a valuation $\nu : \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] \to \{\text{PWL functions } \mathbb{Z}^r \to \mathbb{Z}\}$ given by $\nu(\sum c_\alpha x^\alpha) := \min_{c_\alpha \neq 0} \langle \alpha, - \rangle$.

The **support function** $\psi_P : \mathbb{Z}^r \to \mathbb{R}$ of a polytope P is defined by $\psi_P := \min\{\langle m, - \rangle \mid m \in P\}$.

We have that $\mathbb{C}[X_P] = \bigoplus_{k=0}^{\infty} \{f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \mid \nu(f) \geq \psi_{kP}\}$.

Detropicalization of a polyptych lattice

Assume that $\mathcal{M}$ is **dualizable**. Roughly, this means there is a p-lattice $\mathcal{N}$ and a pair of bijections $\mathfrak{N} : \mathcal{N} \to \mathrm{Sp}(\mathcal{M})$ and $\mathfrak{M} : \mathcal{M} \to \mathrm{Sp}(\mathcal{N})$.

Let $\mathcal{P}_{\mathcal{N}}$ be the semialgebra generated by $\mathrm{Sp}(\mathcal{N})$ with respect to the operations $\oplus := \min$ and $\odot := +$.

Definition. Given a domain $\mathcal{A}$, a function $\nu : \mathcal{A} \to \mathcal{P}_{\mathcal{N}}$ is a **valuation** if

- $\nu(f + g) \geq \nu(f) \oplus \nu(g)$,
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A **detropicalization** of $\mathcal{M}$ is a domain $\mathcal{A}$ together with a valuation $\nu : \mathcal{A} \to \mathcal{P}_{\mathcal{N}}$ such that $\mathrm{Sp}(\mathcal{N}) \subseteq \mathrm{im}(\nu)$.

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Example. For the trivial p-lattice of rank r, $\mathcal{M}_\circ^r := (\{\mathbb{Z}^r\}, \{\mathrm{id}\})$, recall that $\mathsf{Sp}(\mathcal{M}_\circ^r) = \mathrm{Hom}(\mathbb{Z}^r, \mathbb{Z})$. The dual is $\mathcal{M}_\circ^r$ and $\mathcal{P}_{\mathcal{M}_\circ^r}$ is the set of piecewise-linear concave functions on $\mathbb{Z}^r$.

The ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ is a detropicalization of $\mathcal{M}_\circ^r$ with valuation $\nu(\sum c_\alpha x^\alpha) := \bigoplus_{c_\alpha \neq 0} \langle \alpha, - \rangle$.

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Example. Let $\mathcal{A} = \mathbb{C}[x_1, x_2, t^{\pm 1}]/\langle x_1 x_2 - 1 - t\rangle$. There exists a valuation ν such that $(\mathcal{A}, \nu)$ is a detropicalization of $\mathcal{M}_2$.

Remark. For each $d, r \in \mathbb{N}$ we give a p-lattice $\mathcal{M}_{d,r}$ together with detropicalization $(\mathcal{A}_{d,r}, \nu_{d,r})$ where
$\mathcal{A}_{d,r} = \mathbb{C}[x_1, \ldots, x_d, t_1^{\pm 1}, \ldots, t_r^{\pm 1}]/\langle x_1 \cdots x_d - t_1 - \cdots - t_r\rangle$.

Compactifications via PL polytopes

2. PL case.

Let $\mathcal{A}$ be a detropicalization of $\mathcal{M}$ with valuation ν and Δ an integral PL polytope.

Let $\mathcal{N}$ be the dual of $\mathcal{M}$ with bijection $\mathfrak{M} : \mathcal{M} \to \mathsf{Sp}(\mathcal{N})$.

The **support function** $\psi_\Delta : \mathcal{N} \to \mathbb{R}$ of Δ is defined by $\psi_\Delta := \min\{\mathfrak{M}(\mathsf{m}) \mid \mathsf{m} \in \Delta \cap \mathcal{M}\}$.

Define the graded algebra $\mathcal{A}_\Delta := \bigoplus_{k=0}^{\infty}\{f \in \mathcal{A} \mid \nu(f) \geq \psi_{k\Delta}\}$.

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Theorem. (Escobar–H–Manon) $X_\Delta := \mathrm{Proj}(\mathcal{A}_\Delta)$ is a compactification of $\mathrm{Spec}(\mathcal{A})$. Moreover, for each $i \in I$, the chart image of Δ in M_i is a Newton–Okounkov body of X_Δ and these polytopes are connected by the PWL bijections μ_{ij}.

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Moreover,

- $\mathcal{A}_\Delta$ is finitely generated.
- If $\mathcal{A}$ is normal, then X_Δ is also normal.
- X_Δ is arithmetically Cohen-Macaullay.
- If $\mathcal{A}$ is a UFD, then X_Δ has a finitely generated class group and a finitely generated Cox ring.

We give a family of rank-2, two-chart examples in (Cook-Escobar-H-Manon), and also give lots of sample computations for this family, e.g. the PL analogue of Gorenstein-Fano polytopes.

Thank you!