Buchstaber Numbers and Toric Wedge Induction

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joint work with Hyeontae Jang and Mathieu Vallée

Workshop on Toric Topology
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References

This talk is based on the following two papers, co-authored with Hyeontae Jang and Mathieu Vallée.

1. *The characterization of $(n-1)$-spheres with $n+4$ vertices having maximal Buchstaber number*, J. Reine Angew. Math. 811 (2024)
2. *Toric wedge induction and toric lifting property for piecewise linear spheres with a few vertices*, arXiv:2404.15600

Moment-angle complex

- K : a simplicial complex on $[m] = \{1, \ldots, m\}$

We define the polyhedral product $(\underline{X}, \underline{Y})^K$ of K with respect to a pair (X, Y) of topological spaces as follows:

$$(\underline{X}, \underline{Y})^K := \bigcup_{\sigma \in K} \{(x_1, \ldots, x_m) \in X^m \mid x_i \in Y \text{ when } i \notin \sigma\}.$$

- $\mathcal{Z}_K := (\underline{D^2}, \underline{S^1})^K$ the moment-angle complex of K
- $\mathbb{R}\mathcal{Z}_K := (\underline{D^1}, \underline{S^0})^K$ the real moment-angle complex of K

$$T^1 = S^1 \curvearrowright (D^2, S^1) \quad \rightsquigarrow \quad T^m = (S^1)^m \curvearrowright \mathcal{Z}_K$$

$$\mathbb{Z}_2 \curvearrowright (D^1, S^0) \quad \rightsquigarrow \quad (\mathbb{Z}_2)^m \curvearrowright \mathbb{R}\mathcal{Z}_K$$

Buchstaber Number

Consider $H^r \subset T^m$ such that $H^r \curvearrowright \mathcal{Z}_K$ freely.

- $s(K) := \max(r : \exists T^r \overset{\textit{freely}}{\curvearrowright} \mathcal{Z}_K)$ the Buchstaber number of K

Similarly,

- $s_{\mathbb{R}}(K) := \max(r : \exists \mathbb{Z}_2^r \overset{\textit{freely}}{\curvearrowright} \mathbb{R}\mathcal{Z}_K)$ the real Buchstaber number of K

In general, $s(K) \neq s_{\mathbb{R}}(K)$; for instance, an r-skeleton of Δ^n. (Ayzenberg, 2011)
However, there is no known such example in the class of PL spheres yet.

Inequality on (real) Buchstaber numbers

- K : a PL $(n-1)$-sphere on $[m]$

Theorem (Erokhovets, 2014)

$$1 \leq s(K) \leq s_{\mathbb{R}}(K) \leq m-n$$

We are especially interested in the maximal case : $s(K) = s_{\mathbb{R}}(K) = m-n$.

- Why a PL sphere and maximal? This covers many important classes of toric spaces such as toric manifolds, quasitoric manifolds, ...
(Davis-Januskiewicz, 1991, Buchstaber-Panov, 2002,. . . .)

Example: Hopf fibration

- $K = \partial\Delta^n$ on $[n+1]$ $(m = n+1)$

$$\begin{aligned}\mathcal{Z}_K &= (\underline{D^2}, \underline{S^1})^K \\ &= \bigcup_{\sigma \neq [n+1]} \left\{ (x_1, \ldots, x_{n+1}) \in (D^2)^{n+1} \mid x_i \in S^1 \text{ when } i \notin \sigma \right\} \\ &= (D^2 \times \cdots \times D^2 \times S^1) \cup \cdots \cup (S^1 \times D^2 \times \cdots \times D^2) \\ &= \partial (D^2)^{n+1} = S^{2n+1}\end{aligned}$$

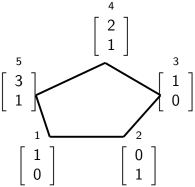
There is the canonical free S^1 action on S^{2n+1}; $s(K) = 1 = m - n$ is maximal.

We have the Hopf fibration $\mathbb{C}P^n = S^{2n+1}/S^1$, while $\mathbb{C}P^n$ admits a well-behaved T^n-action whose orbit space is Δ^n.

Characteristic map

K: a PL $(n-1)$-sphere on $[m]$

A characteristic map over K is a map $\lambda^{\mathbb{C}}\colon [m] \longrightarrow \mathbb{Z}^n$ such that $\lambda^{\mathbb{C}}(\sigma)$ gives a part of a basis of $\mathbb{Z}^n$ for each face σ of K.



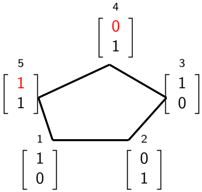
$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Toric colorability

$$K \text{ is toric colorable} \quad := \quad \exists \lambda^{\mathbb{C}} \text{ over } K \quad \Leftrightarrow \quad s(K) = m - n$$

Mod 2 characteristic map

A mod 2 characteristic map over K is a map $\lambda^{\mathbb{R}}\colon [m] \longrightarrow \mathbb{Z}_2^n$ such that $\lambda^{\mathbb{R}}(\sigma)$ gives a part of a basis of $\mathbb{Z}_2^n$ for each face σ of K.



$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$\mathbb{Z}_2^n$-colorability

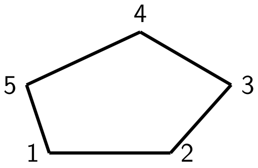
$$K \text{ is } \mathbb{Z}_2^n\text{-colorable} \quad := \quad \exists \lambda^{\mathbb{R}} \text{ over } K \quad \Leftrightarrow \quad s_{\mathbb{R}}(K) = m - n$$

Wedge operation

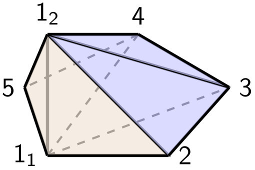
The wedge of K at v is

$$\mathrm{wed}_v(K) := (I * \mathrm{Lk}_K(v)) \cup (\partial I * K \setminus \{v\}).$$

Simplicial complexes which are not wedges are called seeds.



K



$\mathrm{wed}_1(K)$

Recall the $K(J)$-operation (Bahri-Benderski-Cohen-Gitler, 2010); for $J = (j_1, \dots, j_m) \in \mathbb{Z}^m_{>0}$, $K(J)$ is obtained after performing $j_i - 1$ wedges at the vertex i for $i = 1, \dots, m$.

Remark

Any simplicial complex can always be represented as $K(J)$ with K being a seed.

Wedged simplicial complex

- $\operatorname{Pic} K := m - n$ the Picard number of K

Remark

$$\operatorname{Pic} K = \operatorname{Pic} K(J) = m - n$$

Remark

(1) *K is a PL sphere if and only if so is $K(J)$.*

(2) *K is star-shaped if and only if so is $K(J)$.*

(3) *K is polytopal if and only if so is $K(J)$.*

Theorem (Ewald, 1986, BBCG, 2010)

K admits a (mod 2) characteristic map if and only if so does $K(J)$.

Toric colorable seeds with Picard number ≤ 3

Note: The wedge operation preserves

- the Picard number,
- the polytopality (or star-shapedness), and
- the toric colorability.

- $m - n = 1 \quad \Longrightarrow \quad K = \partial(\Delta^n) = \partial(I^1)(J)$
- $m - n = 2 \quad \Longrightarrow \quad K = \partial(\Delta^{n_1} \times \Delta^{n_2}) = \partial(I^2)(J)$

If K is toric colorable, then

- $m - n = 3 \quad \Longrightarrow \quad K = \partial(I^3)(J)$, $\partial P_5(J)$, or $\partial C_7^4(J)$.

For each Pic $K \leq 3$, observe that there are a few toric colorable seeds.

Challenging Picard number 4

Theorem (Choi-Park, 2017)

Let K be a toric colorable seed with $m - n \geq 3$. Then,

$$m \leq 2^{m-n} - 1.$$

Corollary

For fixed $m - n$, there are finitely many toric colorable seeds.

Theorem (Choi-Jang, arXiv:2407.12400)

The above inequality is tight.

For $m - n = 4$, all we need to do is list up all PL spheres for $n \leq 11$ (finite!), and check if they are toric colorable.

Picard number 4

Theorem (Choi-Jang-Vallée, 2024)

The numbers of toric colorable seeds up to $m - n \leq 4$ are as follows.

$p \backslash n$	1	2	3	4	5	6	7	8	9	10	11	> 11	*total*
1	1												1
2		1											1
3		1	1	1									3
4		1	4	21	142	733	1190	776	243	39	4		3153

The list and codes can be found in the following repository:

`https://github.com/MVallee1998/GPU_handle`

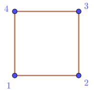
Application

Theorem (Choi-Park, 2016)

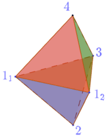
Any characteristic map λ over $\mathrm{wed}_v(K)$ is constructed by two characteristic maps λ_1 and λ_2 over K. In this case, denoted by

$$\lambda = \lambda_1 \wedge_v \lambda_2.$$

If one knows every (real or complex) toric spaces over K, then so over $K(J)$.

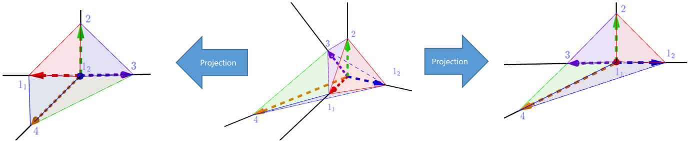


$K = \partial(\Delta^1 \times \Delta^1)$



$\mathrm{Wed}_1 K$

$$\begin{array}{ccccc} 1_1 & 1_2 & 2 & 3 & 4 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & -1 & a \\ 0 & 1 & 0 & -1 & b \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$



Toric wedge induction

$\mathcal{X}$: a set of (real or complex) toric spaces (L, λ) with the following;

(1) L is obtained by a sequence of wedge operations from K,

(2) (L, λ_1), $(L, \lambda_2) \in \mathcal{X}$ with $\lambda = \lambda_1 \wedge_v \lambda_2$ if $(\mathrm{wed}_v(L), \lambda) \in \mathcal{X}$.

$\mathcal{P}$: a statement on $\mathcal{X}$.

Theorem (Toric wedge induction)

Suppose that the following holds:

Basis step

- $\mathcal{P}(K, \lambda)$ *holds for any* $(K, \lambda) \in \mathcal{X}$.

Inductive step

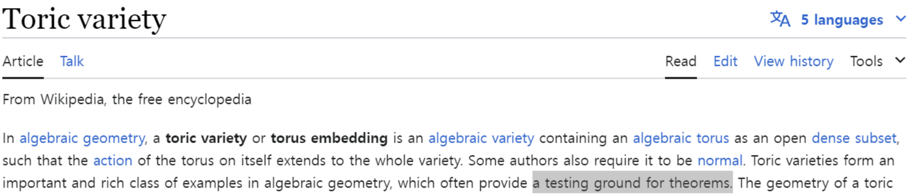
- $\mathcal{P}(L, \lambda_1)$ *and* $\mathcal{P}(L, \lambda_2)$ *hold* $\quad \Rightarrow \quad$ $(\mathrm{wed}_v(L), \lambda_1 \wedge_v \lambda_2)$ *holds*

Then $\mathcal{P}$ *holds on* $\mathcal{X}$.

Applications

Toric wedge induction was used for:

- proving the projectivity of toric manifold with Picard number 3 (originally proved by Kleinschmidt-Sturmfels, 1991, and reproved by Choi-Park, 2016)
- proving the projectivity of toric manifold over $\partial P_k(J)$ (Choi-Park, 2017)
- answering the question of Chen-Fu-Hwang 2014 (Choi-Jang-Vallée, 2024)
- ...


Toric variety

5 languages

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From Wikipedia, the free encyclopedia

In algebraic geometry, a **toric variety** or **torus embedding** is an algebraic variety containing an algebraic torus as an open dense subset, such that the action of the torus on itself extends to the whole variety. Some authors also require it to be normal. Toric varieties form an important and rich class of examples in algebraic geometry, which often provide a testing ground for theorems. The geometry of a toric

Application : Lifting Problem

Recall that there is no known example that $s(K) \neq s_{\mathbb{R}}(K)$ for a PL sphere K.

A stronger question, the lifting problem, was proposed by Zhi Lü at the Osaka toric topology conference in 2011.

Lifting problem

Let K be a polytopal simplicial complex.
Is every small cover M over K the fixed points of conjugation on some quasitoric manifold?

Application : Lifting Problem

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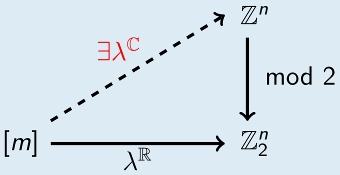
A stronger question, the lifting problem, was proposed by Zhi Lü at the Osaka toric topology conference in 2011.

Lifting problem

Let K be a PL sphere.
For a mod 2 characteristic map $\lambda^{\mathbb{R}}$ over K, is there $\lambda^{\mathbb{C}}$ over K such that the following diagram commute?



If it exists, then $\lambda^{\mathbb{C}}$ is called a lift of $\lambda^{\mathbb{R}}$.

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Stronger version of Lifting problem

Let K be a PL sphere on $[m]$.
Given a subgroup of $\mathbb{Z}_2^m$ acting freely on $\mathbb{R}\mathcal{Z}_K$, is this action induced by a subtorus of T^m freely acting on $\mathcal{Z}_K$?

Note : Since $\mathbb{R}\mathcal{Z}_K$ is the fixed point set by the involution on $\mathcal{Z}_K$ induced by the complex conjugation on $D^2 \subset \mathbb{C}$, the T^m-action on $\mathcal{Z}_K$ induces the $\mathbb{Z}_2^m$-action on $\mathbb{R}\mathcal{Z}_K$.

Application : Lifting Problem

Let K be a PL $(n-1)$-sphere on $[m]$.

Any 3×3 $(0,1)$-matrix has the determinant either 0 or ± 1. Hence, for $n \leq 3$, if

$$\lambda^{\mathbb{C}} \text{ sends to } [m] \text{ to } \{0,1\}\text{-vectors and } \lambda^{\mathbb{R}} \equiv \lambda^{\mathbb{C}} (\text{mod } 2),$$

then $\lambda^{\mathbb{C}}$ is a lift of $\lambda^{\mathbb{R}}$.

Since similar arguments can be applicable to the dual characteristic map, the lifting property holds for the case where $n \leq 3$ or $m-n \leq 3$.

Goal

Solve the lifting problem for $n = 4$ or $m - n = 4$.

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Since similar arguments can be applicable to the dual characteristic map, the lifting property holds for the case where $n \leq 3$ or $m - n \leq 3$.

Goal

Solve the lifting problem for ~~$n = 4$ or~~ $m - n = 4$.

Modified toric wedge induction

If $\lambda^{\mathbb{R}} = \lambda_1^{\mathbb{R}} \wedge_v \lambda_2^{\mathbb{R}}$ and $\lambda_1^{\mathbb{R}} \neq \lambda_2^{\mathbb{R}}$, then $\lambda^{\mathbb{R}}$ is said to be irreducible.

Note : The irreducibility of $\lambda^{\mathbb{R}}$ over $K(J)$ can be determined by the injectivity of the dual characteristic map $\overline{\lambda^{\mathbb{R}}}$.

Theorem (Modified toric wedge induction)

Suppose that the following holds:

Basis step

- $\mathcal{P}(K(J), \lambda^{\mathbb{R}})$ *holds for any irreducible* $(K(J), \lambda^{\mathbb{R}}) \in \mathcal{X}$.

Inductive step

- $\mathcal{P}(L, \lambda^{\mathbb{R}})$ *holds* $\quad\Rightarrow\quad$ $(\mathrm{wed}_v(L), \lambda^{\mathbb{R}} \wedge_v \lambda^{\mathbb{R}})$ *holds.*

Then $\mathcal{P}$ holds on $\mathcal{X}$.

Application : Lifting Problem

Goal

Solve the lifting problem for $m - n = 4$.

Inductive step

Proposition

If $\lambda^{\mathbb{R}}$ over K has a lift, then $\lambda^{\mathbb{R}} \wedge_v \lambda^{\mathbb{R}}$ over $\operatorname{wed}_v K$ has a lift.

In particular, we regard the cases when $\lambda_1^{\mathbb{R}} \wedge_v \lambda_2^{\mathbb{R}}$ for distinct $\lambda_1^{\mathbb{R}}$ and $\lambda_2^{\mathbb{R}}$ as the basis step. (This is still finite!)

Application : Lifting Problem

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Solve the lifting problem for $m - n = 4$.

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Application : Lifting Problem

$\mathcal{M}_A$: the binary matroid represented by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$\mathcal{M}_A$ has 840 facets:

- 835 facets with determinants ± 1,
- 5 facets with determinants ± 3.

A dual characteristic map $\overline{\lambda^{\mathbb{R}}}$ is injective.
$\Longrightarrow$ $\overline{\lambda^{\mathbb{R}}}$ can be regarded as an embedding $\overline{K} \to \mathcal{M}_A$.
$\Longrightarrow$ if $\overline{\lambda^{\mathbb{R}}}(\sigma)$ is not one of the 5 facets for any facet σ of K, then $\overline{\lambda^{\mathbb{R}}}$ has a lift.

Application : Lifting Problem

Basis step

Lemma

If K has facets fewer than 168, then any $\lambda^{\mathbb{R}}$ has a lift.

Proof. $\exists g \in GL(4; \mathbb{Z}_2)$ such that

$$\overline{\lambda^{\mathbb{R}}}(\overline{K}) \subset \mathcal{M}_{gA} \cong \mathcal{M}_A \text{ does not contain any of the 5 facets.}$$

□

There are only 6 pairs $(K, \lambda^{\mathbb{R}})$ for which K has more than 167 facets and $\overline{\lambda^{\mathbb{R}}}$ is injective.

Theorem (Choi-Jang-Vallée, arXiv:2404.15600)

Every mod 2 characteristic map over a PL sphere with $m - n = 4$ has a lift.

Application : Lifting Problem

Basis step

Lemma

If K has facets fewer than 168, then any $\lambda^{\mathbb{R}}$ has a lift.

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There are only 6 pairs $(K, \lambda^{\mathbb{R}})$ for which K has more than 167 facets and $\overline{\lambda^{\mathbb{R}}}$ is injective.

Theorem (Choi-Jang-Vallée, arXiv:2404.15600)

Let K be a PL $(n-1)$-sphere on $[m]$ with $m - n \leq 4$.
Then, every subgroup of $\mathbb{Z}_2^m$ freely acting on $\mathbb{R}\mathcal{Z}_K$ is induced from a subtorus of T^m freely acting on $\mathcal{Z}_K$.

Thank you for your attention!

Appendix : PL spheres of Picard number 4

Intuitively try to find all PL spheres, and compute their (real) Buchstaber numbers. However, it is hopeless when we consider high dimensions.

We could obtain up to $n = 6$, but it seems to take too long to finish for bigger n.

n	2	3	4	5	6
PL spheres	1	5	39	337	6257
$s_{\mathbb{R}} = 4$	1	5	37	281	2353
$s_{\mathbb{R}} = 3$	0	0	2	56	3904
seeds	1	4	23	194	4237
$s_{\mathbb{R}} = 4$	1	4	21	142	733
$s_{\mathbb{R}} = 3$	0	0	2	52	3504

Appendix : Ideas for GPU-friendly coding

$\mathcal{M}_A$: the binary matroid represented by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We observe the following facts;

1. (Gale duality) If K is a $\mathbb{Z}_2^n$-colorable seed, then $\overline{K}$ is embedded into $\mathcal{M}_A$, where $\overline{K}$ is the simplicial complex whose cofacets are facets of K.
2. (Linear algebra) Note that $\mathcal{M}_A$ has 840 facets. If we regard K as a $\mathbb{Z}_2^{840}$-vector, then it is in the kernel of the ridge-facet incidence matrix M of the dual of $\mathcal{M}_A$.
3. (Weak pseudo-manifold condition) For each element of the kernel of M, we have to check whether every ridge is included in exactly two facets.