Cluster algebras and Monotone Lagrangian tori

(joint work with Myungho Kim, Yoosik Kim, Euiyong Park)

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Workshop on Toric Topology, Fields Institute

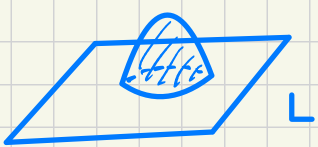
Contents of this talk :

(or $\mathbb{Q}$-factorial)
" Monotone Lagrangian tori in a smooth Fano variety
birational to a cluster variety "

- How to construct ? Via toric degenerations [Harada - Kaveh]
  - Gröbner degeneration (in an algorithmic way)
  - Newton - Okounkov body [Anderson]

- How to distinguish ?
  - Counting invariants (disk counting) called a potential ftn.

(or disk potential, superpotential, Landau - Ginzburg model ...)



(Lagrangian Floer theory)

Story goes as follows : Why studying monotone Lagrangians ?

- Many theories on counting is unobstructed
- characterises Fano varieties

Monotone Symplectic mfd (including smooth Fano varieties)

- closed curve (sphere, or Riemann surface) counting ⟵ GW-inv, or LG-model
- open curve (disk) counting ⟵ open GW-inv

(Lagrangian boundary condition is necessary)

Mirror symmetry

I. Monotone Lagrangian submanifold

- **Maslov index** $\quad L \subset (X, \omega)$: Lagrangian submfd $\leftrightarrow$ $\dim L = \frac{1}{2} \dim M$, $\omega|_L \equiv 0$

A **Maslov index** is a homomorphism $\pi_2(X, L) \xrightarrow{\mu} \mathbb{Z}$ defined by :

pick $\alpha \in \pi_2(X, L)$

$\rightsquigarrow$ $u : (D, \partial D) \to (X, L)$ with $[u] = \alpha$

$$\rightsquigarrow \quad \begin{array}{ccc} u^* TX & \simeq & D \times \mathbb{C}^n \\ \cup\!\!\!| & & \cup\!\!\!| \\ (u|_{\partial D})^* TL & \hookrightarrow & \partial D \times \mathbb{C}^n \end{array}$$

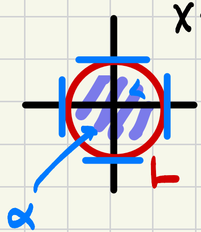
$$\begin{tikzcd} \partial D \simeq S^1 \arrow[r] \arrow[dr, "\phi"'] & U(n)/O(n) \arrow[d, "\det^2"] \\ & S^1 \end{tikzcd}$$

: Lagrangian Grassmannian

$\rightsquigarrow$ $\mu(\alpha) := \text{degree } \phi$

Example

$X = \mathbb{C}$



$\mu(\alpha) = 2$

More generally, $u : D \to (S^1)^n \subseteq \mathbb{C}^n$

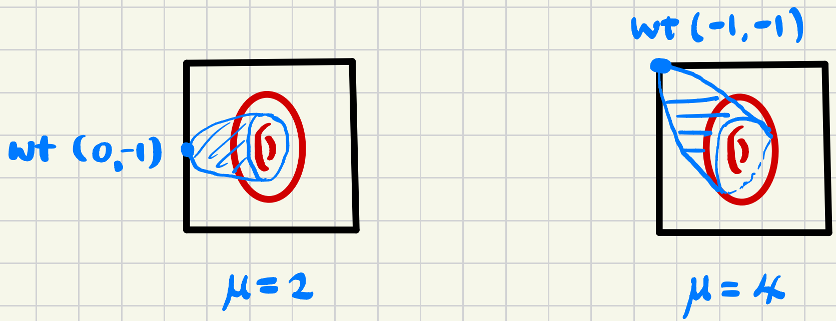
$$z \mapsto (\underbrace{z, \cdots, z}_{k}, 1, \cdots, 1)$$

has Maslov index $2k$

Thm [C.-Kim] Let (X, ω) be a Hamiltonian S^1-mfd and $L \subset (X, \omega)$ an S^1-invariant Lagrangian lying on some level set. For a free S^1-orbit $C := S^1 \cdot p \subset L$, let u: gradient hol. disk attained at a fixed point z_0. Then

$$\mu(u) = 2 \times (\#\text{ negative wts at } z_0)$$

Example Symplectic toric mfd $(X,\omega) \to P$



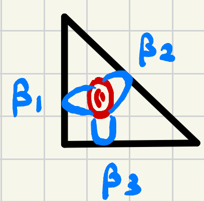
Thm [Cho] Let L: torus fiber on a compact symplectic mfd. Then

$$\{\text{Maslov index two disks bounded by } L\} \overset{1:1}{\longleftrightarrow} \{\text{facets of } P\}$$

called "basic disks"

Note In toric case, $\pi_2(X,L) \simeq \pi_2(X) \oplus \pi_1(L)$ via long exact seq.

$$\pi_2(L)=0 \to \pi_2(X) \to \pi_2(X,L) \to \pi_1(L) \to \pi_1(X)=0$$



: $\beta_1+\beta_2+\beta_3$ is a spherical class

• Monotone symplectic & monotone Lagrangian

Recall that

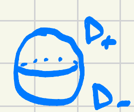
- (X, ω) is **monotone** if $c_1(TX) = \lambda \cdot [\omega]$ for some $\lambda > 0$

  ($\Leftrightarrow$ volume and Chern number have the same ratio λ for any surfaces)

- $L \subset X$ is **monotone** if $\mu(\alpha) = \delta \cdot \omega(\alpha)$ for some $\delta > 0$

  $(\alpha \in \pi_2(X, L))$

Thm
- Every smooth Fano variety is monotone.
- If $X \subset \mathbb{P}^N$ is simply connected and $L \subset (X, \omega)$: monotone, then X is Fano. Moreover, $\delta = 2\lambda$



: $c_1(TX) = 2[\omega_{FS}]$, $\mu(D_+) = 2 = 4 \cdot (\frac{1}{2})$

- **Toric Fano varieties**

$$\{\text{Toric Fano varieties}\} \overset{1:1}{\longleftrightarrow} \{\underline{\text{reflexive polytopes}}\}$$

:= P and P^* are both lattice polytopes

More generally,

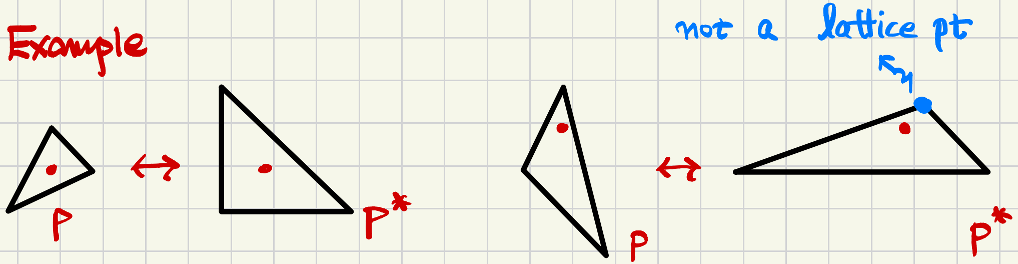
$$\{\text{Toric } \mathbb{Q}\text{-Fano varieties}\} \overset{1:1}{\longleftrightarrow} \{(\text{dual})\ \underline{\text{Fano polytopes}}\}$$

lattice polytope P s.t.

i) $0 \in \operatorname{int} P$

ii) $v \in \operatorname{Vert} P$ is primitive

Example



reflexive

Fano

dual Fano

Thm Every smooth toric Fano variety has a monotone Lagrangian torus L_0

↳ 

- L_0 : fiber at 0
- Every facet is expressed by $a_1x_1 + \cdots + a_nx_n + 1 = 0$
- Each basic disk is of Maslov index 2
- " " is of area 1

same ratio $2:1$

Conjecture Every smooth, not necessarily toric, Fano variety has a monotone Lagrangian torus

$\updownarrow \simeq$

Existence of Landau-Ginzburg models

(mirror symmetry of smooth Fano varieties)

- **Distinguishing Lagrangians** (potential functions)

Two Lagrangians L, L' are symplectic / Hamiltonian isotopic if

$\exists \{\phi_t\} \subseteq \mathrm{Symp}(X,\omega) / \mathrm{Ham}(X,\omega)$ s.t $L' = \phi_1(L)$

Question How to distinguish Lagrangians ? $\rightsquigarrow$ Counting disks

- $\mathcal{M}(X,\beta) := \{ u : (D, \partial D) \to (X, L) : [u] = \beta \} / \sim$: virtual dimension : $n-3+\mu(\beta)$

$\rightsquigarrow \mathcal{M}_1(X,\beta) = \{ (u,z) : u \in \mathcal{M}(X,\beta), z \in \partial D \} \xrightarrow{ev} L$

$(u,z) \longmapsto u(z)$

dim $n-2+\mu(\beta)$

If $\mu(\beta) = 2$, $\dim \mathcal{M}_1(X,\beta) = n$ $\rightsquigarrow$ "degree of ev" $= \#\, u$'s passing through a generic pt

Defn Let L : monotone Lagrangian torus. Then

$$W_L(z) := \sum_{\substack{\beta \in \pi_2(X,L) \\ \mu(\beta)=2}} n_\beta z^{\partial\beta} \quad \text{where}$$

- n_β : # disks of class β
- $\partial\beta \in H_1(L;\mathbb{Z}) \simeq \mathbb{Z}^n$

  ($\rightsquigarrow z^{(a_1,\cdots,a_n)} := z_1^{a_1} \cdots z_n^{a_n}$)

is a Laurent polynomial called a potential function

- If $W_L(z) \nsim W_{L'}(z)$, then $L \nsim L'$.

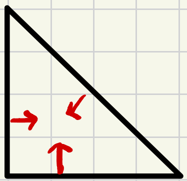
  (E.g., sum of coef's is invariant under $\sim$)

  basis change of $H_1(X;\mathbb{Z})$

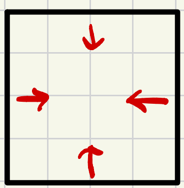
  (or monomial change)

Example [Cho-Oh] P : reflexive polytope, $L_0 \subset X_P$: monotone Lag. fiber

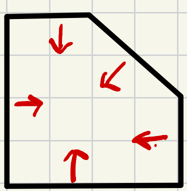
Then $W_{L_0}(z) = \sum_{F:\text{ facet}} z^{v_F}$, v_F : inward facet normal



$$W_L(z) = z_1 + z_2 + \frac{1}{z_1 z_2}$$



$$W_L(z) = z_1 + z_2 + \frac{1}{z_1} + \frac{1}{z_2}$$



$$W_L(z) = z_1 + z_2 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_1 z_2}$$

• Construction of monotone Lagrangian tori : toric degenerations [Harada-Kaveh]

$X \subset \mathbb{P}^N$: projective variety

Defn A toric degeneration is a flat family of varieties

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d, "\pi"'] & \mathbb{P}^N \times \mathbb{C} \arrow[dl] \\ \mathbb{C} & \end{tikzcd}$$

s.t. i) $\pi^{-1}(\mathbb{C}^*) \simeq X \times \mathbb{C}^*$

ii) $\pi^{-1}(0) = X_0$ is an irreducible toric variety.

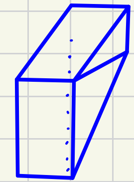
Example Flag variety $\mathcal{F}\ell(3) = \{ V_\bullet : 0 \subset V_1 \subset V_2 \subset \mathbb{C}^3 , \dim_{\mathbb{C}} V_i = i \}$

$\rightsquigarrow$ Plücker embedding $\mathcal{F}\ell(3) \hookrightarrow \underset{\text{hypersurface}}{\mathbb{P}^2 \times \mathbb{P}^2}$

$\rightsquigarrow$ Homog. coordinate $([x_1, x_2, x_3], [x_{12}, x_{23}, x_{31}])$ + Relation $\langle x_1 x_{23} - x_2 x_{31} + x_3 x_{12} \rangle$

$\rightsquigarrow$ Toric degeneration

$$\langle x_1 x_{23} - x_2 x_{31} + t \cdot x_3 x_{12} \rangle$$

$X_0 \sim$ 

- Monotone Lagrangian tori from toric degenerations

Thm [C.-M. Kim - Y. Kim - Park]
Let $X \subset \mathbb{P}^N$ be a smooth Fano variety. Suppose that there is a toric degeneration $\pi : \mathfrak{X} \to \mathbb{C}$ s.t X_0 is a normal $\mathbb{Q}$-Fano. Then $\exists$ monotone Lagrangian torus

↳ i) By [Harada-Kaveh], $\exists$ continuous map $\phi : X \to X_0$ s.t.

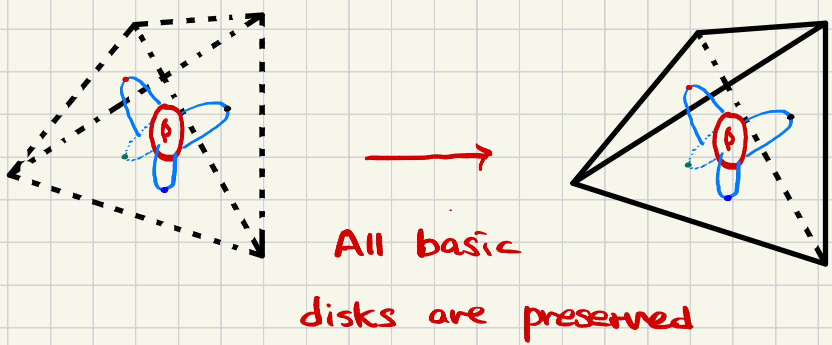
$$\phi^{-1}(X_0^{sm}) \xrightarrow{\phi} X_0^{sm} \quad \text{is a symplectomorphism}$$

ii) X_0^{sm} contains $\mu^{-1}(\operatorname{int} P \cup \bigcup_{F:\text{facet}} \operatorname{int} F)$ where $\mu : X_0 \to P$ moment map

iii) Basic disks 

are over $\operatorname{int} P \cup \bigcup_{F:\text{facet}} \operatorname{int} F$ and hence pulled-back to X via ϕ

$$\begin{array}{ccc} X & \xrightarrow{\quad\phi\quad} & X_0 \\ \rotatebox{90}{$\subseteq$} & & \parallel \end{array}$$



In conclusion, to construct a monotone Lagrangian torus, it is enough to construct a normal $\mathbb{Q}$-Fano toric degeneration

- Toric degeneration via Newton - Okounkov bodies

For $X \subseteq \mathbb{P}^N$ and D : prime divisor of X, define a valuation

$$\nu_D : \mathbb{C}(X)^* \to \mathbb{Z}$$
$$f \mapsto \nu_D(f) := \text{order of vanishing of } f \text{ on } D$$

(If $D = \mathbb{V}(g)$ and $f = g^k \cdot h$ where h: non-vanishing on D, $\nu_D(f) = k$.)

Example. If $f = x^2 y$ and $D = \{x=0\} \subseteq \mathbb{C}^2$, then $\nu_D(f) = 2$

For $Y_\bullet := \{ X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n = \{p\} \;:\; \operatorname{codim} Y_i = i ,\ Y_i\text{'s are smooth at } p \}$

$$\rightsquigarrow \nu_{Y_\bullet} : \mathbb{C}(X)^* \to \mathbb{Z}^n$$
$$f \mapsto \nu_{Y_\bullet}(f) = (a_1, \cdots, a_n)$$

where $f = f_1^{a_1} g_1$ with $Y_1 = \mathbb{V}(f_1)$ and $g_1 \neq 0$ on Y_1

$g_1 = f^{a_2} g_2$ " $Y_2 = \mathbb{V}(f_2)$ in Y_1 and $g_2 \neq 0$ on Y_2

Now, restrict $\nu_{Y_\bullet}$ to $\mathbb{C}[X] \simeq \bigoplus_{m \geq 0} H^0(X, \mathcal{L}^m) \subset \mathbb{C}(X)$ and extend it to

$$\boxed{\tilde{\nu}_{Y_\bullet}(f_m) =: (m, \nu_{Y_\bullet}(f_k)) \in \mathbb{N} \times \mathbb{Z}^n}$$

Defn The image $S_{Y_\bullet} := \tilde{\nu}_{Y_\bullet}(\mathbb{C}[X] \setminus \{0\})$ is called **valuation semigroup**

Thm [Anderson] If $S_{Y_\bullet}$ is finitely generated, then $\exists$ toric degeneration of X to a toric variety $\mathrm{Proj}\, \mathbb{C}[S_{Y_\bullet}]$ over $\Delta_{Y_\bullet}$ where

$$\Delta_{Y_\bullet} := \overline{\mathrm{Cone}\, S_{Y_\bullet}} \cap \{1\} \times \mathbb{R}^n \subset \mathbb{N} \times \mathbb{R}^n$$

- X_0 is projectively normal (and hence) normal if $S_{Y_\bullet}$ is saturate

Made with Goodnotes

Cluster variety

- Cluster algebra : $\mathbb{C}$-algebra generated by "seed"

i) A seed is a pair $s = (\mathbb{x}, \varepsilon)$ consisting of

$$\mathbb{x} = (x_1, \cdots, x_n) \qquad \varepsilon = \begin{pmatrix} \varepsilon_{11} & \cdots & \varepsilon_{1n} \\ \vdots & & \vdots \\ \varepsilon_{n1} & \cdots & \varepsilon_{nn} \end{pmatrix} : \text{skew-symmetric}$$

⇝ called "cluster variables"

⇝ called "exchange matrix"

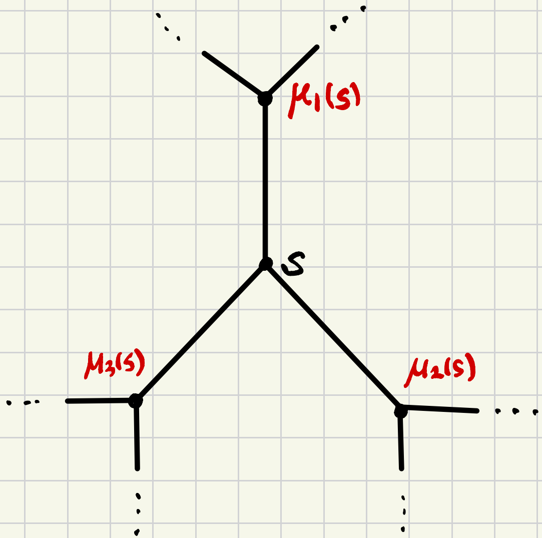
ii) For each $k = 1, \cdots, n$, define $\mu_k(s) := s' = (\mathbb{x}', \varepsilon')$ where

called "mutation"

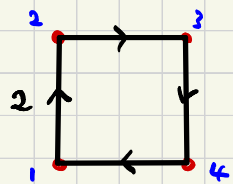
$$\mathbb{x}' := \begin{cases} x_1' = x_1 \\ \quad \vdots \\ x_k' = x_k^{-1} \cdot \left(\prod\limits_{\varepsilon_{ik} > 0} x_i^{\varepsilon_{ik}} + \prod\limits_{\varepsilon_{ik} < 0} x_i^{-\varepsilon_{ik}} \right) \\ \quad \vdots \\ x_n' = x_n \end{cases} \qquad \varepsilon_{ij}' = \begin{cases} -\varepsilon_{ij} & (i = k \text{ or } j = k) \\ \varepsilon_{ij} \pm \varepsilon_{ik} \cdot \varepsilon_{kj} & (\varepsilon_{ik}, \varepsilon_{kj} \gtrless 0) \\ \varepsilon_{ij} & (\text{o.w}) \end{cases}$$

Exchange graph : vertex – seed

(n-valent) edge – mutation



Note In skew-symmetric case, the exchange matrix ε can be replaced by a "quiver"



$$\leftrightarrow \begin{pmatrix} 0 & 2 & 0 & -1 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

Thm (Laurent phenomenon)

Fix an initial seed $s := (\mathbf{x}, \varepsilon)$. For any seed $s' = (\mathbf{x}', \varepsilon')$, each x_i' is a Laurent polynomial in $(x_1, \cdots, x_n)$

Defn A cluster algebra is a subalgebra of $\mathbb{C}(x_1, \cdots, x_n)$ generated by $\bigcup_s \mathbf{x}_s$

To construct a geometric counterpart, we take $\mathbb{C}[\mathbf{x}_s^{\pm 1}]$ instead of $\mathbb{C}[\mathbf{x}_s]$ and glue $\operatorname{Spec} \mathbb{C}[\mathbf{x}_s^{\pm}]$ along mutations.

$\simeq (\mathbb{C}^*)^n$ regular on each tori

$\bigcup (\mathbb{C}^*)^n_s$ is called a cluster variety

s: seed

- Gross - Hacking - Keel - Kontsevich constructed a toric degeneration of a (partially) compactified cluster variety.

(In our case, cluster variety $= U^-_{w_0} := U^- \cap B\tilde{w}_0 B$

compactified variety $= G/B \hookleftarrow$ open embedding)

- Later, Fujita - Oya proved that GHKK's polytope can be realized as a NOBY :

i) Each seed $(\mathbf{x}, \varepsilon)$ defines a valuation on $\mathbb{C}(X)^*$

- $\mathbf{x}$ defines "monomials"
- ε defines an ordering on $\mathbb{Z}^n$ $a \leq_\varepsilon b \Leftrightarrow a = b + \varepsilon v$ for some $v \in \mathbb{Z}^n$

ii) $\Delta(X(w), \mathcal{L}_\lambda, \nu_s) \simeq \Delta_{\underline{i}}(\lambda)$

Schubert variety

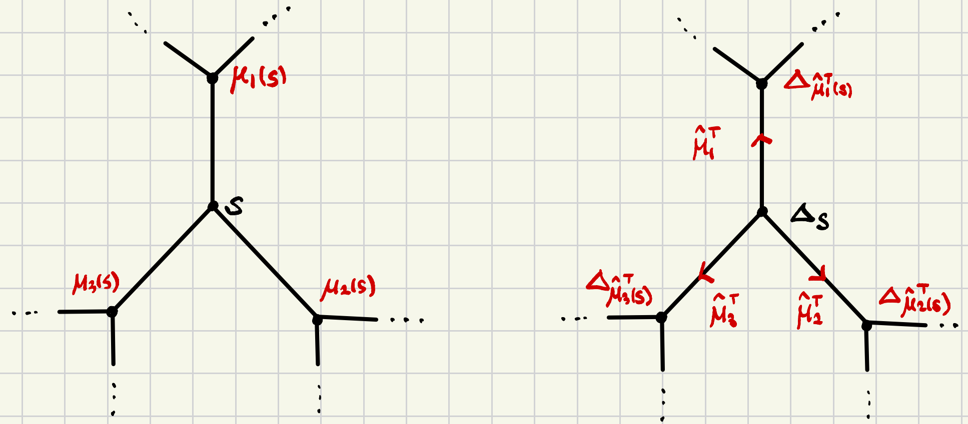
λ : dominant wt

string polytope

Main result

- <u>Combinatorial cluster structures</u> :

Given exchange graph $\mathbb{T}$, consider $\{\Delta_s : s \in \mathbb{T}\}$ s.t.



$[a]_+ := \max(a, 0)$

$$\hat{\mu}_i^T : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0} \times \mathbb{R}^n$$

$$(r, u) \mapsto (r, \mu_i^T(u))$$, μ_i^T is a tropicalized version of μ_i :

$$\mu_i^T : u = (u_j) \mapsto u' = (u_j') \text{ where } u_j' = \begin{cases} -u_j & (j=k) \\ u_j + [\varepsilon_{k,j}]_+ u_k & (j \neq k,\ u_k \geq 0) \\ u_j - [-\varepsilon_{k,j}]_+ u_k & (j \neq k,\ u_k \leq 0) \end{cases}$$

piecewise linear

Then we say that $\{\Delta_s : s \in \mathbb{T}\}$ has a combinatorial cluster structure

(I.e., $\mu_i^{\mathbb{T}}(\Delta_s) = \Delta_{\mu_i(s)}$)

Proposition [CKKP] Suppose that $\{\Delta_s : s \in \mathbb{T}\}$ is a family of NOBY's of $X \subseteq \mathbb{P}^N$ having a combinatorial cluster structure.

Fix an initial seed s_0.

i). If Δ_{s_0} is a dual Fano polytope, then so is Δ_s for $\forall s \in \mathbb{T}$.

ii) If the semigroup S_{s_0} is saturated, then so is S_s for $\forall s \in \mathbb{T}$

Corollary Each seed $s \in \mathbb{T}$ produces a monotone Lagrangian torus $L_s \subseteq X$.

If the cluster algebra is of infinite type, then this process produces infinitely many monotone Lagrangian tori

Thm [CKKP] Suppose that $\{\Delta_t\}$ satisfies the conditions in the proposition.

If $\exists$ sequence of seeds $\{S_i = (\mathbf{x}_i, \varepsilon_i)\}$ s.t. $|\varepsilon_i| \to \infty$ as $i \to \infty$,

then $\exists$ infinitely many distinct monotone Lagrangian tori

(up to symplectic isotopy)

↳ main idea : the growth of $|\varepsilon_i|$ ~ the growth of lattice pts in $\Delta^\vee_{S_i}$

Thm [CKKP] Let G be a simple complex Lie group other than $A_1, A_2, A_3, A_4, B_2 = C_2$

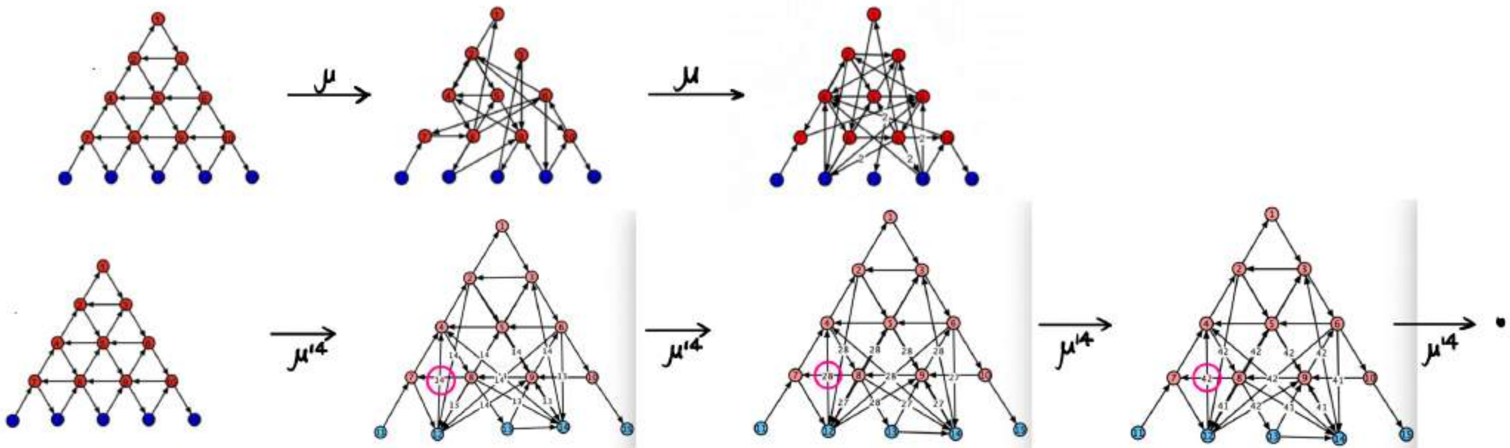
Then G/B has infinitely many monotone Lagrangian tori

(w.r.t. KKS Kähler form $\omega_{2\rho}$)

↳ main idea : i) Take initial seed S_0 s.t $\Delta_{S_0} \simeq \Delta_{\mathbf{i}_0}(2\rho)$ ⇝ known that it is dual Fano

ii) Using quiver description, we can find (S_i) s.t

$|\varepsilon_i| \to \infty$



grows linearly.

Thank You!