Homology of real toric varieties associated with the Weyl groups of types E_7 and E_8

Seonghyeon Yu

joint work with Suyoung Choi and Younghan Yoon
(Ajou Univ.)

2024.08.20.

2024 Workshop on Toric Topology
at Fields institute

Background : Root system

Root system and reflections

A **root system** Φ in a finite dimensional Euclidean vector space E is a finite set of non-zero vectors that satisfy the following conditions:

1. The roots span E.
2. The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α it self and $-\alpha$.
3. For every root $\alpha \in \Phi$, the set Φ is closed under reflection through the hyperplane perpendicular to α.
4. For any two roots $\alpha, \beta \in \Phi$, the number $2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

Background : Weyl group and Weyl chambers

Weyl group

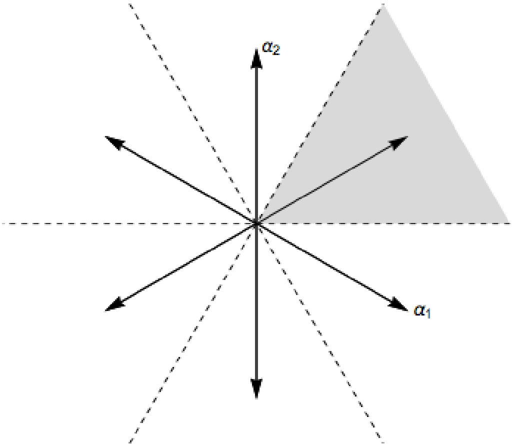
The Weyl group W_R of type R is the group of the reflections associated to the roots of type R.

Weyl chamber

The complement of the set of hyperplanes perpendicular to simple roots is disconnected, and each connected component is called a Weyl chamber.

Background : Root system and Weyl chambers

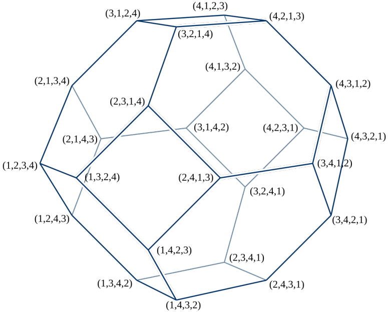
Example: Type A_2



Background : Weyl group and Coxeter complex

Example: Type A_n

$$\mathfrak{S}_{n+1}$$



Weyl group W_{A_n} $\Longleftrightarrow$ Coxeter complex K_{A_n}

symmetric group — dual of permutohedron

Background : Real toric space associated with the Weyl group

Proposition [1991, C. Procesi]

For a Weyl group W_R of type R, the corresponding Coxeter complex K_R and its co-weight lattice induce a complete non-singular fan. So, the toric variety X_R over this fan is smooth and compact.

Def) The real toric variety associated with the Weyl group

The real locus $X_R^{\mathbb{R}}$ of X_R is called the *real toric variety associated with the Weyl group of type* R.

Background : Real toric space associated with the Weyl group

Remark

The fan Σ_R associated with the Weyl group W_R of type R is corresponding to the pair (K_R, Λ_R)

$$\Sigma_R \Longleftrightarrow (K_R, \Lambda_R),$$

where Λ_R is the $(n \times m)$-matrix such that n is the dimension of Φ_R and m is the number of vertices and each column is the coordinate of each vertex in the co-weight lattice.

Background : Real toric space associated with Weyl group

Lemma [2019, Cho, Choi and Kaji]

The Weyl group W_R acts on K_R and preserves kerΛ_R.

Theorem [2019, Cho, Choi and Kaji]

For any root system Φ_R of type R, let W_R be the Weyl group of Φ_R. Then, there is a W_R-module isomorphism

$$H_*(X_R^{\mathbb{R}}) \cong \bigoplus_{S \in Row(\Lambda_R)} \widetilde{H}_{*-1}(K_S),$$

where K_S is the induced subcomplex of K_R with respect to S.

Background : History

The $\mathbb{Q}$-Betti numbers of $X_R^{\mathbb{R}}$ were computed for

- Type A (2012, Henderson)
- Type B (2017, Choi, Park and Park)
- Type C and D (2019, Choi, Kaji and Park)
- Type G_2, F_4 and E_6 (2019, Cho, Choi and Kaji)

Background : Types E_7 and E_8

The number of facets of K_R

- G_2 : 12
- F_4 : $1,152$
- E_6 : $51,840$

K_R for types E_7 and E_8

	$R = E_7$	$R = E_8$
# vertices of K_R	17,642	881,760
# facets of K_R	2,903,040	696,729,600

Remark

$$K_S \cong K_{gS} \text{ for } S \in Row(\Lambda_R) \text{ and } g \in W_R.$$

W_R-orbits in Row(Λ_R) for types E_7 and E_8

	$R = E_7$			$R = E_8$	
W_R-orbit types of S	S_1	S_2	S_3	S_4	S_5
# elements of orbit	63	63	1	120	135

Computing K_S

Lemma

For type R, let K_ω be a subcomplex of K_R induced by the set $\{h \cdot \Omega \mid h \in H_\omega\}$. , where Ω is the fundamental Weyl chamber and H_ω is the isotropy subgroup of W_R to a fixed fundamental coweight ω. Then there is a decomposition of the Coxeter complex K_R as follows:

$$K_R = \bigsqcup_{g \in W_R / H_\omega} K^g,$$

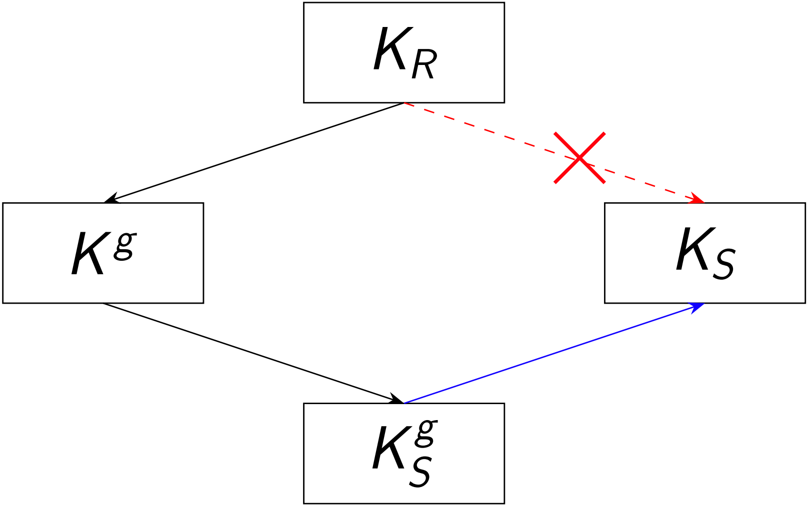
where $K^g = g \cdot K_\omega$.

Lemma

Let $g, h \in W_R/H_\omega$. If $g \cdot V_S^h = V_S^{gh}$, then $g \cdot K_S^h = K_S^{gh}$, where $K_S^g := K^g \cap K_S$ and V_S^g is the set of its vertices.

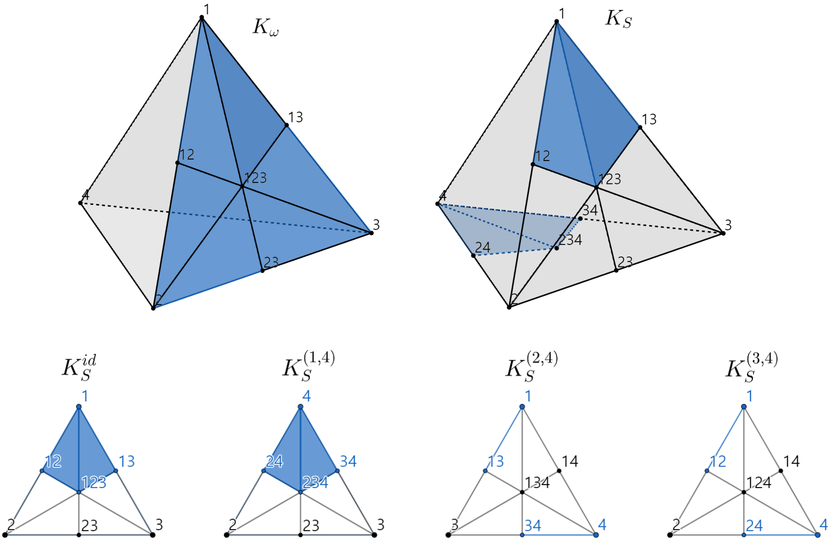
Computing K_S

Sketch for computing K_S



Computing K_S

Example : type A_3



Reduced simplicial complex : background

Definition

- $Lk_K(v) = \{\sigma \in K \mid v \notin \sigma \text{ and } \{v\} \cup \sigma \in K\}$
- $St_K(v) = \{\sigma \in K \mid \{v\} \cup \sigma \in K\}$

Mayer-Vietoris Sequence

Let X be a topological space and A, B be two subspaces whose interiors cover X and $A \cap B$ is nonempty. Then the sequence

$$\cdots \to \widetilde{H}_k(A \cap B) \to \widetilde{H}_k(A) \oplus \widetilde{H}_k(B) \to \widetilde{H}_k(X) \to \widetilde{H}_{k-1}(A \cap B) \to \cdots$$

is exact for all positive integer k.

Reduced simplicial complex : idea

Vertex Removing sequence

- $A = K \setminus v$
- $B = St_k(v)$

$$\cdots \to \widetilde{H}_k(Lk_K(v)) \to \widetilde{H}_k(K \setminus v) \oplus \widetilde{H}_k(St_K(v)) \to \widetilde{H}_k(K)$$

$$\to \widetilde{H}_{k-1}(Lk_K(v)) \to \cdots$$

if $Lk_K(v) \neq \emptyset$, where $K \setminus v = \{\sigma \setminus \{v\} \in K \mid \sigma \in K\}$.

Therefore, if $\widetilde{H}_{k-1}(Lk_K(v)) = \widetilde{H}_k(Lk_K(v)) = 0$, then $\widetilde{H}_k(K \setminus v) \cong \widetilde{H}_k(K)$ as groups.

Reduced simplicial complex

The number of vertices

E_7	$S = S_1$	$S = S_2$	$S = S_3$
K_S	9,176	8,672	4,664
$\widehat{K}_S$	408	928	4,664

E_8	$S = S_4$	$S = S_5$
K_S	432,944	451,200
$\widehat{K}_S$	9,328	15,488

Propositions

1. K_{S_1} and K_{S_4} have two connected components; the other K_S are connected.
2. For $S = S_1, S_4$, two components of K_S are isomorphic.
3. All $\widehat{K}_S$ are pure simplicial complexes.
4. Each component of $\widehat{K}_{S_1}$ is isomorphic to some induced subcomplex of K_{D_6}.
5. Each component of $\widehat{K}_{S_4}$ is isomorphic to $\widehat{K}_{S_3}$.

Results

Reduced Betti numbers of K_S

$\widetilde{\beta}_k(K_S)$	$S = S_1$	$S = S_2$	$S = S_3$
$k = 0$	1	0	0
$k = 1$	0	129	0
$k = 2$	1,622	0	28,855
$k = 3$	0	1,952	0
# orbit	63	63	1

$\widetilde{\beta}_k(K_S)$	$S = S_4$	$S = S_5$
$k = 0$	1	0
$k = 1$	0	769
$k = 2$	57,710	0
$k = 3$	0	177,280
# orbit	120	135

Results

Theorem

The kth Betti numbers β_k of $X^{\mathbb{R}}_{E_7}$ and $X^{\mathbb{R}}_{E_8}$ are as follows.

$$\beta_k(X^{\mathbb{R}}_{E_7};\mathbb{Q}) = \begin{cases} 1, & \text{if } k = 0 \\ 63, & \text{if } k = 1 \\ 8{,}127, & \text{if } k = 2 \\ 131{,}041, & \text{if } k = 3 \\ 122{,}976, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta_k(X^{\mathbb{R}}_{E_8};\mathbb{Q}) = \begin{cases} 1, & \text{if } k = 0 \\ 120, & \text{if } k = 1 \\ 103{,}815, & \text{if } k = 2 \\ 6{,}925{,}200, & \text{if } k = 3 \\ 23{,}932{,}800, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

Results

Remark

The Euler characteristic numbers of $\chi(X^{\mathbb{R}}_{E_7})$ and $\chi(X^{\mathbb{R}}_{E_8})$ are

$$\chi(X^{\mathbb{R}}_{E_7}) = 0, \text{ and}$$

$$\chi(X^{\mathbb{R}}_{E_8}) = 17,111,296$$

Thank you for attention