Real toric manifolds and permutations derived from chordal nestohedra

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Joint work with Suyoung Choi
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The Fields Institute

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0. Preview

Definition

Let $x = x_1 x_2 \cdots x_n$ be a (one-line notation) permutation. For each $1 \leq i \leq n-1$, i is a **descent** of x if $x_i > x_{i+1}$.

Example 1

For each $0 \leq k \leq 3$, count the number of permutations on $\{1, 2, 3, 4\}$ with exactly k descents.

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#descents	0		1		2		3
permutations	1234						
count	1						

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permutations	1234		4123				
count	1		1				

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#descents	0		1		2		3
permutations	1234		4123,3124				
count	1		2				

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permutations	1234	4123,3124, 2134,3412, 2413,2314, 1423,1324, 2341,1342, 1243	3214,4213, 4312,2143, 3142,4132, 3241,4231, 1432,2431, 3421	4321
count	1	11	11	1

0. Preview

Proposition

For a $2n$-dimensional **permutohedral variety** X_{A_n},

$$\dim H_k(X_{A_n};\mathbb{Q}) = \begin{cases} \left|\{x \in \mathfrak{S}_{n+1} : \#\text{descents of } x = \frac{k}{2}\}\right|, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

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Example 1 (revisist)

Compute the rational Betti numbers $\dim H_*(X_{A_3};\mathbb{Q})$ of X_{A_3}.

k	0	1	2	3	4	5	6
permutations	1234		4123, 3124, 2134, 3412, 2413, 2314, 1423, 1324, 2341, 1342, 1243		3214, 4213, 4312, 2143, 3142, 4132, 3241, 4231, 1432, 2431, 3421		4321
$\dim H_k(X_{A_3};\mathbb{Q})$	1	0	11	0	11	0	1

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Definition

A permutation $x = x_1 x_2 \cdots x_n$ is **alternating** if

$$x_1 > x_2 < x_3 > \cdots .$$

Example 2

For each $0 \leq k \leq 2$, count #alternating permutations on $2k$-subsets of $\{1, 2, 3, 4\}$.

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k	0	1	2
alternating permutations on $2k$-subsets of {1,2,3,4}	( )	43,42,41, 32,31,21	4231,4132, 3241,3142, 2143
count	1	6	5

0. Preview

Theorem(A. Henderson, 2012)

Let $X^{\mathbb{R}}_{A_n}$ be an n-dimensional **real permutohedral variety**, that is, the real locus of a $2n$-dimensional permutohedral variety $X^{\mathbb{C}}_{A_n}$. Then

$$\dim H_k(X^{\mathbb{R}}_{A_n};\mathbb{Q}) = \#\text{alternating permutations on } 2k\text{-subsets of } \{1,2,\ldots,n+1\}.$$

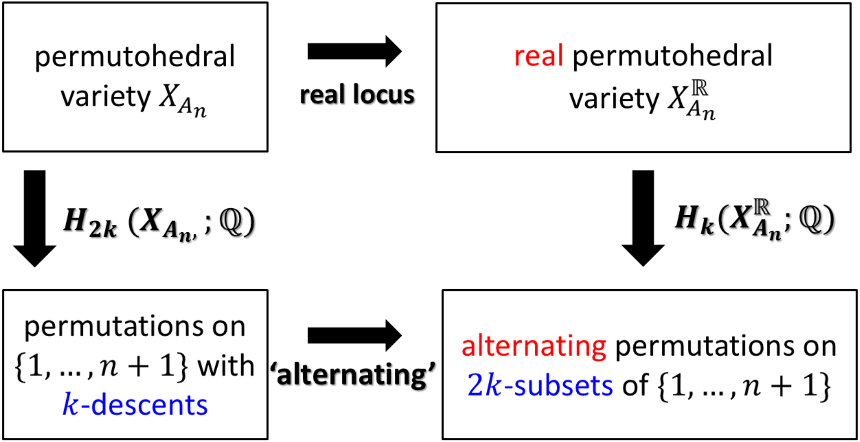
Example 2 (revisist)

Compute the rational Betti numbers $\dim H_*(X^{\mathbb{R}}_{A_3};\mathbb{Q})$ of $X^{\mathbb{R}}_{A_3}$.

k	0	1	2
alternating permutations on $2k$-subsets of {1,2,3,4}	( )	43,42,41, 32,31,21	4231,4132, 3241,3142, 2143
$\dim H_k(X^{\mathbb{R}}_{A_3};\mathbb{Q})$	1	6	5

0. Preview

Summary



1. Introduction

Definition

A **building set** $\mathcal{B}$ is a collection of nonempty subsets of $\{1, \ldots, n+1\}$ that satisfies the following condtions:

1. $\{i\} \in \mathcal{B}$ for $1 \leq i \leq n$, and
2. if $\mathrm{I}, \mathrm{J} \in \mathcal{B}$ and $\mathrm{I} \cap \mathrm{J} \neq \emptyset$, $\mathrm{I} \cup \mathrm{J} \in \mathcal{B}$.

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3. $\{1, \ldots, n+1\} \in \mathcal{B}$

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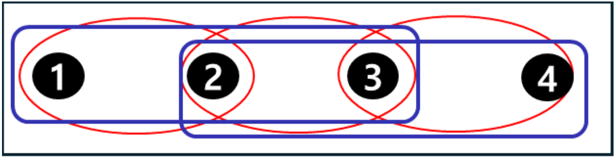
Example

Consider a (connected) building set $\mathcal{B}$ on $\mathrm{I} = \{1, 2, 3, 4\}$ defined by

$$\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}.$$



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Definition

For a building set $\mathcal{B}$, a **nestohedron** $P_{\mathcal{B}}$ is the minkowski sum

$$P_{\mathcal{B}} = \sum_{\mathrm{I} \in \mathcal{B}} \text{convex hull}(\{e_i : i \in \mathrm{I}\}).$$

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Remark

The family of nestohedra include the following polytopes:

- Permutohedra
- Associahedra
- Stellohedra
- Stanley-Pitman polytopes
- Hochschild polytopes

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Definition

A smooth compact toric variety X is called a **toric manifold**.

Theorem (Fundamental theorem of toric geometry)

Category of toric manifolds $\overset{equiv}{\longleftrightarrow}$ Category of smooth complete fans

Proposition

The normal fan of each nestohedron is a smooth complete fan.

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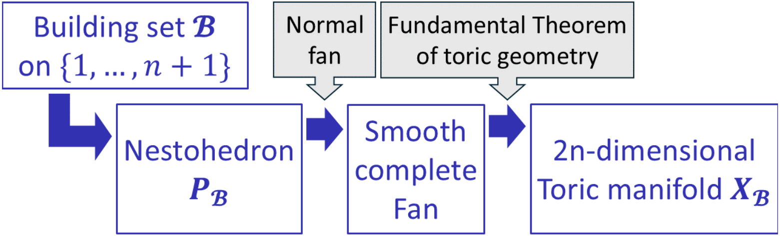

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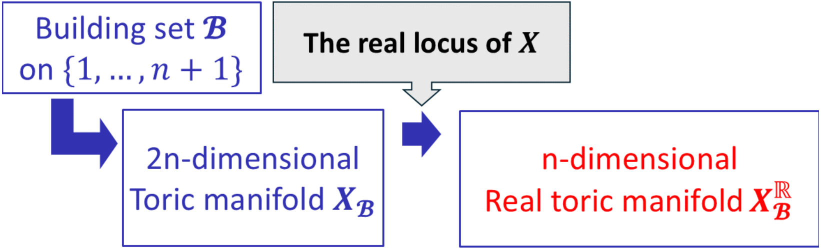
Definition

For a toric manifold X, a **real toric manifold** $X^{\mathbb{R}}$ is the fixed point set of X by the canonical involution induced from a complex conjugation.

Remark



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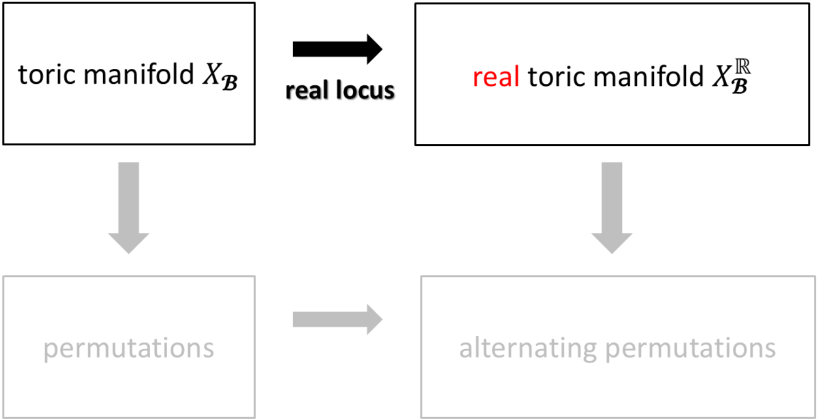

2. Chordal nestohedra and $\mathcal{B}$-permutations

Definition

A building set $\mathcal{B}$ is **chordal** if for each $\{i_1 < \cdots < i_k\} \in \mathcal{B}$,

$$\{i_2, \ldots, i_k\} \in \mathcal{B} \ .$$

If $\mathcal{B}$ is chordal, $P_{\mathcal{B}}$ is a **chordal nestohedron**.

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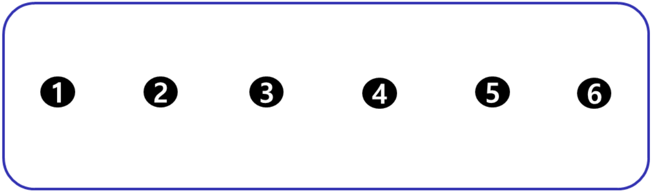
Example

Consider a (connected) building set $\mathcal{B}$ on $\{1, 2, 3, 4, 5, 6\}$.

$$\mathcal{B} = \{1, 2, 3, 4, 5, 6, 123456\}.$$



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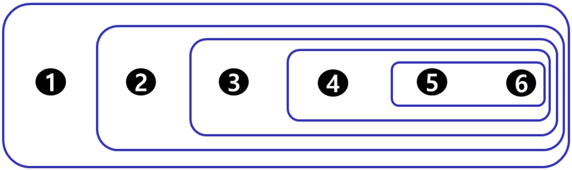
If $\mathcal{B}$ is chordal, $P_{\mathcal{B}}$ is a **chordal nestohedron**.

Example

Consider a (connected) building set $\mathcal{B}$ on $\{1,2,3,4,5,6\}$. If $\mathcal{B}$ is chordal, then

$$\mathcal{B} = \{1,2,3,4,5,6,56,456,3456,23456,123456\}.$$



2. Chordal nestohedra and $\mathcal{B}$-permutations

Definition

For a subset I of $\{1, 2, \ldots, n\}$,

$$\mathcal{B}|_{\mathrm{I}} = \{\mathrm{J} \in \mathcal{B} \colon \mathrm{J} \subset \mathrm{I}\}$$

is a **restricted building set** of $\mathcal{B}$ to I.

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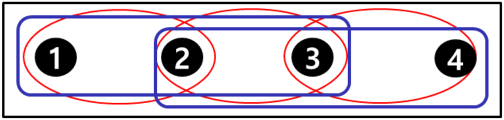
Example

A building set $\mathcal{B}$ is as follows:

$$\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$$

$\mathcal{B}$



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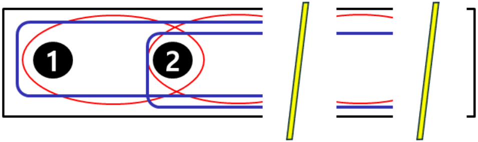
Example

The restricted building set $\mathcal{B}|_{\{1,2\}}$ is as follows:

$$\mathcal{B}|_{\{1,2\}} = \{1, 2, \cancel{3}, \cancel{4}, 12, \cancel{23}, \cancel{34}, \cancel{123}, \cancel{234}, \cancel{1234}\}$$

$\mathcal{B}|_{\{1,2\}}$ 

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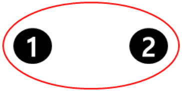
$$\mathcal{B}|_{\{1,2\}} = \{1, 2, 12\}$$

$\mathcal{B}|_{\{1,2\}}$



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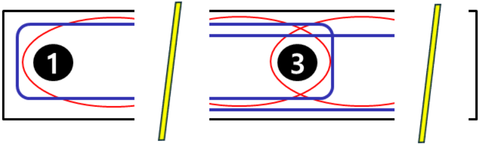
Example

The restricted building set $\mathcal{B}|_{\{1,3\}}$ is as follows:

$$\mathcal{B}|_{\{1,3\}} = \{1, \cancel{2}, 3, \cancel{4}, \cancel{12}, \cancel{23}, \cancel{34}, \cancel{123}, \cancel{234}, \cancel{1234}\}$$

$\mathcal{B}|_{\{1,3\}}$ 

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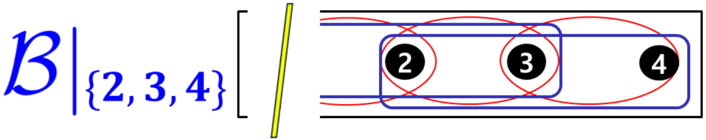
Example

The restricted building set $\mathcal{B}|_{\{2,3,4\}}$ is as follows:

$$\mathcal{B}|_{\{2,3,4\}} = \{\cancel{1}, 2, 3, 4, \cancel{12}, 23, 34, \cancel{123}, 234, \cancel{1234}\}$$



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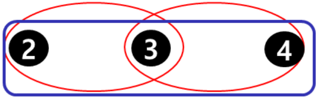
The restricted building set $\mathcal{B}|_{\{2,3,4\}}$ is as follows:

$$\mathcal{B}|_{\{2,3,4\}} = \{2, 3, 4, 23, 34, 234\}$$

$$\mathcal{B}|_{\{2,3,4\}}$$



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Definition

For a building set $\mathcal{B}$, a permutation $x_1 \cdots x_n$ is a $\mathcal{B}$-**permutation** if, for each $1 \leq i \leq n$, there is $\mathrm{J}_i \in \mathcal{B}|_{\{x_1, x_2, \ldots, x_i\}}$ such that

$$\{x_i, \max\{x_1, x_2, \ldots, x_i\}\} \subset \mathrm{J}_i,$$

and $\mathfrak{S}(\mathcal{B})$ denotes the set of $\mathcal{B}$-permutations.

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Example

If a building set $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$, then **1432** is a $\mathcal{B}$-permutation.

$$\because \begin{cases} \{\mathbf{1}, \max\{\mathbf{1}\}\} \subset \mathrm{J}_1 = \{1\} \in \mathcal{B}|_{\{1\}}, \\ \{\mathbf{4}, \max\{1, \mathbf{4}\}\} \subset \mathrm{J}_2 = \{4\} \in \mathcal{B}|_{\{1,4\}}, \\ \{\mathbf{3}, \max\{1, \mathbf{4}, 3\}\} \subset \mathrm{J}_3 = \{3, 4\} \in \mathcal{B}|_{\{1,4,3\}}, \\ \{\mathbf{2}, \max\{1, \mathbf{4}, 3, 2\}\} \subset \mathrm{J}_4 = \{1, 2, 3, 4\} \in \mathcal{B}. \end{cases}$$

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Example

If a building set $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$, then **1423** is **not** a $\mathcal{B}$-permutation.

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and $\mathfrak{S}(\mathcal{B})$ denotes the set of $\mathcal{B}$-permutations.

Example

If a building set $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$, then **1423** is **not** a $\mathcal{B}$-permutation. Because, there is no J_3 in

$$\mathcal{B}|_{\{1,4,2\}} = \{1, 2, 4, 12\}$$

such that $\{\mathbf{2}, \max\{1, \mathbf{4}, 2\}\} = \{2, 4\} \in \mathrm{J}_3$.

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Example 3

A building set $\mathcal{B}$ is as follows:

$$\{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$$

For each $0 \leq k \leq 3$, count the number of $\mathcal{B}$-permutations on $\{1, 2, 3, 4\}$ with exactly k descents.

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count	1		6		6		1

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count	1	6	6	1

3. Toric manifolds associated with chordal nestohedra

Theorem (A. Postnikov, V. Reiner, L. Williams, 2008)

For a chordal building set $\mathcal{B}$ on $\{1, 2, \ldots, n+1\}$, the h-vector of the nestohedron $P_{\mathcal{B}}$ is as follows:

$$h_k(P_{\mathcal{B}}) = \begin{cases} \left|\{x \in \mathfrak{S}(\mathcal{B}) \colon \#\text{descents of } x = \frac{k}{2}\}\right|, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

3. Toric manifolds associated with chordal nestohedra

Corollary (A. Postnikov, V. Reiner, L. Williams, 2008)

For a chordal building set $\mathcal{B}$ on $\{1, \ldots, n\}$,

$$\dim H_k(X_{\mathcal{B}}; \mathbb{Q}) = \begin{cases} \left|\{x \in \mathfrak{S}(\mathcal{B})\colon \#\text{descents of } x = \frac{k}{2}\}\right|, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

3. Toric manifolds associated with chordal nestohedra

Corollary (A. Postnikov, V. Reiner, L. Williams, 2008)

For a chordal building set $\mathcal{B}$ on $\{1, \ldots, n\}$,

$$\dim H_k(X_{\mathcal{B}}; \mathbb{Q}) = \begin{cases} \left|\{x \in \mathfrak{S}(\mathcal{B}) \colon \#\text{descents of } x = \frac{k}{2}\}\right|, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

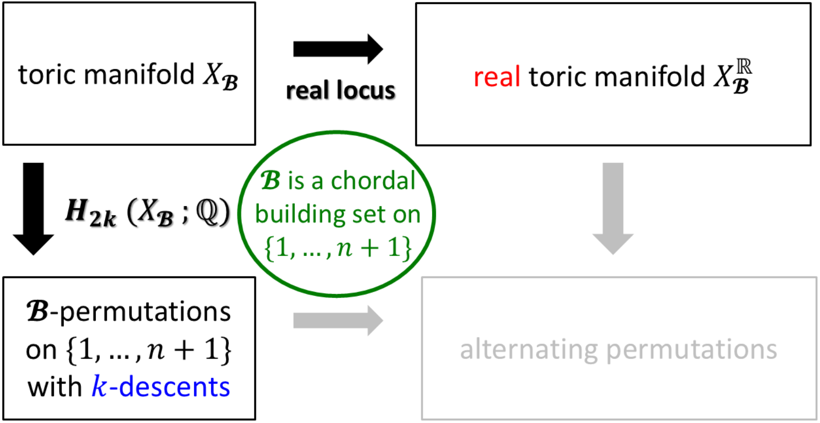
Example 3 (revisit)

Let $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. Compute the rational Betti numbers $\dim H_*(X_{\mathcal{B}}; \mathbb{Q})$ of $X_{\mathcal{B}}$.

k	0	1	2	3	4	5	6
$\mathcal{B}$-permutations	1234		2134,2314, 1324,2341, 1342,1243		3214,2143, 3241,1432, 2431,3421		4321
$\dim H_k(X_{\mathcal{B}}; \mathbb{Q})$	1	0	6	0	6	0	1

3. Toric manifolds associated with chordal nestohedra

Summary



4. Real toric manifolds associated with chordal nestohedra

Theorem (S. Choi and Y. Yoon, arXiv:2407.11313)

For a chordal building set $\mathcal{B}$ on $[n+1] = \{1, 2, \ldots, n+1\}$,

$$\dim H_k(X_{\mathcal{B}}^{\mathbb{R}}; \mathbb{Q}) = \sum_{\mathrm{I}} \#\text{alternating } \mathcal{B}|_{\mathrm{I}}\text{-permutations},$$

where I is a $2k$-subset of $[n+1]$.

4. Real toric manifolds associated with chordal nestohedra

Theorem (S. Choi and Y. Yoon, arXiv:2407.11313)

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4. Real toric manifolds associated with chordal nestohedra

Example 4

Let $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. Compute the rational Betti numbers $\dim H_*(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$ of $X^{\mathbb{R}}_{\mathcal{B}}$.

4. Real toric manifolds associated with chordal nestohedra

Example 4

Let $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. Compute the rational Betti numbers $\dim H_*(X_{\mathcal{B}}^{\mathbb{R}}; \mathbb{Q})$ of $X_{\mathcal{B}}^{\mathbb{R}}$.

k	0	1	2
alternating permutations on $2k$–subsets of $\{1, \ldots, n+1\}$	( )	43,42,41, 32,31,21	4231,4132, 3241,3142,2143
count	1	6	5

4. Real toric manifolds associated with chordal nestohedra

Example 4

Let $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. Compute the rational Betti numbers $\dim H_*(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$ of $X^{\mathbb{R}}_{\mathcal{B}}$.

k	0	1	2
alternating $\mathcal{B}\|_I$-permutations ($I \subset \{1, \ldots, n+1\}$, $\|I\| = 2k$)	( )	43,42,41, 32,31,21	4231,4132, 3241,3142,2143
count = $\dim H_k(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$	1		

4. Real toric manifolds associated with chordal nestohedra

Example 4

Let $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. Compute the rational Betti numbers $\dim H_*(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$ of $X^{\mathbb{R}}_{\mathcal{B}}$.

k	0	1	2
alternating $\mathcal{B}\|_I$-permutations ($I \subset \{1, \dots, n+1\}, \|I\| = 2k$)	( )	**43**,42,41, 32,31,21	4231,4132, 3241,3142,2143
count = $\dim H_k(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$	1	1	

- Since

$$\begin{cases} \{\mathbf{4}, \max\{\mathbf{4}\}\} \subset J_1 = \{4\} \in \mathcal{B}|_{\{4\}} = \{\{4\}\} \\ \{\mathbf{3}, \max\{\mathbf{4}, 3\}\} \subset J_2 = \{4, 3\} \in \mathcal{B}|_{\{4,3\}} = \{\{3\}, \{4\}, \{3, 4\}\}, \end{cases}$$

the permutation **43** is an alternating $\mathcal{B}|_{\{4,3\}}$-permutation.

4. Real toric manifolds associated with chordal nestohedra

Example 4

Let $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. Compute the rational Betti numbers $\dim H_*(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$ of $X^{\mathbb{R}}_{\mathcal{B}}$.

k	0	1	2
alternating $\mathcal{B}\vert_I$-permutations ($I \subset \{1, \dots, n+1\}, \vert I\vert = 2k$)	( )	**43**,~~42~~,**41, 32,31,21**	~~4231~~,**4132, 3241,3142,2143**
count = $\dim H_k(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$	1	1	

- Since there is no J_2 such that

$$\{\mathbf{2}, \max\{\mathbf{4}, 2\}\} \subset \mathrm{J}_2 \in \mathcal{B}|_{\{4,2\}} = \{2, 4\},$$

both **42** and **4231** are **not** alternating $\mathcal{B}|_{\{4,2\}}$-permutations.

4. Real toric manifolds associated with chordal nestohedra

Example 4

Let $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. Compute the rational Betti numbers $\dim H_*(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$ of $X^{\mathbb{R}}_{\mathcal{B}}$.

k	0	1	2
alternating $\mathcal{B}\|_I$-permutations $(I \subset \{1, \ldots, n+1\}, \|I\| = 2k)$	( )	**43**,~~42~~,~~41~~, 32,31,21	~~4231~~,~~4132~~, 3241,3142,2143
count = $\dim H_k(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$	1	1	

- Since there is no J_2 such that

$$\{\mathbf{1}, \max\{\mathbf{4}, 1\}\} \subset J_2 \in \mathcal{B}|_{\{4,1\}} = \{1, 4\},$$

both **41** and **4132** are **not** alternating $\mathcal{B}|_{\{4,1\}}$-permutations.

4. Real toric manifolds associated with chordal nestohedra

Example 4

Let $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. Compute the rational Betti numbers $\dim H_*(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$ of $X^{\mathbb{R}}_{\mathcal{B}}$.

k	0	1	2
alternating $\mathcal{B}\vert_I$-permutations ($I \subset \{1, \dots, n+1\}, \vert I\vert = 2k$)	( )	**43**,~~42~~,~~41~~, **32**,**31**,**21**	~~4231~~,~~4132~~, **3241**,**3142**,**2143**
count = $\dim H_k(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$	1	2	

- Both **32** and **3241** are alternating $\mathcal{B}|$-permutations.

$$\because \begin{cases} \{\mathbf{3}, \max\{\mathbf{3}\}\} \subset J_1 = \{3\} \in \mathcal{B}|_{\{3\}} = \{\{3\}\} \\ \{\mathbf{2}, \max\{\mathbf{3}, 2\}\} \subset J_2 = \{3, 2\} \in \mathcal{B}|_{\{3,2\}} = \{\{2\}, \{3\}, \{2, 3\}\} \\ \{\mathbf{4}, \max\{3, 2, \mathbf{4}\}\} \subset J_3 = \{4\} \in \mathcal{B}|_{\{3,2,4\}} = \{\{2\}, \{3\}, \{4\}, \cdots\} \\ \{\mathbf{1}, \max\{3, 2, \mathbf{4}, 1\}\} \subset J_4 = \{1, 2, 3, 4\} \in \mathcal{B}, \end{cases}$$

4. Real toric manifolds associated with chordal nestohedra

Example 4

Let $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. Compute the rational Betti numbers $\dim H_*(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$ of $X^{\mathbb{R}}_{\mathcal{B}}$.

k	0	1	2
alternating $\mathcal{B}\|_I$-permutations ($I \subset \{1, \ldots, n+1\}, \|I\| = 2k$)	( )	43, ~~42~~, ~~41~~, 32, ~~31~~, 21	~~4231~~, ~~4132~~, 3241, ~~3142~~, 2143
count = $\dim H_k(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$	1	2	

- Since there is no J_2 such that

$$\{\mathbf{1}, \max\{\mathbf{3}, 1\}\} \subset J_2 \in \mathcal{B}|_{\{3,1\}} = \{1, 3\},$$

both **31** and **3142** are **not** alternating $\mathcal{B}|_{\{3,1\}}$-permutations.

4. Real toric manifolds associated with chordal nestohedra

Example 4

Let $\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. Compute the rational Betti numbers $\dim H_*(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$ of $X^{\mathbb{R}}_{\mathcal{B}}$.

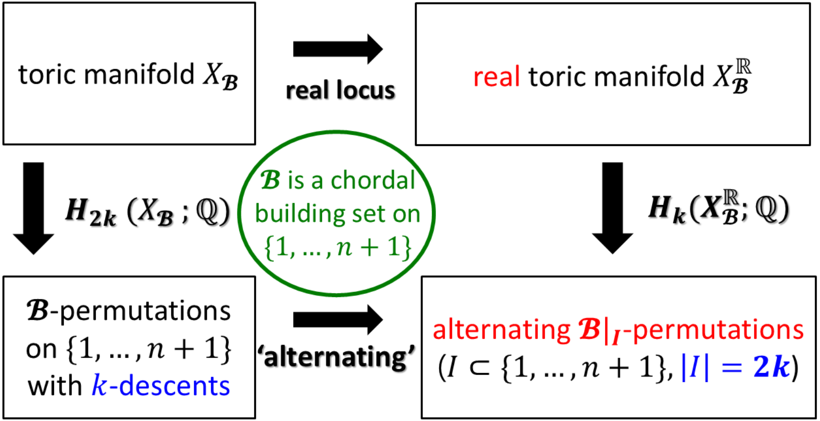
k	0	1	2
alternating $\mathcal{B}\|_I$-permutations $(I \subset \{1, \dots, n+1\}, \|I\| = 2k)$	( )	43,~~42~~,~~41~~, 32,~~31~~,21	~~4231~~,~~4132~~, 3241,~~3142~~,2143
count = $\dim H_k(X^{\mathbb{R}}_{\mathcal{B}}; \mathbb{Q})$	1	3	2

- Both **21** and **2143** are alternating $\mathcal{B}|_{\{4,3\}}$-permutations.

$$\because \begin{cases} \{\mathbf{2}, \max\{\mathbf{2}\}\} \subset J_1 = \{2\} \in \mathcal{B}|_{\{2\}} = \{\{2\}\} \\ \{\mathbf{1}, \max\{\mathbf{2}, 1\}\} \subset J_2 = \{1, 2\} \in \mathcal{B}|_{\{2,1\}} = \{\{1\}, \{2\}, \{1, 2\}\} \\ \{\mathbf{4}, \max\{2, 1, \mathbf{4}\}\} \subset J_3 = \{4\} \in \mathcal{B}|_{\{2,1,4\}} = \{\cdots, \{4\}, \{1, 2\}\} \\ \{\mathbf{3}, \max\{2, 1, \mathbf{4}, 3\}\} \subset J_4 = \{3, 4\} \in \mathcal{B}, \end{cases}$$

4. Real toric manifolds associated with chordal nestohedra

Summary



5. Examples

Definition

For a finite simple graph G, the set of connected induced subgraphs of G is the **graphical building set** $\mathcal{B}(G)$ of G.
The nestohedron $P_{\mathcal{B}(G)}$ of $\mathcal{B}(G)$ is called a **graph associahedron** of a graph G.

Example

Let P_4 be a path graph as follows:

Then $\mathcal{B}(P_4) = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$.

5. Examples

Remark

Let G be a finite simple graph G. In 2015, S. Choi and H. Park define the **graph a-number** $a(G)$, that is, a purely graph invariant which corresponds the rational Betti numbers of its associated real toric manifold $X^{\mathbb{R}}_{\mathcal{B}(G)}$.

Proposition

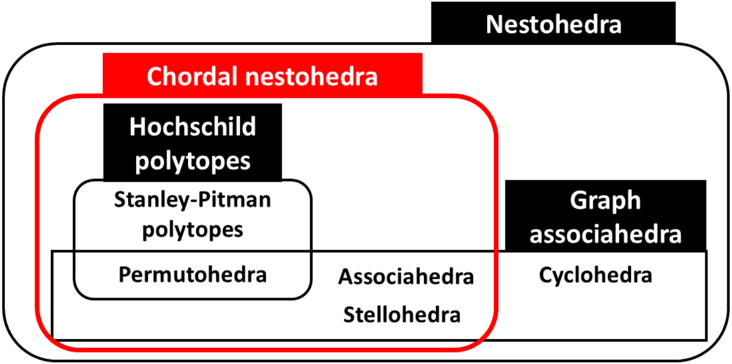
For a chordal graph G with a perfect elimination ordering, the graphical building set $\mathcal{B}(G)$ is a chordal building set.

Remark

For a chordal graph G, a combinatorial interpretation of the graph a-number $a(G)$ is the number of alternating $\mathcal{B}(G)$-permutations.

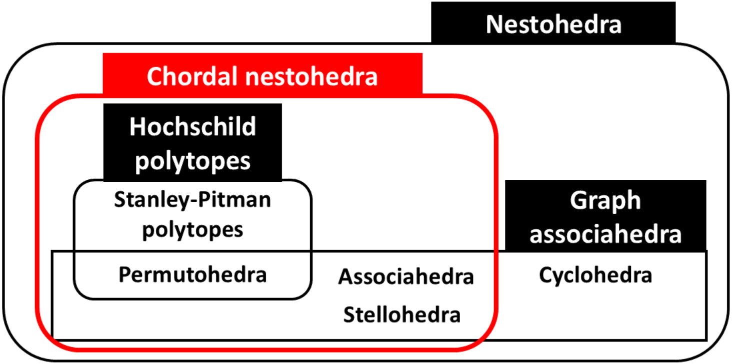
5. Examples

Category of nestohedra



5. Examples

Category of nestohedra



Conjecture

If $\mathcal{B}$ is a chordal or graphical building set, then the sequence $\{\dim H_k(X^{\mathbb{R}}_{\mathcal{B}};\mathbb{Q})\}_{k\geq 0}$ is unimodal.

Thank you for your attention!

-Younghan Yoon (Ajou univ.)