Gelfand-Cetlin systems and analogous integrable systems

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1. Gelfand-Cetlin system

• Let $G = U(n)$. The Gelfand-Cetlin (or Gelfand-Zeitlin) system is a collection of functions on a (co)adjoint orbit of $U(n)$ (in other words a flag manifold).

• Let U be a Hermitian $n \times n$ matrix, so that $\sqrt{-1}U$ is an element of the Lie algebra of $U(n)$. The orbits of the adjoint action of G on the space of Hermitian matrices are the collections of Hermitian matrices with fixed eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$.

• The functions $F_{i,j}$ $(1 \leq i \leq n, 1 \leq j \leq n-i)$ are the eigenvalues of the square $(n-i) \times (n-i)$ submatrix in the top left corner.

Guillemin and Sternberg showed that the above functions $F_{i,j}$ Poisson commute and their Hamiltonian vector fields are linearly independent and span the tangent space to the flag manifold almost everywhere, but flag manifolds are not toric manifolds.

2. Flag manifolds are not in general toric manifolds

It is known that the only flag manifolds that are toric varieties are products of complex projective spaces.

This combines:

- Beilinson-Bernstein localization theorem
- Partial flag varieties are $\mathcal{D}$-affine, in other words they have a deformation quantization with a $\mathbf{C}^*$ action with respect to which they are affine. (Jesper Funch Thomsen, 1997)
- The only projective toric varieties that are $\mathcal{D}$-affine are products of projective spaces. (N. Hemelsoet 2022)

However there are open dense subsets of flag manifolds which are homeomorphic to $(C^*)^N$ (K. Kaveh 2017; M. Harada-K. Kaveh 2018)

Example: $SU(3)/T$ is a flag manifold (where T is the maximal torus). The Cox ring is not a polynomial ring so it is not a toric manifold See Y. Hu, S. Keel, “Mori dream spaces and GIT” (2000) See also “The Cox ring of a spherical embedding”, G. Gagliardi (2014)

3. Gelfand-Cetlin system

• The functions $F_{i,j}$ satisfy the Gelfand-Cetlin pyramid:

$$\lambda_1 \geq F_{1,n-1} \geq \lambda_2 \geq F_{1,n-2} \geq \lambda_3 \geq F_{1,n-3} \geq \cdots$$

and so on. Because the rank $(n-2)$ square matrix is a submatrix of the rank $n-1$ square matrix, we get

$$F_{1,n-1} \geq F_{2,n-2} \geq F_{1,n-2} \geq \cdots$$

This provides the second row of the pyramid.

• The *Gelfand-Cetlin polytope* is the polytope described by the above inequalities (where $\lambda_1, \ldots, \lambda_n$ are fixed).

• This polytope is not the Delzant polytope associated to any toric manifold, because at some vertices the number of edges is more than the dimension of the polytope. (Jongbaek Song, private communication; Presnova–Smirnov, 2023)

• See p. 4 of the following paper for a figure (Figure 1) displaying the pyramid:

D. Bouloc, E. Miranda, N.T. Zung, Singular fibres of the Gelfand-Cetlin system on $u(n)^*$, Phil. Trans. Roy. Soc. London A (2018)

4. Moduli spaces of flat connections

• It is not known how to classify similar examples. One that is known is the moduli space of equivalence classes of flat $SU(2)$ connections on an orientable compact 2-manifold Σ, also called *character varieties* (LJ– J. Weitsman 1992).

• In the language of flat connections and gauge transformations, the symplectic form is (Atiyah-Bott 1983)

$$\omega(a, b) = \int_{\Sigma} \mathrm{Trace}(a \wedge b).$$

where a, b are elements in the tangent space to the space of all connections, in other words Lie algebra valued 1-forms (think of these as matrices).

• Goldman 1984 section 1.4: the space is smooth if the representation of π into G has finite stabilizer (under conjugation).

• In this situation, the dimension of the space is the dimension of G times the absolute value of the Euler characteristic of the surface Σ.

Theorem (Goldman 1984 section 1.7): Suppose Σ has no boundary. The

2-form is a closed nondegenerate 2-form on the open subset of $\mathrm{Hom}(\pi, G)/G$ consisting of points where this space is smooth.

• Goldman 1984 used Atiyah-Bott's infinite-dimensional description of the form to conclude ω was closed.

Karshon (1992) gave the first proof that ω is closed using only group cohomology.

• For a surface Σ, the 2-form is closed only if Σ has no boundary components. Otherwise the character variety is Poisson, and the symplectic leaves are the subsets where the holonomy of the connections around each boundary component is a constant. These spaces are called *moduli spaces of parabolic bundles.*

• The Poisson bracket of these functions is as follows. (Goldman 1986)

These functions are obtained from a pants decomposition of the 2-manifold. The number of boundary circles is half the dimension of the moduli space. The functions are θ_j $(j = 1, \ldots, 3g - 3)$ where the holonomy of the connection around the j-th boundary circle is conjugate to a diagonal matrix with eigenvalues $e^{\pm i\theta_j}$ $(0 \leq \theta_j \leq \pi$. These satisfy inequalities

$$|\theta_1 - \theta_2| \leq \theta_3 \leq \theta_1 + \theta_2, \theta_1 + \theta_2 + \theta_3 \leq 2\pi.$$

• These inequalities define a tetrahedron, inscribed in the cube of side length π with vertices

$$(0,0,0), (\pi,\pi,0), (\pi,0,\pi), (0,\pi,\pi).$$

If $\theta_1, \theta_2, \theta_3$ correspond to the three boundary circles which form the boundary of a trinion (pair of pants), then these variables must satisfy these inequalities.

• The moment polytope for these circle actions is obtained by imposing these inequalities for every trinion in the trinion decomposition.

• There are $3g - 3$ boundary circles for any pants decomposition. Different pants decompositions give different polyhedra, but the volume of all polyhedra is the same. It is equal to the symplectic volume of the character variety.

• These flows Poisson commute. So we have $(3g-3)\dim(T)$ Poisson commuting flows. The dimension of the character variety is $(2g-2)\text{dim}(G)$. The number of flows is half the dimension of the moduli space only if $\text{dim}G = 3\text{dim}T$, in other words $SU(2), SO(4), SL(2,R), SL(2,C)$.

• The flow associated to a curve C is

$$\text{Hol}_C(A) \cong \text{diag}(e^{i\theta}, e^{-i\theta})$$

(the diagonal matrix with these eigenvalues).

• Goldman (1986) uses $\text{TraceHol}_C(A)$, which leads to periodic flows (when G is compact) but for which the period is not constant. So these flows do not come from a Hamiltonian torus action.

• To recover a Hamiltonian torus action we must use θ instead of $\text{TraceHol}_C(A)$.

For $SU(2)$, the Hamiltonian flow of the function θ is only well defined when the value of the holonomy around a curve C is not $\pm I$. Thus these functions are well defined on an open dense set. This suffices to determine the symplectic volume of the moduli space and the image of the moment map. The moment map is continuous but not differentiable at the values described above.

This is because $\theta = \cos^{-1}\mathrm{Trace}(\mathrm{Hol}_C A)/2)$ and the function $\cos^{-1}$ is not differentiable when $\cos(\theta) = \pm 1$. in other words
$\theta = \cos^{-1}\mathrm{Trace}(\mathrm{Hol}_C(A)/2) = 0, \pi$

5. Poisson structures

• Flag manifolds are orbits of the (co)adjoint action on the Lie algebra. The Poisson bracket comes from the Kirillov-Kostant-Souriau form on the orbits.

• Character varieties:

The Poisson bracket of these functions is as follows. (Goldman 1986)

The Poisson structure is given by the 2-form described above.

• Suppose α, β are based loops in Σ, giving rise to elements $[\alpha]$, $[\beta]$ in $\pi_1(\Sigma)$.

WLOG these based loops intersect transversely.

• Let $\alpha \cap \beta$ denote the set of intersections.

• Let $\epsilon(\pi, \alpha, \beta)$ be the intersection number at $p \in \alpha \cap \beta$.

• Let $\rho : \pi \to G$ be a representation.

• Then define $f_\alpha(\rho) = f(\rho(\alpha))$, where $f : G \to \mathbf{C}$.

• Let $A \in G$ and let $f : G \to \mathbf{R}$ be a map which is invariant under conjugation. Define $F(A)$ (an element of the Lie algebra of G) by

$$< F(A), V >= \frac{d}{dt}|_{t=0} f\Big((\exp tV)A\Big)$$

where $< \cdot, \cdot >$ is an Ad -invariant inner product on the Lie algebra of G and V is an element of the Lie algebra of G.

(Goldman, 1986)

• Here for $G = U(n)$ and $G = SU(n)$ and $f(A) = \mathrm{Trace}(A)$, it turns out that $F(A) = \frac{1}{2}(A - A^{-1})$. It can be shown that this is an element of the Lie algebra of G.

• When ρ is a homomorphism from π to G, the Poisson bracket is defined by

$$\{f_\alpha(\rho), g_\beta(\rho)\} = \sum_p \epsilon(p, \alpha, \beta) < F(\rho(\alpha)), G(\rho(\beta)) > .$$

• Here we sum over points p where α and β intersect.

6. Quantization

• These Gelfand-Cetlin-like systems have been quantized using a real polarization.

• Flag manifolds: Guillemin-Sternberg quantized flag manifolds using a real polarization by selecting integer values of the variables $F_{i,j}$ (Bohr-Sommerfeld quantization).

• Hamilton [Ham] studied a quantization using a real polarization, which did not include those Bohr-Sommerfeld points which were singular fibers in the foliation of the symplectic manifold by Lagrangian submanifolds. Both Guillemin-Sternberg and J-Weitsman included some singular Bohr-Sommerfeld fibers.

• Guillemin and Sternberg show that the number of such integer values (the number of entries in the Gelfand-Cetlin pyramid) is equal to the dimension of the quantization of the flag manifold using a Kähler polarization (Bott-Borel-Weil theorem). Their argument proceeds using a theorem of Hermann Weyl about restrictions of representations of $U(k)$ to $U(k-1)$.

• The Hamiltonian flow of the $F_{i,j}$ is not well defined if any eigenvalues of square submatrices are equal, The reason is that the function that sends a matrix to its eigenvalues is not differentiable when any eigenvalues are equal.

Moduli spaces of flat connections: LJ- J. Weitsman quantized the moduli space by selecting integer values of the variables θ_j provided these variables lie within the moment polytope. This quantization leads to the Verlinde formula, the known formula for quantization of this space (Beauville-Laszlo, Faltings, Kumar, Tsuchiya-Ueno-Yamada, Zagier).

• J. Uren (2011) wrote his PhD thesis under my supervision on the subject of the toric manifolds whose Gelfand-Cetlin polytope would coincide with the moment polytope for tbe Hamiltonian torus actions on character varieties studied by J-Weitsman . He found that in most cases these toric varieties are singular. For example many of these polytopes are not Delzant polytopes.

• Uren found for example that the polytope associated to a genus 2 surface with a pants decomposition consisting of two one-holed tori is a square pyramid. This has 4 edges coming from the apex of the pyramid, so it is not a Delzant polytope.

• By contrast, the polytope associated to a genus 2 wurface with the other pants decomposition for genus 2 is a tetrahedron. This is the moment polytope of complex projective space.

• For $SU(2)$, the Hamiltonian flow of the function θ is only well defined when the value of the holonomy around a curve C is not $\pm I$. Thus these functions are well defined on an open dense set. This suffices to determine the symplectic volume of the moduli space and the image of the moment map. The moment map is continuous but not differentiable at the values described above.

• We must include integer values on the boundary of the mmoment polytope, where the Hamiltonian flows are not necessarily well defined.

• The same is true for Guillemin-Sternberg's construction (they also include boundary values of the moment polytope).

• The Hamiltonian vector fields of these functions are undefined only when the group element is in more than one maximal torus (in other words the group element corresponds to a matrix with some repeated eigenvalues). In this situation it is impossible to define the Hamiltonian vector field. This occurs on a subset of measure 0.

For $SU(2)$ this happens when any of the matrices take the value I or $-I$ (where I is the identity matrix).

- Goldman instead uses functions $f_C(A) = \mathrm{TraceHol}_C(A)$. These functions are defined everywhere, and their Hamiltonian flows are periodic, but the period is not constant. It depends on the conjugacy class of $\mathrm{Hol}_C(A)$.

- J. Weitsman and P. Crooks (*J. Geom. Phys.* 2023) studied the quantization of the cotangent bundle of $U(n)$ using two distinct real polarizations. The quantization of the first led to the square integrable functions. The quantization on the other led to the direct sum of the endomorphisms of all irreducible representations of $U(n)$. The Peter-Weyl theorem identifies these two quantizations, giving another instance of independence of polarization.

- Lane (*Transformation Groups*, 2018) describes a generalization of Gelfand-Cetlin systems to groups other than $U(n)$ and systems other than coadjoint orbits.

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