Moduli spaces of weighted pointed stable curves and Grassmannians $G_{n,2}$

Svjetlana Terzić

University of Montenegro

based on joint work with Victor M. Buchstaber

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- We have recently studied the equivariant topology of Grassmann manifolds $G_{n,2}$ for the canonical action of the compact torus T^n.
- We constructed, by topological methods, a model $U_n = \Delta_{n,2} \times \mathcal{F}_n$ for $G_{n,2}/T^n$, where $\mathcal{F}_n$ is a compact smooth manifold, together with continuous projection $G_n : U_n \to G_{n,2}/T^n$.
- The projection G_n is a diffeomorphism between dense open subsets in U_n and $G_{n,2}/T^n$ and it preserves combinatorial structure of U_n and $G_{n,2}/T^n$.

- We proved that $\mathcal{F}_n$ is diffeomorphic to the well known Deligne - Mumford compactification $\overline{\mathcal{M}}_{0,n}$ for $\mathcal{M}_{0,n}$ by stable genus 0 curves with n-marked distinct points.
- This theory has been enriched by Hassett by introducing the notion of stable curves of genus g with weighted marked point and their moduli spaces.
- We present our results establishing relations between equivariant topology of Grassmann manifolds and moduli spaces of weighted stable genus 0 curves.

Moduli spaces $\mathcal{M}_{0,n}$

Example

- For $n = 3$ any $(\mathbb{C}P^1, s_1, s_2, s_3) \cong (\mathbb{C}P^1, 0, 1, \infty)$, that is $\mathcal{M}_{0,3}$ is a point.
- For $n = 4$ any $(\mathbb{C}P^1, s_1, s_2, s_3, s_4) \cong (\mathbb{C}P^1, 0, 1, \infty, t)$, where $t \neq 0, 1, \infty$. Thus, $\mathcal{M}_{0,4} = \mathbb{C}P^1 \setminus \{0, 1, \infty\}$.

Generally $(\mathbb{C}P^1, s_1, \ldots, s_n) \cong (\mathbb{C}P^1, 0, 1, \infty, t_1, \ldots, t_{n-3})$, that is

$$\mathcal{M}_{0,n} = \{(t_1, \ldots, t_{n-3}) \in (\mathbb{C}P^1)^{n-3} | t_i \neq 0, 1, \infty, t_i \neq t_j\}$$

- $\mathcal{M}_{0,n}$ is not compact for $n \geq 4$.

Compactification $\overline{\mathcal{M}}_{0,n}$ for $\mathcal{M}_{0,n}$

Problem of "good" compactification of $\mathcal{M}_{0,n}$: it should be itself a moduli space parametrazing natural generalization of the object of $\mathcal{M}_{0,n}$

- Deligne-Mumford ("69) equivalent to Grotendieck-Knudsen compactification ('72, '83) is $\overline{\mathcal{M}}_{0,n}$. It is a smooth manifold.

Stability condition for $(C = \mathbb{C}P^1, s_1, \ldots, s_n)$ requires:

$$K_C + \sum_{i=1}^{n} s_i \text{ is an ample divisor, where } K_C \text{ it the canonical class of } C.$$

Example: $\overline{\mathcal{M}}_{0,4} = \mathbb{C}P^1, \quad \overline{\mathcal{M}}_{0,5}$ - del Pezzo surface of degree 5.

Moduli space $\mathcal{M}_{0,\mathcal{A}}$ of $\mathcal{A}$-weighted stable curves

A curve $(C = \mathbb{C}P^1, S = \{s_1, \ldots, s_n\})$ is said to be $\mathcal{A}$-weighted stable by Hassett ('03) if:

1. It is given a function $\mathcal{A} : S \to \mathbb{Q}$ such that $0 < \mathcal{A}(i) \leq 1$.
2. $K_C + \sum_{i=1}^{n} \mathcal{A}(i) s_i$ is an ample divisor,
3. $\sum_{i \in I} \mathcal{A}(i) \leq 1$ for $I \subset S$ such that s_i pairwise coincide for $i \in I$.

It follows:

- $\sum_i \mathcal{A}(i) > 2$; $\quad \mathcal{M}_{0,\mathcal{A}} = \overline{\mathcal{M}}_{0,n}$ for $\mathcal{A} = (1, \ldots, 1)$
- Proved by Hassett:

  $\overline{\mathcal{M}}_{0,\mathcal{A}}$ is a smooth connected Deligne-Mumford stack proper over $\mathbb{Z}$.

  .

Reduction and forgetting morphisms

Theorem (Hassett '03)

Fix g and n.

- *Let* $\mathcal{A} = (a_1, \ldots, a_n)$ *and* $\mathcal{B} = (b_1, \ldots, b_n)$ *be such that* $b_i \leq a_i$ *for* $i = 1, \ldots n$. *There exists birational reduction morphism*

$$\rho_{\mathcal{B},\mathcal{A}} : \overline{\mathcal{M}}_{0,\mathcal{A}} \to \overline{\mathcal{M}}_{0,\mathcal{B}}.$$

- *Let* $\mathcal{A} = (a_1, \ldots, a_n)$ *and* $\mathcal{A}' = \{a_{i_1}, \ldots, a_{i_r}\} \subset \mathcal{A}$ *such that* $a_{i_1} + \ldots + a_{i_r} > 2$. *There exists natural forgetting morphism*

$$\Phi_{\mathcal{A},\mathcal{A}'} : \overline{\mathcal{M}}_{0,\mathcal{A}} \to \overline{\mathcal{M}}_{0,\mathcal{A}'}.$$

Let $(C, s_1, \ldots s_n) \in \overline{\mathcal{M}}_{0,\mathcal{A}}$. *The points* $\rho_{\mathcal{B},\mathcal{A}}(C, s_1, \ldots, s_n)$ *and* $\Phi_{\mathcal{A},\mathcal{A}'}(C, s_1, \ldots, s_n)$ *are obtained by successively collapsing components of C along which* $K_C + b_1 s_1 + \ldots + b_n s_n$, *that is* $K_C + a_{i_1} s_{i_1} + \ldots + a_{i_r} s_{i_r}$ *fails to be ample.*

Domain of admissible weight data $\mathcal{D}_{0,n}$

$$\mathcal{D}_{0,n} = \{(a_1, \ldots, a_n) \in \mathbb{R}^n | 0 < a_j \leq 1,\ a_1 + \ldots + a_n > 2\}.$$

The coarse chamber decomposition $\{\mathcal{W}_c\}$ of $\mathcal{D}_{0,n}$ is defined by the lattice of hyperplane arrangement

$$\mathcal{W} = \left\{ w_S : \sum_{j \in S} a_j = 1 | S \subset \{1, \ldots, n\}, 2 < |S| < n - 2 \right\}$$

intersected with $\mathcal{D}_{0,n}$. The boundary $\nabla \mathcal{D}_{0,n}$ of $\mathcal{D}_{0,n}$ is defined by

$$\nabla \mathcal{D}_{0,n} = \{(t_1, \ldots, t_n) \in \mathbb{R}^n | 0 < t_i < 1,\ t_1 + \ldots + t_n = 2\}.$$

Note:

$$\nabla \mathcal{D}_{0,n} = \overset{\circ}{\Delta}_{n,2}$$

Relation to Geometric Invariant Theory (GIT) quotients

- Consider $(\mathbb{C}P^1)^n$ with diagonal action of $PGL_2(\mathbb{C})$;
- Let $L_i = \mathcal{O}(-1)$ be a canonical line bundle on i-th factor $\mathbb{C}P^1$ and $\mathcal{L} = (t_1, \ldots, t_n) \in \mathbb{Q}^n$, then $\mathcal{O}(\mathcal{L}) = \otimes_{i=1}^n L_i^{t_i}$ is an ample line bundle and it defines a fractional linearisation of this action.
- We assume it is renormalised such that $t_1 + \ldots + t_n = 2$.
- The (semi) stability condition for a point $(x_1, \ldots, x_n) \in (\mathbb{C}P^1)^n$ can be formulated by: for any $\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$, $x_{i_1}, \ldots, x_{i_r}$ may coincide only when $t_{i_1} + \ldots + t_{i_r} (\leq) < 1$
- $\mathcal{O}(\mathcal{L})$ is typical when all semistable point are stable and it is atypical otherwise.
- Let $\mathcal{G}(\mathcal{O})$ be the GIT-quotient of $(\mathbb{C}P^1)^n$ defined by linearisation $\mathcal{O}$ of $PGL_2(\mathbb{C})$-action.

Theorem (Hassett '03)

For a typical linearisation $\mathcal{O}(\mathcal{L}), \mathcal{L} \in \nabla\mathcal{D}_{0,n}$ there exists an open neighborhood U of $\mathcal{L}$ such that $U \cap \mathcal{D}_{0,n}$ is contained in an open chamber of $\mathcal{D}_{0,n}$. For any $\mathcal{A} \in U \cap \mathcal{D}_{0,n}$, there is a natural isomorphism

$$\overline{\mathcal{M}}_{0,\mathcal{A}} \stackrel{\cong}{\to} \mathcal{G}(\mathcal{O}).$$

Theorem (Hassett '03)

For an atypical linearisation $\mathcal{O}(\mathcal{L}), \mathcal{L} \in \nabla\mathcal{D}_{0,n}$, suppose that $\mathcal{L}$ is in the closure of the chamber associated with $\mathcal{A}$. There exists a natural birational morphism

$$\rho : \overline{\mathcal{M}}_{0,\mathcal{A}} \to \mathcal{G}(\mathcal{O}).$$

Losev-Manin spaces $\overline{L}_{0,S}$

The spaces $\overline{L}_{0,S}$ are defined (Losev & Manin ('00)) as the moduli spaces of

- stable curves of genus 0 endowed with a family of painted by black or white (≥ 2) points labeled by S,
- all white points are pairwise distinct and distinct from black ones, while black points are without restrictions.

Let $S = W \cup B$ and $\mathcal{A}$ a weight data on S:

$$a_s = 1 \text{ for all } s \in W, \text{ and } \sum_{t \in B} a_t \leq 1.$$

Theorem

- $\overline{L}_{0,S} = \overline{\mathcal{M}}_{0,\mathcal{A}}$, *(Manin, '04)*
- $\overline{\mathcal{M}}_{0,\mathcal{A}}$ *does not depend on* $(a_{i_{k+1}}, \ldots, a_{i_n})$ *such that* $0 < a_{i_j} < 1$ *and* $\sum_{j=k+1}^{n} a_{i_j} \leq 1$, *where* $k = |W|$, *(B & T ,'24)*

The structure of the model U_n

- T^n action on $G_{n,2}$ extends to the canonical $(\mathbb{C}^*)^n$-action.
- $\mu : G_{n,2} \to \Delta_{n,2} \subset \mathbb{R}^n$ be the standard moment map

$$\mu(L) = \frac{1}{\sum_I |P^I(L)|^2} \sum_I |P^I(L)|\Lambda_I,$$

$\Lambda_I \in \mathbb{Z}^n$, $\Lambda_I(i) = 1$ iff $i \in I$ otherwise it is zero.

Strata

Strata $\{W_\sigma\}$ on $G_{n,2}$: non-empty sets of the form

$$W_\sigma = \{L \in G_{n,2} | P^I(L) \neq 0 \text{ iff } I \in \sigma\}.$$

where $\sigma \subset \{I \subset \{1, \dots, n\} : |I| = 2\}$ - admissible set.

- $G_{n,2} = \cup W_\sigma$ -disjoint union, W_σ is $(\mathbb{C}^*)^n$-invariant

Proposition (B & T, '19)

- $\mu(W_\sigma) = \overset{\circ}{P}_\sigma$, where $P_\sigma = \text{convhull}(\Lambda_I, I \in \sigma)$

P_σ is called an admissible polytope.

Theorem (B& T, '19)

$F_\sigma = W_\sigma/(\mathbb{C}^*)^n$ is an algebraic variety and

$$W_\sigma / T^n \cong \overset{\circ}{P}_\sigma \times F_\sigma.$$

Main stratum

The main stratum

$$W_n = \{L \in G_{n,2} | P^I(L) \neq 0 \text{ for any } I \subset \{1, \dots, n\},\ |I| = 2\}.$$

It is an open, dense set in $G_{n,2}$ and there is a homeomorphism:

$$W_n/T^n \cong \overset{\circ}{\Delta}_{n,2} \times F_n \ - \ \text{open, dense in } G_{n,2}/T^n,$$

$F_n = W_n/(\mathbb{C}^*)^n = \{((c_{ij} : c_{ij}^{'})) \in (\mathbb{C}P_A^1)^N | c_{ij}c_{ik}^{'}c_{jk} = c_{ij}^{'}c_{ik}c_{jk}^{'}\},$
$N = \binom{n-2}{2}, \quad 3 \leq i < j \leq n,\ \mathbb{C}P_A^1 = \mathbb{C}P^1 \setminus A,$
$A = \{(1:0), (0:1), (1:1)\}.$

In what follows:

We construct the compactification U_n for W_n/T^n which is a model for $G_{n,2}/T^n$ and describe its outgrows.

Chamber decomposition of $\Delta_{n,2}$

Chambers C_ω, $\omega \subset \{\sigma | \sigma \text{ admissible set}\}$ are given by

$$C_\omega = \bigcap_{\sigma\in\omega} \mathring{P}_\sigma \neq \emptyset \text{ and } C_\omega \cap \mathring{P}_\sigma = \emptyset \text{ for } \sigma \notin \omega.$$

- $\hat{\mu} : G_{n,2}/T^n \to \Delta_{n,2}$ - induced by the moment map
- $\hat{C}_\omega = \hat{\mu}^{-1}(C_\omega) \subset G_{n,2}/T^n$.

Proposition (B & T, '22)

$$h_\omega : C_\omega \times F_\omega \cong \hat{C}_\omega, \quad \hat{\mu} \circ h_\omega = pr_1$$

where

- $F_\omega = \cup_{\sigma\in\omega} F_\sigma$ - disjoint union, $F_n \subset F_\omega$,
- F_ω is a compact space - a compactification of F_n.

$$G_{n,2}/T^n = \cup_\omega \hat{C}_\omega = \cup_\omega C_\omega \times F_\omega$$

Universal space of parameters $\mathcal{F}_n$

It is a compactification $\mathcal{F}_n$ for F_n such that:

- there exists the projection $G_n : \Delta_{n,2} \times \mathcal{F}_n \to G_{n,2}/T^n$.
- $G_n^{-1}(W_\sigma/T^n) \cong \overset{\circ}{P}_\sigma \times \tilde{F}_\sigma$ for some $\tilde{F}_\sigma \subset \mathcal{F}_n$, where $\widetilde{F}_\sigma$ is called a virtual space of parameters for W_σ,
- there exists the projection $p_\sigma : \tilde{F}_\sigma \to F_\sigma$ such that

  $$\overset{\circ}{P}_\sigma \times \widetilde{F}_\sigma \xrightarrow{G_n} W_\sigma/T^\sigma \xrightarrow{h_\sigma} \overset{\circ}{P}_\sigma \times F_\sigma$$

  coincides with

  $$\overset{\circ}{P}_\sigma \times \widetilde{F}_\sigma \xrightarrow{(I_d, p_\sigma)} \overset{\circ}{P}_\sigma \times F_\sigma.$$

Wonderful compactification - application to $G_{n,2}$

Note:

The natural compactification $\bar{F}_n$ of F_n in $(\mathbb{C}P^1)^N$ given by $c_{ij}c'_{ik}c_{jk} = c'_{ij}c_{ik}c'_{jk}$, which is a smooth algebraic variety, does not satisfy our conditions on $\mathcal{F}_n$.

- De Concini & Processi ('95), Fulton & MacPherson ('94) and Li ('09) introduced the notion of wonderful compactification of a smooth algebraic variety generated by a building set of its smooth subvarieties.

Theorem (B & T, '23)

There exists a building set $\mathcal{G}_n$ in $\bar{F}_n$ such that the smooth, compact manifold $\mathcal{F}_n$ obtained as the wonderful compactification of $\bar{F}_n$ generated by $\mathcal{G}_n$ is the universal space of parameters for $G_{n,2}$.

Virtual spaces of parameters

- $\mathcal{F}_n = \cup_\sigma \tilde{F}_\sigma, \quad \tilde{F}_n = F_n,$
- there exists a projection $p_\sigma : \tilde{F}_\sigma \to F_\sigma$ for any σ,

Theorem (B & T, '22)

For any $C_\omega \subset \overset{\circ}{\Delta}_{n,2}$ it holds

$$\bigcup_{\sigma\in\omega} \tilde{F}_\sigma = \mathcal{F}_n.$$

This union is disjoint, which implies that it is defined the projection

$$p_\omega : \mathcal{F}_n \to F_\omega \ \text{ by } \ p_\omega(y) = p_\sigma(y), \ \ y \in \tilde{F}_\sigma.$$

The proof of the theorem uses that $F_\omega = \cup_{\sigma\in\omega} F_\sigma$ - disjoint union.

Model for $G_{n,2}/T^n$

It is $(U_n = \Delta_{n,2} \times \mathcal{F}_n, G_n)$, where $G_n : U_n \to G_{n,2}/T^n = \cup \hat{C}_\omega$ is given by

$$G_n(x, y) = h_\omega(x, p_\omega(y)) \text{ for } x \in C_\omega.$$

On $\partial\Delta_{n,2} \times \mathcal{F}_n$ the projection G_n is defined successively using the following:

- $\hat{\mu}^{-1}(\partial\Delta_{n,2}) = (\cup_{q=1}^{n} G_{n-1,2}/T^{n-1}(q)) \cup (\cup_{q=1}^{n} \Delta^{n-2}(q))$
- Over $\Delta^{n-2}(q)$ contracts into a point.
- $\mathcal{F}_{n-1,q} \cong \mathcal{F}_{n-1}$ is universal space of parameters for $G_{n-1,2}/T^{n-1}(q)$.

Proposition (B & T, '23)

The space $\mathcal{F}_{n-1,q}$ is the image of the projection $r_q : \mathcal{F}_n \to \mathcal{F}_{n-1}$ defined by forgetting the coordinates indexed by q:

$$\mathcal{F}_{n-1,q} = \mathcal{F}_n|_{(c_{ij}:c'_{ij}),i,j\neq q},$$

Summary

The structural data for the model (U_n, G_n) are:

- $\mathcal{F}_n$ - universal space of parameters,
- $\tilde{F}_\sigma$ - virtual spaces of parameters,
- F_σ and F_ω - spaces of parameters of strata and over chambers
- $p_\sigma : \tilde{F}_\sigma \to F_\sigma, \;\; p_\omega : \mathcal{F}_n \to F_\omega, \quad r_q : \mathcal{F}_n \to \mathcal{F}_{n-1,q}$ - projections

We present the realization of this model in terms of the moduli spaces of weighted curves and morphisms between them.

Spaces of parameters as moduli spaces of weighted curves

We prove:

Proposition

$$F_\sigma \cong \mathcal{M}_{0,m},\ 3 \leq m \leq n-1$$

Theorem (B & T, '23)

$$\mathcal{F}_n \text{ is diffeomorphic to } \overline{\mathcal{M}}_{0,n} = \overline{\mathcal{M}}_{0,\mathcal{A}_0},$$

where $\mathcal{A}_0 = (1, \ldots, 1)$.

From the results of D. Mumford on categorical quotients and F. Kirwan on GIT-quotients it follows:

Proposition

$$F_\omega \cong \mathcal{G}(\mathcal{O}),$$

where $\mathcal{O} = \mathcal{O}(\mathcal{L})$, for $\mathcal{L} = (t_1, \ldots, t_n) \in C_\omega$

Corollary (B & T, '24)

- If $\dim C_\omega = n - 1$ there exists $\mathcal{A} \in \mathcal{D}_{0,n}$ such that $F_\omega \cong \overline{\mathcal{M}}_{0,\mathcal{A}}$
- If $\dim C_\omega < n - 1$ there exists $\mathcal{A} \in \mathcal{D}_{0,n}$ and birational morphism $\overline{\mathcal{M}}_{0,\mathcal{A}} \to F_\omega$.

Proposition (B & T, '24)

Any $\tilde{F}_\sigma$ is homeomorphic to $\mathcal{F}_s \times \mathcal{F}_l \times \hat{F}_q$, where:

- $\hat{F}_q \cong F_m,\ m \leq q$ or
- $\hat{F}_q$ is a wonderful compactification of $\{(c_{ij} : c^{'}_{ij}) \in (\mathbb{C}P^1)^{N_q} | c_{ij}c^{'}_{il}c_{jl} = c^{'}_{ij}c_{il}c^{'}_{jl},\ c_{ij}, c^{'}_{ij} \neq 0\}$ for $N_q = \binom{q-2}{2}$, with the building set induced from $\mathcal{G}_n$,

  $3 \leq p, q, l \leq n-2$ and $p+q+l = n+4$.

Let

- $\mathcal{C}_{n,2}$ be the family of chambers in $\overset{\circ}{\Delta}_{n,2}$,
- $C_{\mathcal{H}}$ be the family of chambers of $\mathcal{W}_c$ in $\mathcal{D}_{0,n}$.

Lemma

There exists injective map $\xi : \mathcal{C}_{n,2} \to \mathcal{C}_{\mathcal{H}}$ defined by

$$\xi(C_\omega) = D_\omega,\ C_\omega \subset \overline{D}_\omega,$$

where D_ω is a such of smallest dimension.

Theorem (B & T, '24)

The projection $r_q : \mathcal{F}_n \to \mathcal{F}_{n-1,q}$ *and forgetting map* $\Phi_{\mathcal{A},\mathcal{A}'} : \overline{\mathcal{M}}_{0,\mathcal{A}} \to \overline{\mathcal{M}}_{0,\mathcal{A}'}$ *coincide,*
where $\mathcal{A} = (1,\dots,1)$ *and* $\mathcal{A}' = \pi^q(\mathcal{A})$ *- forgetting q-th coordinate.*

Theorem (B & T, '24)

The projection $p_\omega : \mathcal{F}_n \to F_\omega$, $C_\omega \subset \overset{\circ}{\Delta}_{n,2}$, $\dim C_\omega = n-1$ *and reduction map* $\rho_{\mathcal{A}_0,\mathcal{B}} : \overline{\mathcal{M}}_{0,\mathcal{A}_0} \to \overline{\mathcal{M}}_{0,\mathcal{B}}$ *coincide, where* $\mathcal{A}_0 = (1,\dots,1)$ *and* $\mathcal{B} = (b_1,\dots,b_n) \in D_\omega$*, where* $D_\omega = \xi(C_\omega)$.

Losev-Manin spaces and spaces of parameters

$L_{0,n,k} \cong \overline{\mathcal{M}}_{0,\mathcal{A}}$ for $\mathcal{A} \in D_{0,n,k}$ - a chamber in $\mathcal{D}_{0,n}$ defined by

$$x_i + x_l > 1,\ 1 \leq i \leq k, 1 \leq 1 \leq n, l \neq i,\ x_{k+1} + \ldots + x_n < 1.$$

Lemma

The chamber $D_{0,n,k}$ is not in the image of the map $\xi : \mathcal{C}_{n,2} \to \mathcal{C}_{\mathcal{H}}$.

We prove:

Proposition

There exists $0 < b_1, \ldots, b_k < 1$ such that:

- $\mathcal{B} = (b_1, \ldots, b_k, a_{k+1}, \ldots, a_n) \in \mathcal{D}_{0,n}$,
- for D_ω such that $\mathcal{B} \in D_\omega$ there exists $C_\omega \subset \{x_{k+1} + \ldots + x_n < 1\}$, $\dim C_\omega = n - 1$ such that $D_\omega = \xi(C_\omega)$.
- $\overline{\mathcal{M}}_{0,\mathcal{B}} \cong F_\omega$

Then reduction $\rho_{\mathcal{A},\mathcal{B}}$ theorem gives:

Corollary

There exists $C_\omega \subset \{x_{k+1} + \ldots + x_n \leq 1\}$, $\dim C_\omega = n-1$ and birational morphism $\overline{L}_{0,n,k} \to F_\omega$.

This can be generalized :

Theorem (B& T', 24)

For any $C_\omega \subset \overset{\circ}{\Delta}_{n,2}$ such that

- $\dim C_\omega = n-1$ *and* $C_\omega \subset \{x_{k+1} + \ldots + x_n \leq 1\}$ *there exists a Losev-Manin space* $\overline{L}_{0,n,k}$ *and a birational morphism* $\overline{L}_{0,n,k} \to F_\omega$.
- $\dim C_\omega \leq n-2$ *there exists a Losev-Manin space* $\overline{L}_{0,n,k}$ *and a birational morphism* $\overline{L}_{0,n,k} \to F_\omega$.

Losev-Manin spaces and wonderful compactification

There exist:

- natural blown-down birational morphism $\mathcal{F}_n \to \overline{F}_n$,
- natural reduction birational morphism $\mathcal{F}_n \to \overline{L}_{0,n,2}$.

Theorem (B & T, 24)

The divisors for these morphisms coincide, that is

$$\overline{L}_{0,n,2} \text{ is isomorphic to } \bar{F}_n \subset (\mathbb{C}P^1)^N,\ N = \binom{n-2}{2}.$$

Demonstration $n = 5$

1.
   - Let $\mathcal{A} = (1, 1, \frac{5}{18}, \frac{5}{18}, \frac{5}{18})$, $\mathcal{A} \in D_{0,5,2} \subset \mathcal{D}_{0,5}$, $\overline{\mathcal{M}}_{0,\mathcal{A}} = \bar{L}_{0,5,2} \cong \bar{F}_5$

   $$D_{0,5,2} : t_1 + t_i > 1,\ t_2 + t_i > 1,\ i \neq 1, 2,\ t_3 + t_4 + t_5 < 1.$$

2.
   - Let $\mathcal{O} = (\frac{3}{6}, \frac{4}{6}, \frac{5}{18}, \frac{5}{18}, \frac{5}{18})$, $\mathcal{O} \in \overset{\circ}{\Delta}_{5,2}$, it is typical linearisation,
   - $\mathcal{O} \in C_\omega \subset \overset{\circ}{\Delta}_{5,2}$: $t_1 + t_2 > 1$, and $t_i + t_j < 1$ for others.
   - $F_\omega \cong \mathcal{G}(\mathcal{O}) \cong \overline{\mathcal{M}}_{0,\mathcal{B}}$ for $\mathcal{B} \in U(\mathcal{O}) \cap \mathcal{D}_{0,5}$

It holds:

Lemma

$F_\omega \cong \bar{F}_5 \subset (\mathbb{C}P^1)^3$.

Corollary

$\overline{L}_{0,5,2}$ is contained in the orbit space $G_{5,2}/T^5$.

Toric varieties and spaces of weighted genus 0 curves

Recall (Manin ('04), Kapranov ('93), Gelfand and Serganova ('87)):

- $\overline{L}_{0,n,2}$ is a smooth, projective toric variety $\mathcal{X}(P_e^{n-3})$ over the permutahedron P_e^{n-3} - permutohedral variety
- $\overline{L}_{0,n,2}$ can be obtained as the closure of a principal $(\mathbb{C}^*)^{n-3}$-orbit in $Fl(n-2)$ - complete complex flag manifolds.

Corollary

$\mathcal{X}(P_e^{n-3})$ can be mapped by birational morphism to a variety in the orbit space $G_{n,2}/T^n$, $n \geq 6$.

Graph associahedra toric varieties do not much intersect with Hassett spaces:

Theorem (Da Rosa, Jensen, Ranganathan ('16))

A toric variety $\mathcal{X}(\mathcal{P}\Gamma)$ is isomorphic to a moduli space of stable genus 0 curves if and only if Γ is an iterated cone over a discrete set.

Corollary

The graph associahedron toric variety for a graph $\Gamma = \mathrm{Cone}^{n-k-2}(\cup_{i=1}^{k} v_i)$ on $(n-2)$-vertices, can be mapped by birational morphism to a variety in $G_{n,2}/T^n$.

$G_{n,2}/T^n$ as universal object for $\{\overline{\mathcal{M}}_{0,\mathcal{A}}\}$

The reduction morphisms $\rho_{\mathcal{A}_0,\mathcal{A}}$ give:

Lemma

For a fixed n all moduli spaces $\overline{\mathcal{M}}_{0,\mathcal{A}}$, $|\mathcal{A}| = n$, are birationally equivalent.

Note: There are no birational morphisms between them in general.

Lemma

For a fixed n and any $2 \leq k_1 < k_2 \leq n-2$ there exists a birational morphism $\overline{L}_{0,n,k_2} \to \overline{L}_{0,n,k_1}$.

The existence of birational morphisms $\overline{L}_{0,n,k} \to F_\omega$, for $C_\omega \subset \overset{\circ}{\Delta}_{n,2}$ implies:

Proposition

For a fixed n, all spaces of parameters F_ω of the chambers $C_\omega \subset \overset{\circ}{\Delta}_{n,2}$ are birationally equivalent.

Note: There are no birational morphisms between them in general.

Altogether:

Theorem (B& T, '24)

For any moduli space $\overline{\mathcal{M}}_{0,\mathcal{A}}$ there exists a birational morphism to F_ω for some chamber $C_\omega \subset \overset{\circ}{\Delta}_{n,2}$.

Geometric complexity of a torus action

The cases of effective actions of T^k on M^{2n} with a moment $\mu : M^{2n} \to P^k$ and the induced moment map $\hat{\mu} : M^{2n}/T^k \to P^k$ one can divide as follows:

1. $k = n$ and $\hat{\mu}^{-1}(x)$ is a point for $x \in P^k$

   Ex: toric and quasitoric manifolds

2. $k = n - 1$ and $\hat{\mu}^{-1}(x) \cong \hat{\mu}^{-1}(y)$ for any $x, y \in \overset{\circ}{P^k}$,

   Ex: T^4-action on $G_{4,2}$, T^3-action on $F_3 = U(3)/T^3$, Hamiltonian T^{n-1}-action on a symplectic manifold M^{2n}

3. $k \leq n - 2$ and there are $x, y \in \overset{\circ}{P^k}$ such that $\mu^{-1}(x) \ncong \mu^{-1}(x)$,

   Ex: T^n action on $G_{n,2}$, $n \geq 5$

Problem: Classify the spaces $\hat{\mu}^{-1}(x)$, $x \in \overset{\circ}{P^k}$ up to homeomorphism or birational equivalence.

According to our results, it is for Grassmannians $G_{n,2}$ equivalent to:

Problem: Formulate classification criterion for F_ω in terms of Hassett and Losev-Manin theory.

- For $(\mathbb{C}^*)^k$ -action on M^{2m}, $k \leq m$, the complexity is defined by $d = m - k$.
- For $M^{2m} = G_{n,2}$, $m = 2(n-2)$ with the canonical $(\mathbb{C}^*)^n$ - action $d = n - 1$ and

$$F_\omega = \cup W_\sigma / (\mathbb{C}^*)^n, \ \ \dim \cup W_\sigma = 2(n-2).$$

Our research is related to developing toric topology of positive complexity.

- V. M. Buchtaber, S. Terzić, *Moduli spaces of weighted pointed stable curves and toric topology of Grassmann manifolds*, preprint, 2024
- V. M. Buchstaber, S. Terzić, *Resolution of Singularities of the Orbit Spaces $G_{n,2}/T^n$*, Proc. Steklov Inst. Math., **317** (2022), 21–54.
- V. M. Buchstaber, S. Terzić, *The orbit spaces $G_{n,2}/T^n$ and the Chow quotients $G_{n,2}//(\mathbb{C}^*)^n$ of the Grassmann manifolds $G_{n,2}$*, Mat. Sbornik (2023)
- V. M. Buchstaber, S. Terzić, *The foundations of $(2n, k)$-manifolds*, Mat. Sbornik, **210**, Iss 4, (2019), 508–549
- B. Hassett, *Moduli spaces of weighted pointed stable curves*, Advances in Mathematics, **173**, Iss. 2, (2003), 316-352.
- A. Losev, Y. Manin, *New moduli spaces of pointed curves and pencils of flat connections*, Michigan Journ. of Math., **48** (Fulton's Festschrift), (2000), 443–472.