Model Theory Postdoctoral Position

I expect to have a postdoctoral position available starting either in January or July. Please tell any plausible candidate you know about this and ask them to write to me at `f.tall@utoronto.ca`.

I work on applications of set theory and/or topology to model theory, and applications of model theory to analysis, but don't know enough model theory or analysis, so I am looking for a collaborator. The ideal candidate would know model theory, some set theory, and some functional analysis.

On the Undefinability of Pathological Banach Spaces

Clovis Hamel & Franklin D. Tall

September 5, 2024 version

Overview

Definition

The Banach space of sequences of real numbers converging to 0 is denoted by c_0. The Banach space of sequences $\{x_n\}_{n<\omega}$ of real numbers such that $\sum_{n<\omega} |x_n|^p < \infty$ is denoted by ℓ^p.

Overview

Definition

The Banach space of sequences of real numbers converging to 0 is denoted by c_0. The Banach space of sequences $\{x_n\}_{n<\omega}$ of real numbers such that $\sum_{n<\omega} |x_n|^p < \infty$ is denoted by ℓ^p.

Classical Banach spaces all include an isomorphic copy of either c_0 or some ℓ^p.

Overview

Definition

The Banach space of sequences of real numbers converging to 0 is denoted by c_0. The Banach space of sequences $\{x_n\}_{n<\omega}$ of real numbers such that $\sum_{n<\omega} |x_n|^p < \infty$ is denoted by ℓ^p.

Classical Banach spaces all include an isomorphic copy of either c_0 or some ℓ^p. Tsirelson implicitly(!) defined a Banach space that didn't. Gowers asked whether explicitly defined ones do.

Overview

Definition

The Banach space of sequences of real numbers converging to 0 is denoted by c_0. The Banach space of sequences $\{x_n\}_{n<\omega}$ of real numbers such that $\sum_{n<\omega} |x_n|^p < \infty$ is denoted by ℓ^p.

Classical Banach spaces all include an isomorphic copy of either c_0 or some ℓ^p. Tsirelson implicitly(!) defined a Banach space that didn't. Gowers asked whether explicitly defined ones do. Casazza and Iovino showed that to be the case for sufficiently expressive compact logics, in particular for first order continuous logic.

Overview

Definition

The Banach space of sequences of real numbers converging to 0 is denoted by c_0. The Banach space of sequences $\{x_n\}_{n<\omega}$ of real numbers such that $\sum_{n<\omega} |x_n|^p < \infty$ is denoted by ℓ^p.

Classical Banach spaces all include an isomorphic copy of either c_0 or some ℓ^p. Tsirelson implicitly(!) defined a Banach space that didn't. Gowers asked whether explicitly defined ones do. Casazza and Iovino showed that to be the case for sufficiently expressive compact logics, in particular for first order continuous logic. We extend their results to logics satisfying much weaker conditions, employing the topology of function spaces. A particular concrete example is continuous $L_{\omega_1,\omega}$.

Overview

Definition

The Banach space of sequences of real numbers converging to 0 is denoted by c_0. The Banach space of sequences $\{x_n\}_{n<\omega}$ of real numbers such that $\sum_{n<\omega} |x_n|^p < \infty$ is denoted by ℓ^p.

Classical Banach spaces all include an isomorphic copy of either c_0 or some ℓ^p. Tsirelson implicitly(!) defined a Banach space that didn't. Gowers asked whether explicitly defined ones do. Casazza and Iovino showed that to be the case for sufficiently expressive compact logics, in particular for first order continuous logic. We extend their results to logics satisfying much weaker conditions, employing the topology of function spaces. A particular concrete example is continuous $L_{\omega_1,\omega}$. This work dramatically improves our previous papers on this subject.

Definitions

Definition

For X a completely regular Hausdorff topological space, $C_p(X)$ denotes the space of real-valued continuous functions on X, endowed with the subspace topology inherited from the product topology on $\mathbb{R}^X$.

Definitions

Definition

Let X be a topological space.

(a) A subset $A \subseteq X$ is *relatively compact* (in X) if $\overline{A}$ is compact.

Definitions

Definition

Let X be a topological space.

(a) A subset $A \subseteq X$ is *relatively compact* (in X) if $\overline{A}$ is compact.

(b) The space X is *countably compact* if every infinite subset $Y \subseteq X$ has a *limit point*, i.e., a point $x \in X$ such that every open set about x contains infinitely many points of Y.

Definitions

Definition

Let X be a topological space.

(a) A subset $A \subseteq X$ is *relatively compact* (in X) if $\overline{A}$ is compact.

(b) The space X is *countably compact* if every infinite subset $Y \subseteq X$ has a *limit point*, i.e., a point $x \in X$ such that every open set about x contains infinitely many points of Y.
For $A \subseteq B \subseteq X$, the subset A is *countably compact in B* if every infinite subset of A has a limit point in B. The subset A is *relatively countably compact* if its closure in X is countably compact.

Definitions

Definition

Let X be a topological space.

(a) A subset $A \subseteq X$ is *relatively compact* (in X) if $\overline{A}$ is compact.

(b) The space X is *countably compact* if every infinite subset $Y \subseteq X$ has a *limit point*, i.e., a point $x \in X$ such that every open set about x contains infinitely many points of Y.
For $A \subseteq B \subseteq X$, the subset A is *countably compact in B* if every infinite subset of A has a limit point in B. The subset A is *relatively countably compact* if its closure in X is countably compact.

(Just think of "countably compact" vs. "compact")

Grothendieck's Theorem

Proposition

A T_1 space X is countably compact if and only if every countable open cover of X includes a finite subcover.

Grothendieck's Theorem

Proposition

A T_1 space X is countably compact if and only if every countable open cover of X includes a finite subcover.

Grothendieck's Theorem

Let X be a countably compact topological space, and $A \subseteq C_p(X)$. Then A is relatively compact (in $C_p(X)$) if and only if it is countably compact in $C_p(X)$.

Grothendieck's Theorem

Proposition

A T_1 space X is countably compact if and only if every countable open cover of X includes a finite subcover.

Grothendieck's Theorem

Let X be a countably compact topological space, and $A \subseteq C_p(X)$. Then A is relatively compact (in $C_p(X)$) if and only if it is countably compact in $C_p(X)$.

Grothendieck's Theorem for compact X is the topology behind the results of Casazza-Iovino. We use more sophisticated topology to extend their results.

L-structures

Definition

Given a language L, an *L-structure* $\mathfrak{M}$ is a set M, called the *universe* of $\mathfrak{M}$, together with an *interpretation* of elements of L, i.e., a function which assigns an element of M to each constant symbol, a function from M^n to M for each n-ary function symbol, and a subset of M^n to each n-ary relation symbol.
Given an L-formula φ, we denote by $\varphi^{\mathfrak{M}}$ the interpretation of φ in $\mathfrak{M}$.

L-structures

Definition

Let $\mathrm{Str}(L)$ be the set of all equivalence classes under elementary equivalence of L-structures. For each theory T, let $[T] = \{\widetilde{\mathfrak{M}} \in \mathrm{Str}(L) : \mathfrak{M} \models T\}$ (where $\widetilde{\mathfrak{M}}$ is the equivalence class of $\mathfrak{M}$). The collection of all sets of the form $[T]$ constitutes a basis for the closed sets of the topology on $\mathrm{Str}(L)$ known as the *space of L-structures*.

L-structures

Definition

Let $\mathrm{Str}(L)$ be the set of all equivalence classes under elementary equivalence of L-structures. For each theory T, let $[T] = \{\widetilde{\mathfrak{M}} \in \mathrm{Str}(L) : \mathfrak{M} \models T\}$ (where $\widetilde{\mathfrak{M}}$ is the equivalence class of $\mathfrak{M}$). The collection of all sets of the form $[T]$ constitutes a basis for the closed sets of the topology on $\mathrm{Str}(L)$ known as the *space of L-structures*.

(A logic satisfies the Compactness Theorem iff its structure space is compact.)

Metric structures

When doing continuous logic, one doesn't deal with all possible structures but rather only with *metric structures*. We follow the formulation in Chris Eagle's papers.

Metric structures

When doing continuous logic, one doesn't deal with all possible structures but rather only with *metric structures*. We follow the formulation in Chris Eagle's papers.

The continuous first-order formulas are constructed just as in the discrete setting except for the following addition: if $f : [0,1]^n \to [0,1]$ is a continuous function and $\varphi_0, \ldots, \varphi_{n-1}$ are L-formulas, then $f(\varphi_0, \ldots, \varphi_{n-1})$ is also an L-formula.

Metric structures

When doing continuous logic, one doesn't deal with all possible structures but rather only with *metric structures*. We follow the formulation in Chris Eagle's papers.

The continuous first-order formulas are constructed just as in the discrete setting except for the following addition: if $f : [0,1]^n \to [0,1]$ is a continuous function and $\varphi_0, \ldots, \varphi_{n-1}$ are L-formulas, then $f(\varphi_0, \ldots, \varphi_{n-1})$ is also an L-formula.

It is customary to identify formulas φ of arity n with functions $\mathfrak{M}^n \to [0,1]$. This allows an easy way to define the satisfaction relation, i.e., if φ is an L-formula, $\mathfrak{M}$ an L-structure, and $a \in \mathfrak{M}^n$, then $\mathfrak{M} \models \varphi(a)$ if and only if $\varphi(a) = 1$.

Definition

(a) If (M, d) and (N, ρ) are metric spaces and $f\colon M^n \to N$ is uniformly continuous, a *modulus of uniform continuity of* f is a function $\delta\colon (0,1) \cap \mathbb{Q} \to (0,1) \cap \mathbb{Q}$ such that whenever $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_n) \in M^n$ and $\varepsilon \in (0,1) \cap \mathbb{Q}$, $\sup\{d(a_i, b_i) : 1 \le i \le n\} < \delta(\varepsilon)$ implies $\rho(f(\boldsymbol{a}), f(\boldsymbol{b})) < \varepsilon$. Similarly define a modulus of uniform continuity for a predicate.

Definition

(a) If (M, d) and (N, ρ) are metric spaces and $f\colon M^n \to N$ is uniformly continuous, a *modulus of uniform continuity of* f is a function $\delta\colon (0,1) \cap \mathbb{Q} \to (0,1) \cap \mathbb{Q}$ such that whenever $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n) \in M^n$ and $\varepsilon \in (0,1) \cap \mathbb{Q}$, $\sup\{d(a_i, b_i) : 1 \leq i \leq n\} < \delta(\varepsilon)$ implies $\rho(f(\boldsymbol{a}), f(\boldsymbol{b})) < \varepsilon$. Similarly define a modulus of uniform continuity for a predicate.

(b) A *language* for metric structures is a set L which consists of constants, functions with an associated arity, and a modulus of uniform continuity; predicates with an associated arity and a modulus of uniform continuity; and a symbol d for a metric.

Definition

(a) If (M, d) and (N, ρ) are metric spaces and $f\colon M^n \to N$ is uniformly continuous, a *modulus of uniform continuity of* f is a function $\delta\colon (0,1) \cap \mathbb{Q} \to (0,1) \cap \mathbb{Q}$ such that whenever $\boldsymbol{a} = (a_1, ..., a_n)$, $\boldsymbol{b} = (b_1, ..., b_n) \in M^n$ and $\varepsilon \in (0,1) \cap \mathbb{Q}$, $\sup\{d(a_i, b_i) : 1 \leq i \leq n\} < \delta(\varepsilon)$ implies $\rho(f(\boldsymbol{a}), f(\boldsymbol{b})) < \varepsilon$. Similarly define a modulus of uniform continuity for a predicate.

(b) A *language* for metric structures is a set L which consists of constants, functions with an associated arity, and a modulus of uniform continuity; predicates with an associated arity and a modulus of uniform continuity; and a symbol d for a metric.

(c) An *L-metric structure* $\mathfrak{M}$ is a metric space (M, d^M) together with interpretations for each symbol in L: $c^{\mathfrak{M}} \in M$ for each constant $c \in L$; $f^{\mathfrak{M}} : M^n \to M$ a uniformly continuous function for each n-ary function symbol $f \in L$; $P^{\mathfrak{M}} : M^n \to [0,1]$ a uniformly continuous function for each n-ary predicate symbol $P \in L$. Assume for the sake of notational simplicity that all metric structures have diameter 1.

Definition

(a) If (M, d) and (N, ρ) are metric spaces and $f\colon M^n \to N$ is uniformly continuous, a *modulus of uniform continuity of f* is a function $\delta\colon (0,1)\cap\mathbb{Q} \to (0,1)\cap\mathbb{Q}$ such that whenever $\boldsymbol{a} = (a_1, ..., a_n)$, $\boldsymbol{b} = (b_1, ..., b_n) \in M^n$ and $\varepsilon \in (0,1) \cap \mathbb{Q}$, $\sup\{d(a_i, b_i) : 1 \leq i \leq n\} < \delta(\varepsilon)$ implies $\rho(f(\boldsymbol{a}), f(\boldsymbol{b})) < \varepsilon$. Similarly define a modulus of uniform continuity for a predicate.

(b) A *language* for metric structures is a set L which consists of constants, functions with an associated arity, and a modulus of uniform continuity; predicates with an associated arity and a modulus of uniform continuity; and a symbol d for a metric.

(c) An *L-metric structure* $\mathfrak{M}$ is a metric space (M, d^M) together with interpretations for each symbol in L: $c^{\mathfrak{M}} \in M$ for each constant $c \in L$; $f^{\mathfrak{M}} : M^n \to M$ a uniformly continuous function for each n-ary function symbol $f \in L$; $P^{\mathfrak{M}} : M^n \to [0,1]$ a uniformly continuous function for each n-ary predicate symbol $P \in L$. Assume for the sake of notational simplicity that all metric structures have diameter 1.

(d) The space of L-structures in a given language L is the family $\mathrm{Str}(L)$ of all L-metric structures endowed with the topology generated by the following basic closed sets: $[\varphi] = \{\mathfrak{M} : \mathfrak{M} \models \varphi\}$ where φ is an L-sentence.

Definition

(b) A *language* for metric structures is a set L which consists of constants, functions with an associated arity, and a modulus of uniform continuity; predicates with an associated arity and a modulus of uniform continuity; and a symbol d for a metric.

(c) An *L-metric structure* $\mathfrak{M}$ is a metric space (M, d^M) together with interpretations for each symbol in L: $c^{\mathfrak{M}} \in M$ for each constant $c \in L$; $f^{\mathfrak{M}} : M^n \to M$ a uniformly continuous function for each n-ary function symbol $f \in L$; $P^{\mathfrak{M}} : M^n \to [0,1]$ a uniformly continuous function for each n-ary predicate symbol $P \in L$. Assume for the sake of notational simplicity that all metric structures have diameter 1.

(d) The space of L-structures in a given language L is the family $\mathrm{Str}(L)$ of all L-metric structures endowed with the topology generated by the following basic closed sets: $[\varphi] = \{\mathfrak{M} : \mathfrak{M} \models \varphi\}$ where φ is an L-sentence.

Actually, in order to get Hausdorffness and so as not to deal with proper classes, we define $\mathrm{Str}(L)$ to be the *set* of elementary equivalence classes of metric structures. This turns out to be a useful trick in proving the undefinability results for Lindelöf logics. Except where it's useful, we will be sloppy and write $\mathfrak{M}$ rather than $\widetilde{\mathfrak{M}}$.

Ultralimits

One of the many equivalent definitions of *stability* in compact logics is definability; another one, introduced by Iovino, is a double limit condition.

Ultralimits

One of the many equivalent definitions of *stability* in compact logics is definability; another one, introduced by Iovino, is a double limit condition. We pursued a detailed analysis of the topological conditions on a logic that enable the equivalence between definability and double limit conditions to hold.

Ultralimits

We pursued a detailed analysis of the topological conditions on a logic that enable the equivalence between definability and double limit conditions to hold.

Definition

Let X be a topological space. Given an ultrafilter $\mathcal{U}$ on a regular cardinal κ, and a κ-sequence $\{x_\alpha\}_{\alpha<\kappa}$ in X, we say that

$$\lim_{\alpha\to\mathcal{U}} x_\alpha = x$$

if and only if for every open neighbourhood U about x we have $\{\alpha < \kappa : x_\alpha \in U\} \in \mathcal{U}$.

Ultralimits

Definition

Let X be a topological space. Given an ultrafilter $\mathcal{U}$ on a regular cardinal κ, and a κ-sequence $\{x_\alpha\}_{\alpha<\kappa}$ in X, we say that

$$\lim_{\alpha \to \mathcal{U}} x_\alpha = x$$

if and only if for every open neighbourhood U about x we have $\{\alpha < \kappa : x_\alpha \in U\} \in \mathcal{U}$.

It follows from the fact that an ultrafilter cannot contain disjoint sets that if such an ultralimit exists in a Hausdorff space, it is unique. The following classical result relates ultralimits and compactness:

Ultralimits

Definition

Let X be a topological space. Given an ultrafilter $\mathcal{U}$ on a regular cardinal κ, and a κ-sequence $\{x_\alpha\}_{\alpha<\kappa}$ in X, we say that

$$\lim_{\alpha\to\mathcal{U}} x_\alpha = x$$

if and only if for every open neighbourhood U about x we have $\{\alpha < \kappa : x_\alpha \in U\} \in \mathcal{U}$.

It follows from the fact that an ultrafilter cannot contain disjoint sets that if such an ultralimit exists in a Hausdorff space, it is unique. The following classical result relates ultralimits and compactness:

Theorem

A space X is compact if and only if every ultralimit in X exists.

Double Ultralimits

Definition

Let X be a topological space and $A \subseteq C_p(X, [0,1])$. We write $\mathrm{DULC}(A, X)$ if:

for every pair of sequences $\{f_n\}_{n<\omega} \subseteq A$ and $\{x_m\}_{m<\omega} \subseteq X$, and ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on ω, the following double limits agree

$$\lim_{n\to\mathcal{U}} \lim_{m\to\mathcal{V}} f_n(x_m) = \lim_{m\to\mathcal{V}} \lim_{n\to\mathcal{U}} f_n(x_m),$$

whenever all these limits exist.

Double Ultralimits

Definition

Let X be a topological space and $A \subseteq C_p(X, [0,1])$. We write $\mathrm{DULC}(A, X)$ if:

for every pair of sequences $\{f_n\}_{n<\omega} \subseteq A$ and $\{x_m\}_{m<\omega} \subseteq X$, and ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on ω, the following double limits agree

$$\lim_{n\to\mathcal{U}} \lim_{m\to\mathcal{V}} f_n(x_m) = \lim_{m\to\mathcal{V}} \lim_{n\to\mathcal{U}} f_n(x_m),$$

whenever all these limits exist.

We write $\mathrm{DULC}(X)$ if $\mathrm{DULC}(A, X)$ holds for all $A \subseteq C_p(X, [0,1])$. We say X satisfies the *double ultralimit condition* if for each $A \subseteq C_p(X, [0,1])$, $\mathrm{DULC}(A, X)$ is equivalent to A being relatively countably compact.

Double limits

Definition

Let X be a topological space and $A \subseteq C_p(X, [0,1])$. We write $\mathrm{DLC}(A, X)$ if:

for every pair of sequences $\{f_n\}_{n<\omega} \subseteq A$ *and* $\{x_m\}_{m<\omega} \subseteq X$, *the following double limits agree whenever they exist*

$$\lim_{n\to\infty} \lim_{m\to\infty} f_n(x_m) = \lim_{m\to\infty} \lim_{n\to\infty} f_n(x_m).$$

Double limits

Definition

Let X be a topological space and $A \subseteq C_p(X, [0,1])$. We write $\mathrm{DLC}(A, X)$ if:

for every pair of sequences $\{f_n\}_{n<\omega} \subseteq A$ *and* $\{x_m\}_{m<\omega} \subseteq X$, *the following double limits agree whenever they exist*

$$\lim_{n\to\infty} \lim_{m\to\infty} f_n(x_m) = \lim_{m\to\infty} \lim_{n\to\infty} f_n(x_m).$$

We write $\mathrm{DLC}(X)$ if $\mathrm{DLC}(A, X)$ holds for all $A \subseteq C_p(X, [0,1])$. We say X satisfies the *double limit condition* if for each $A \subseteq C_p(X, [0,1])$, $\mathrm{DLC}(A, X)$ is equivalent to A being relatively countably compact.

Double limits

Definition

Let X be a topological space and $A \subseteq C_p(X, [0,1])$. We write $\mathrm{DLC}(A, X)$ if:

for every pair of sequences $\{f_n\}_{n<\omega} \subseteq A$ and $\{x_m\}_{m<\omega} \subseteq X$, the following double limits agree whenever they exist

$$\lim_{n\to\infty} \lim_{m\to\infty} f_n(x_m) = \lim_{m\to\infty} \lim_{n\to\infty} f_n(x_m).$$

We write $\mathrm{DLC}(X)$ if $\mathrm{DLC}(A, X)$ holds for all $A \subseteq C_p(X, [0,1])$. We say X satisfies the *double limit condition* if for each $A \subseteq C_p(X, [0,1])$, $\mathrm{DLC}(A, X)$ is equivalent to A being relatively countably compact.

Theorem

If X satisfies the double limit condition, then it satisfies the double ultralimit condition.

Grothendieck spaces

Definition

A topological space X is called a *g-space* if every $A \subseteq X$ which is countably compact in X is relatively compact. We say that X is a *hereditary g-space* (a.k.a. *angelic*) if every subspace of X is a g-space.

Grothendieck spaces

The following definitions are the starting point for generalizing Grothendieck's Theorem to a broader class of spaces:

Grothendieck spaces

The following definitions are the starting point for generalizing Grothendieck's Theorem to a broader class of spaces:

Definition (Arhangel'skiĭ)

A topological space X is *Grothendieck* if $C_p(X)$ is a hereditary g-space. X is *weakly Grothendieck* if $C_p(X)$ is a g-space.

Grothendieck spaces

Definition (Arhangel'skiĭ)

A topological space X is *Grothendieck* if $C_p(X)$ is a hereditary g-space. X is *weakly Grothendieck* if $C_p(X)$ is a g-space.

From these definitions, we get a generalization of Grothendieck's Theorem.

Grothendieck spaces

Definition (Arhangel'skiĭ)

A topological space X is *Grothendieck* if $C_p(X)$ is a hereditary g-space. X is *weakly Grothendieck* if $C_p(X)$ is a g-space.

From these definitions, we get a generalization of Grothendieck's Theorem.

Lemma

If X is weakly Grothendieck and satisfies the double ultralimit condition, then a subset $A \subseteq C_p(X, [0,1])$ satisfying $\mathrm{DULC}(A, X)$ is relatively compact.

Grothendieck spaces

Definition (Arhangel'skiĭ)

A topological space X is *Grothendieck* if $C_p(X)$ is a hereditary g-space. X is *weakly Grothendieck* if $C_p(X)$ is a g-space.

Lemma

If X is weakly Grothendieck and satisfies the double ultralimit condition, then a subset $A \subseteq C_p(X, [0,1])$ satisfying $\mathrm{DULC}(A, X)$ is relatively compact.

Now we can follow Casazza-Iovino and get their undefinability results by just finding familiar (at least to topologists) classes of spaces which are weakly Grothendieck and satisfy the double ultralimit condition.

Grothendieck spaces

Lemma

If X is weakly Grothendieck and satisfies the double ultralimit condition, then a subset $A \subseteq C_p(X, [0,1])$ satisfying $\mathrm{DULC}(A, X)$ is relatively compact.

Now we can follow Casazza-Iovino and get their undefinability results by just finding familiar (at least to topologists) classes of spaces which are weakly Grothendieck and satisfy the double ultralimit condition.

Continuous logics whose *type spaces* satisfy these conditions will not be able to define Tsirelson's space. You can find the usual definition of type spaces in any introductory model theory text. It involves looking at maximal consistent sets of formulas. For continuous logic, there is an equivalent definition which we will use. But first, a technicality.

Pairs of Structures

A key technical concept introduced by Casazza and Iovino and used to talk about a pair of Banach space norms is the following:

Pairs of Structures

A key technical concept introduced by Casazza and Iovino and used to talk about a pair of Banach space norms is the following:

Definition

Let L be a language.

(a) We say that $L' \supseteq L$ is a *language for pairs of structures from* L, if L' includes two disjoint copies of L and there is a map $\mathrm{Str}(L) \times \mathrm{Str}(L) \to \mathrm{Str}(L')$ which assigns to every pair of L-structures $\mathfrak{M}, \mathfrak{N}$ an L'-structure $\langle \mathfrak{M}, \mathfrak{N} \rangle$.

Pairs of Structures

Definition

Let L be a language.

(a) We say that $L' \supseteq L$ is a *language for pairs of structures from* L, if L' includes two disjoint copies of L and there is a map $\mathrm{Str}(L) \times \mathrm{Str}(L) \to \mathrm{Str}(L')$ which assigns to every pair of L-structures $\mathfrak{M}, \mathfrak{N}$ an L'-structure $\langle \mathfrak{M}, \mathfrak{N} \rangle$.

(b) Let L' be a language for pairs of structures from L, and X, Y function symbols from L. We say that a formula $\varphi(X, Y)$ is a *formula for pairs of structures from* L, if

$$(\mathfrak{M}, \mathfrak{N}) \mapsto \varphi(X^{\mathfrak{M}}, Y^{\mathfrak{N}}) = \mathrm{Val}(\varphi(X, Y), \langle \mathfrak{M}, \mathfrak{N} \rangle)$$

is separately continuous on $\mathrm{Str}(L) \times \mathrm{Str}(L)$. For simplicity, we write $\varphi(\mathfrak{M}, \mathfrak{N})$ instead of $\mathrm{Val}(\varphi(X, Y), \langle \mathfrak{M}, \mathfrak{N} \rangle)$.

Types

Definition

Let L be a language, φ an L-formula for pairs of structures, and $\mathfrak{M} \in \mathrm{Str}(L)$. The *left φ-type of $\mathfrak{M}$* is the function $\mathrm{ltp}_{\varphi,\mathfrak{M}} : \mathrm{Str}(L) \to [0,1]$ given by $\mathrm{ltp}_{\varphi,\mathfrak{M}}(\mathfrak{N}) = \varphi(\mathfrak{M},\mathfrak{N})$.

Types

Definition

Let L be a language, φ an L-formula for pairs of structures, and $\mathfrak{M} \in \mathrm{Str}(L)$. The *left φ-type of $\mathfrak{M}$* is the function $\mathrm{ltp}_{\varphi,\mathfrak{M}} : \mathrm{Str}(L) \to [0,1]$ given by $\mathrm{ltp}_{\varphi,\mathfrak{M}}(\mathfrak{N}) = \varphi(\mathfrak{M},\mathfrak{N})$.

The *space of left φ-types*, denoted S^l_φ is the closure of $\{\mathrm{ltp}_{\varphi,\mathfrak{M}} : \mathfrak{M} \in \mathrm{Str}(L)\}$ in $C_p(\mathrm{Str}(L),[0,1])$. For a subset $C \subseteq \mathrm{Str}(L)$, we denote by $S^l_\varphi(C)$ the closure of the restricted functions $\{\mathrm{ltp}_{\varphi,\mathfrak{M}}|_C : \mathfrak{M} \in C\}$ in $C_p(C,[0,1])$ and it is called the *space of left φ-types over C*. $C_p(X,[0,1])$ is a closed subspace of $C_p(X)$ so has all the relevant properties that $C_p(X)$ has.

Types

Definition

Let L be a language, φ an L-formula for pairs of structures, and $\mathfrak{M} \in \mathrm{Str}(L)$. The *left φ-type of $\mathfrak{M}$* is the function $\mathrm{ltp}_{\varphi,\mathfrak{M}} : \mathrm{Str}(L) \to [0,1]$ given by $\mathrm{ltp}_{\varphi,\mathfrak{M}}(\mathfrak{N}) = \varphi(\mathfrak{M}, \mathfrak{N})$.

The *space of left φ-types*, denoted S^l_φ is the closure of $\{\mathrm{ltp}_{\varphi,\mathfrak{M}} : \mathfrak{M} \in \mathrm{Str}(L)\}$ in $C_p(\mathrm{Str}(L), [0,1])$. For a subset $C \subseteq \mathrm{Str}(L)$, we denote by $S^l_\varphi(C)$ the closure of the restricted functions $\{\mathrm{ltp}_{\varphi,\mathfrak{M}}\big|_C : \mathfrak{M} \in C\}$ in $C_p(C, [0,1])$ and it is called the *space of left φ-types over C*. $C_p(X, [0,1])$ is a closed subspace of $C_p(X)$ so has all the relevant properties that $C_p(X)$ has.

This definition of (left) φ-types is equivalent to the usual definition for a compact logic but makes it easier to exploit the topological richness of spaces of continuous functions. For more on this, see the first draft of this paper: [HT] ArXiv:3401.10459. The point is that $(\mathrm{ltp}_{\varphi,\mathfrak{M}})^{-1}\{1\} = \{\mathfrak{N} : \varphi(\mathfrak{M}, \mathfrak{N})\}$. I am working on a new, expanded draft which should be ready next month.

Four wide classes of spaces that are weakly Grothendieck and satisfy the double ultralimit condition.

We have already mentioned countably compact spaces.

Four wide classes of spaces that are weakly Grothendieck and satisfy the double ultralimit condition.

We have already mentioned countably compact spaces.

Definition

A space is *separable* if it has a countable dense set.

Four wide classes of spaces that are weakly Grothendieck and satisfy the double ultralimit condition.

Definition

The *tightness* $t(X)$ *of* X is the smallest infinite cardinal such that for every $A \subseteq X$ and $x \in \overline{A}$, there exists a $B \subseteq A$ such that $|B| \leq t(X)$ and $x \in \overline{B}$. When $t(X) = \aleph_0$, we say that X is *countably tight*.

Four wide classes of spaces that are weakly Grothendieck and satisfy the double ultralimit condition.

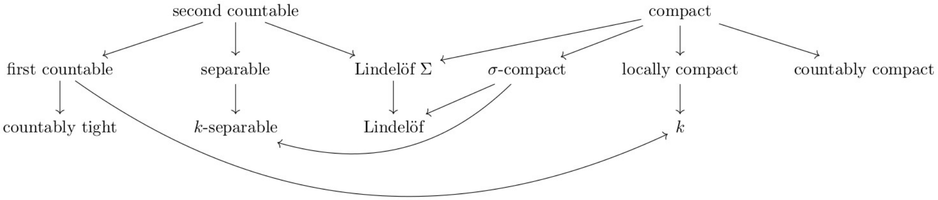
Definition

The *tightness* $t(X)$ *of* X is the smallest infinite cardinal such that for every $A \subseteq X$ and $x \in \overline{A}$, there exists a $B \subseteq A$ such that $|B| \leq t(X)$ and $x \in \overline{B}$. When $t(X) = \aleph_0$, we say that X is *countably tight*.

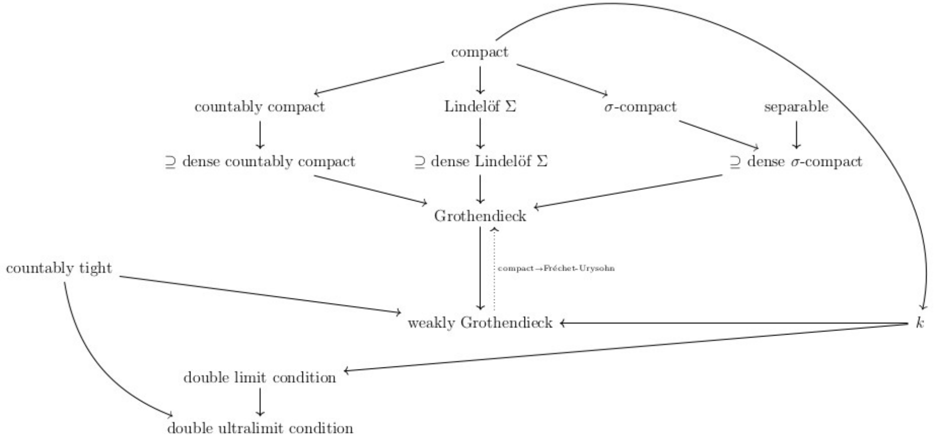
Definition

A topological space X is called a *k-space* when each subset $Y \subseteq X$ is closed if and only if its intersection with every compact subspace of X is closed.

Topology I


second countable
compact
first countable
separable
Lindelöf Σ
σ-compact
locally compact
countably compact
countably tight
k-separable
Lindelöf
k

Topology II


compact
countably compact
Lindelöf Σ
σ-compact
separable
⊇ dense countably compact
⊇ dense Lindelöf Σ
⊇ dense σ-compact
Grothendieck
compact→Fréchet-Urysohn
countably tight
weakly Grothendieck
k
double limit condition
double ultralimit condition

The proofs that various spaces are weakly Grothendieck are all due to Arhangel'skiĭ and appear in his book on C_p-theory, *Topological Function Spaces*, and his paper on Grothendieck spaces, *On a theorem of Grothendieck*.

The proof that the double limit condition implies the double ultralimit condition is in our paper.

The proofs that various topological properties imply the double (ultra) limit condition are in our paper and are not difficult but depend heavily on a 1987 paper of H. König and N. Kuhn: *Angelic spaces and the double limit relation*. Their paper is also not difficult.

Duality Theorems

Without worrying about pairs of structures, there are four topological spaces of interest in continuous logic: $\mathrm{Str}(L)$, $C_p(\mathrm{Str}(L), [0,1])$, $C_p(C_p(\mathrm{Str}(L), [0,1]))$, and the space of types, which is a closed subspace of $C_p(\mathrm{Str}(L), [0,1])$. With pairs of structures, one needs to replace $\mathrm{Str}(L)$ by $\mathrm{Str}(L) \times \mathrm{Str}(L)$.

Duality Theorems

Without worrying about pairs of structures, there are four topological spaces of interest in continuous logic: $\mathrm{Str}(L)$, $C_p(\mathrm{Str}(L), [0,1])$, $C_p(C_p(\mathrm{Str}(L), [0,1]))$, and the space of types, which is a closed subspace of $C_p(\mathrm{Str}(L), [0,1])$. With pairs of structures, one needs to replace $\mathrm{Str}(L)$ by $\mathrm{Str}(L) \times \mathrm{Str}(L)$.

For compact logics, i.e. logics satisfying the Compactness Theorem, the structure and type spaces are compact. Almost all of the topological properties we deal with are closed-hereditary, which simplifies matters. $C_p(X, [0,1])$ is a closed subspace of $C_p(X)$.

Duality Theorems

Without worrying about pairs of structures, there are four topological spaces of interest in continuous logic: $\mathrm{Str}(L)$, $C_p(\mathrm{Str}(L), [0,1])$, $C_p(C_p(\mathrm{Str}(L), [0,1]))$, and the space of types, which is a closed subspace of $C_p(\mathrm{Str}(L), [0,1])$. With pairs of structures, one needs to replace $\mathrm{Str}(L)$ by $\mathrm{Str}(L) \times \mathrm{Str}(L)$.

For compact logics, i.e. logics satisfying the Compactness Theorem, the structure and type spaces are compact. Almost all of the topological properties we deal with are closed-hereditary, which simplifies matters. $C_p(X, [0,1])$ is a closed subspace of $C_p(X)$. A large part of C_p-theory is concerned with proving duality theorems, i.e. proving that X has property P if and only if $C_p(X)$ has property Q. For example:

Theorem (Arhangel'skiĭ-Pytkeev)

All finite powers of X are Lindelöf if and only if $C_p(X)$ is countably tight.

If a continuous logic's type spaces are weakly Grothendieck and satisfy the double ultralimit condition, the [CI] proof works to show Tsirelson's space is not explicitly definable. We have many topological properties that imply this conjunction, but it is not necessarily easy to see if a particular logic's type spaces satisfy such properties. We know compact logics have compact type spaces, but it is not clear whether countably compact logics have countably compact type spaces.

If a continuous logic's type spaces are weakly Grothendieck and satisfy the double ultralimit condition, the [CI] proof works to show Tsirelson's space is not explicitly definable. We have many topological properties that imply this conjunction, but it is not necessarily easy to see if a particular logic's type spaces satisfy such properties. We know compact logics have compact type spaces, but it is not clear whether countably compact logics have countably compact type spaces. However,

L countably compact $\implies \mathrm{Str}(L)$ countably compact $\implies$
$C_p(\mathrm{Str}(L), [0,1])$ is weakly Grothendieck and satisfies the double limit condition.

But $C_p(\mathrm{Str}(L), [0,1])$ is L's type space!

Continuous $L_{\omega_1,\omega}$

This language is obtained by extending $L_{\omega_1,\omega}$ (which allows countable conjunctions and disjunctions) in the same way one extends first order logic to continuous logic.

Continuous $L_{\omega_1,\omega}$

This language is obtained by extending $L_{\omega_1,\omega}$ (which allows countable conjunctions and disjunctions) in the same way one extends first order logic to continuous logic.

If something is definable in (continuous) $L_{\omega_1,\omega}$, it's actually definable in a countable fragment of that language. Morley proved that type spaces of countable fragments of $L_{\omega_1,\omega}$ are separable complete metric spaces and the same proof works for continuous $L_{\omega_1,\omega}$.

Continuous $L_{\omega_1,\omega}$

This language is obtained by extending $L_{\omega_1,\omega}$ (which allows countable conjunctions and disjunctions) in the same way one extends first order logic to continuous logic.

If something is definable in (continuous) $L_{\omega_1,\omega}$, it's actually definable in a countable fragment of that language. Morley proved that type spaces of countable fragments of $L_{\omega_1,\omega}$ are separable complete metric spaces and the same proof works for continuous $L_{\omega_1,\omega}$.

These type spaces are then weakly Grothendieck and satisfy the double ultralimit condition because they are separable. Alternatively, because they are first countable.

Continuous $L_{\omega_1,\omega}$

This language is obtained by extending $L_{\omega_1,\omega}$ (which allows countable conjunctions and disjunctions) in the same way one extends first order logic to continuous logic.

If something is definable in (continuous) $L_{\omega_1,\omega}$, it's actually definable in a countable fragment of that language. Morley proved that type spaces of countable fragments of $L_{\omega_1,\omega}$ are separable complete metric spaces and the same proof works for continuous $L_{\omega_1,\omega}$.

These type spaces are then weakly Grothendieck and satisfy the double ultralimit condition because they are separable. Alternatively, because they are first countable.

So Tsirelson's space cannot be defined in continuous $L_{\omega_1,\omega}$, which is very far from being compact.

New Results (not in arXiv version)

Definition

A logic is *countably compact* if countable sets of formulas are satisfiable whenever all their finite subsets are satisfiable. A logic is *Lindelöf* if sets of formulas are satisfiable whenever all their countable subsets are satisfiable.

New Results (not in arXiv version)

Definition

A logic is *countably compact* if countable sets of formulas are satisfiable whenever all their finite subsets are satisfiable. A logic is *Lindelöf* if sets of formulas are satisfiable whenever all their countable subsets are satisfiable.

Clearly a logic is compact if and only if it is countably compact and Lindelöf.

New Results (not in arXiv version)

Definition

A logic is *countably compact* if countable sets of formulas are satisfiable whenever all their finite subsets are satisfiable. A logic is *Lindelöf* if sets of formulas are satisfiable whenever all their countable subsets are satisfiable.

Clearly a logic is compact if and only if it is countably compact and Lindelöf.

Proposition (Folklore)

A logic is compact (resp. countably compact, resp. Lindelöf) if and only if its structure space is compact (resp. countably compact, resp. Lindelöf).

We already knew that the undefinability results Casazza and Iovino proved for compact logics actually also hold for countably compact logics. Our new result: they hold for Lindelöf logics as well!

We already knew that the undefinability results Casazza and Iovino proved for compact logics actually also hold for countably compact logics. Our new result: they hold for Lindelöf logics as well!

Sketch of proof. It suffices to show that closed subspaces of $C_p(\mathrm{Str}(L))$ are countably tight. Countable tightness is closed-hereditary so it suffices to show $C_p(\mathrm{Str}(L))$ is countably tight if L is Lindelöf. Recall:

Lemma (Arhangel'skiĭ-Pytkeev)

$C_p(X)$ is countably tight if and only if X^n is Lindelöf, for all $n < \omega$.

Lemma (Arhangel'skiĭ-Pytkeev)

$C_p(X)$ is countably tight if and only if X^n is Lindelöf, for all $n < \omega$.

Thus we need only prove that:

Lemma

$\mathrm{Str}(L)$ is Lindelöf if and only if $(\mathrm{Str}(L))^n$ is Lindelöf for all $n < \omega$.

This proof is due to Chris Eagle.

Lemma (Arhangel'skiĭ-Pytkeev)

$C_p(X)$ is countably tight if and only if X^n is Lindelöf, for all $n < \omega$.

Thus we need only prove that:

Lemma

$\mathrm{Str}(L)$ is Lindelöf if and only if $(\mathrm{Str}(L))^n$ is Lindelöf for all $n < \omega$.

This proof is due to Chris Eagle.

Theorem

If $\mathrm{Str}(L)$ is Lindelöf, then Tsirelson space is not definable in L.

Theorem

If $\mathrm{Str}(L)$ *is Lindelöf, then Tsirelson space is not definable in* L*.*

Proof Sketch of Lemma.

Add new predicate to L and axioms saying it divides the universe into 2 disjoint pieces. The set of elementary equivalence classes of L-structures satisfying some axioms is a closed subspace of the set of elementary equivalence classes of L-structures.

□

Theorem

If $\mathrm{Str}(L)$ *is Lindelöf, then Tsirelson space is not definable in* L*.*

Proof Sketch of Lemma.

Add new predicate to L and axioms saying it divides the universe into 2 disjoint pieces. The set of elementary equivalence classes of L-structures satisfying some axioms is a closed subspace of the set of elementary equivalence classes of L-structures.

Aside from avoiding proper classes and making the structure space Hausdorff, another advantage of dealing with equivalence classes of structures rather than with structures is that, given 2 classes of structures $\widetilde{\mathfrak{M}}$ and $\widetilde{\mathfrak{N}}$, we can without loss of generality assume M and N are disjoint by simply choosing disjoint representatives M' and N'.

□

Theorem

If $\mathrm{Str}(L)$ *is Lindelöf, then Tsirelson space is not definable in* L*.*

Proof Sketch of Lemma.

Aside from avoiding proper classes and making the structure space Hausdorff, another advantage of dealing with equivalence classes of structures rather than with structures is that, given 2 classes of structures $\widetilde{\mathfrak{M}}$ and $\widetilde{\mathfrak{N}}$, we can without loss of generality assume M and N are disjoint by simply choosing disjoint representatives M' and N'.

Thus we can get a bijection between $\mathrm{Str}(L) \times \mathrm{Str}(L)$ and the topological sum of 2 copies of $\mathrm{Str}(L)$. If $\mathrm{Str}(L)$ is Lindelöf, so is that topological sum. The bijection actually induces a homeomorphism, whence we get that $\mathrm{Str}(L) \times \mathrm{Str}(L)$ is Lindelöf. Of course in general, the square of a Lindelöf space is not Lindelöf. This proof depends crucially on being able to write down axioms that say the universe is divided into two disjoint closed pieces.

□

Theorem

If $\mathrm{Str}(L)$ *is Lindelöf, then Tsirelson space is not definable in* L.

Proof Sketch of Lemma.

Thus we can get a bijection between $\mathrm{Str}(L) \times \mathrm{Str}(L)$ and the topological sum of 2 copies of $\mathrm{Str}(L)$. If $\mathrm{Str}(L)$ is Lindelöf, so is that topological sum. The bijection actually induces a homeomorphism, whence we get that $\mathrm{Str}(L) \times \mathrm{Str}(L)$ is Lindelöf. Of course in general, the square of a Lindelöf space is not Lindelöf. This proof depends crucially on being able to write down axioms that say the universe is divided into two disjoint closed pieces.

It easily follows that $(\mathrm{Str}(L))^n$ is Lindelöf for all $n < \omega$, but then $C_p(\mathrm{Str}(L))$ is countably tight by Arhangel'skiĭ-Pytkeev. Countable tightness is closed-hereditary, so $C_p(\mathrm{Str}(L), [0,1])$ is countably tight. Then the space of left φ-types is countably tight and hence is weakly Grothendieck and satisfies the double ultralimit condition, so we can proceed as usual. □

Conclusion

[CI] not only prove that Tsirelson's space is not explicitly definable in compact continuous logics, but also prove that for such logics, certain nicely defined Banach spaces do have copies of c_0 or some ℓ^p. This latter proof essentially uses the Stone-Weierstrass Theorem, which does not extend nicely to non-compact spaces, but we were able to find a different proof which did generalize.

Conclusion

[CI] not only prove that Tsirelson's space is not explicitly definable in compact continuous logics, but also prove that for such logics, certain nicely defined Banach spaces do have copies of c_0 or some ℓ^p. This latter proof essentially uses the Stone-Weierstrass Theorem, which does not extend nicely to non-compact spaces, but we were able to find a different proof which did generalize.

Just as with the undefinability result, we replaced the compactness assumption by the assumption of weakly Grothendieck plus the double ultralimit condition. However, I emphasize that the analysis part of our proofs is identical to that in [CI].

Conclusion

[CI] not only prove that Tsirelson's space is not explicitly definable in compact continuous logics, but also prove that for such logics, certain nicely defined Banach spaces do have copies of c_0 or some ℓ^p. This latter proof essentially uses the Stone-Weierstrass Theorem, which does not extend nicely to non-compact spaces, but we were able to find a different proof which did generalize.

Just as with the undefinability result, we replaced the compactness assumption by the assumption of weakly Grothendieck plus the double ultralimit condition. However, I emphasize that the analysis part of our proofs is identical to that in [CI].

Future work: Exchanging double limits is a popular pastime among analysts; can our methods be applied in other areas of analysis?

Appendix: statements of definability and undefinability theorems

Definition

c_{00} is the space of sequences of real numbers that are eventually zero.

We introduce the continuous logic formula for pairs of structures used in [CI]: for norms $\|\|_1$ and $\|\|_2$, define

$$D(\|\|_1, \|\|_2) = \sup \left\{ \frac{\|x\|_1}{\|x\|_2} : \|x\|_{\ell^1} = 1 \right\}.$$

Then define

$$\varphi(\|\|_1, \|\|_2) = 1 - \frac{\log D(\|\|_1, \|\|_2)}{1 + \log D(\|\|_1, \|\|_2)}. \qquad (\star)$$

Definition

Suppose L is a language for pairs of structures and φ is a formula for pairs of structures. Let $C \subseteq \mathrm{Str}(L)$. A function $\tau : S^r_\varphi(C) \to [0,1]$ is a *global left φ-type over C* if there is a κ-sequence $\{\mathfrak{M}_\alpha\}_{\alpha<\kappa} \subseteq C$, and an ultrafilter $\mathcal{U}$ on κ, such that for every type $t \in S^r_\varphi(C)$, say $t = \lim_{\beta\to\mathcal{V}} \mathrm{rtp}_{\varphi,\mathfrak{N}_\beta}$, we have

$$\tau(t) = \lim_{\alpha\to\mathcal{U}} \lim_{\beta\to\mathcal{V}} \varphi(\mathfrak{M}_\alpha, \mathfrak{N}_\beta).$$

We say that τ is *explicitly definable* if it is continuous. If a left φ-type is given by $p = \lim_{\alpha\to\mathcal{U}} \mathrm{ltp}_{\varphi,\mathfrak{M}_\alpha}$, we say that p is *explicitly definable* if the respective τ is continuous.

Theorem

Let L be a language for pairs of structures, and φ a formula for pairs of structures. Suppose $C \subseteq \mathrm{Str}(L)$ is such that $S^l_\varphi(C)$ is weakly Grothendieck and satisfies the double ultralimit condition. Then the following are equivalent:

(i) Whenever a left type over C is given by $t = \lim_{i \to \mathcal{U}} \mathrm{ltp}_{\varphi, \mathfrak{M}_i}$, and $\{\mathfrak{N}_j\}_{j<\omega} \subseteq C$ is a sequence in C, and $\mathcal{V}$ is an ultrafilter on ω, then

$$\lim_{i \to \mathcal{U}} \lim_{j \to \mathcal{V}} \varphi(\mathfrak{M}_i, \mathfrak{N}_j) = \lim_{j \to \mathcal{V}} \lim_{i \to \mathcal{U}} \varphi(\mathfrak{M}_i, \mathfrak{N}_j).$$

(ii) If τ is a global left φ-type over C, then τ is continuous.

Definition

A structure $(c_{00}, \|\|_{\ell^1}, \|\|, e_0, e_1, \ldots)$ where $\|\|$ is an arbitrary norm and $\{e_n\}_{n<\omega}$ is the standard vector basis of c_{00} is called a *structure based on* c_{00}.

Definition

Let $\mathcal{C}$ be a family of structures which are normed spaces based on c_{00}, φ a formula for a pair of structures, and $\|\|_*$ a norm on c_{00}.

(a) If $\{\|\|_i : i < \omega\}$ is a family of norms on c_{00} we say that $\{\mathrm{ltp}_{\varphi,\|\|_i} : i < \omega\}$ *determines* $\|\|_*$ *uniquely* if, for every ultrafilter $\mathcal{U}$ on ω, the type $t = \lim_{i\to\mathcal{U}} \mathrm{ltp}_{\varphi,\|\|_i}$ is realized, and $\|\|_*$ is its unique realization.

(b) We say that $\|\|_*$ is *uniquely determined by its* φ*-type over* $\mathcal{C}$ if there is a family of norms $\{\|\|_i : i < \omega\}$ on c_{00} in $\mathcal{C}$ such that $\{\mathrm{ltp}_{\varphi,\|\|_i} : i < \omega\}$ determines $\|\|_*$ uniquely.

Proposition

Let L be a language for pairs of structures, $\mathcal{C}$ a class of structures $(c_{00}, \|\|_{\ell^1}, \|\|)$ such that the norm completion of $(c_{00}, \|\|)$ is a Banach space including ℓ^p or c_0, and let $\varphi(X, Y)$ be the formula defined by $(\star)$ above. Suppose $\{(c_{00}, \|\|_{\ell^1}, \|\|_i) : i < \omega\}$ is a family of structures in $\mathcal{C}$ such that

$$\|\|_1 \leq \|\|_2 \leq \cdots \leq \|\|_n \leq \cdots$$

and the φ-type $t = \lim_{i \to \mathcal{U}} \mathrm{ltp}_{\varphi, \|\|_i}$ is realized by $(c_{00}, \|\|_{\ell^1}, \|\|_*)$ in $\mathrm{Str}(L)$, then $\{\mathrm{ltp}_{\varphi, \|\|_i} : i < \omega\}$ uniquely determines $\|\|_*$ over $\mathcal{C}$. In particular, the Tsirelson norm is uniquely determined by its φ-type over $\mathcal{C}$.

We then prove:

Theorem

Let L be a language for pairs of structures, and $\mathcal{C}$ a subclass of the class of structures $(c_{00}, \|\|_{\ell^1}, \|\|)$ such that the norm completion of $(c_{00}, \|\|)$ is a Banach space including some ℓ^p or c_0, and including the spaces used in the construction of the Tsirelson space. Let $\|\|_T$ be the Tsirelson norm. Let φ be the formula as in $(\star)$ above. If the space $S^l_\varphi(\mathcal{C})$ of left φ-types over $\mathcal{C}$ is weakly Grothendieck and satisfies the double ultralimit condition, then $\|\|_T$ is uniquely determined by its left φ-type over $\mathcal{C}$ and that left φ-type is not explicitly definable over $\mathcal{C}$.

Theorem

Let L be a language for pairs of structures, φ the formula defined by $(\star)$ above, and C a subclass of the class of structures based on c_{00} such that every closed subspace of a space in C includes a copy of c_0 or ℓ^p. Assume that the space of left φ-types over C is weakly Grothendieck and satisfies the double ultralimit condition. If the left φ-type of an $\mathfrak{M} \in \mathrm{Str}(L)$ is explicitly definable from C, then $\mathfrak{M}$ includes a copy of c_0 or some ℓ^p.

Appendix II Countable Tightness and the Double Ultralimit Condition

Lemma

If X is weakly Grothendieck and satisfies the double ultralimit condition, then a subset $A \subseteq C_p(X, [0,1])$ satisfying $\mathrm{DULC}(A, X)$ is relatively compact.

For countably tight spaces, the "if" can be strengthened to "if and only if":

Theorem

Let X be countably tight. A subset A of $C_p(X, [0,1])$ is relatively compact in $C_p(X, [0,1])$ if and only if $\mathrm{DULC}(A, X)$.

Let $\overline{A}$ denote the closure of A in $[0,1]^X$.

Proof of backward direction.

Suppose $\overline{A} \cap C_p(X)$ is not compact in $C_p(X)$. Then $\overline{A} \cap C_p(X)$ is closed but it is not countably compact in $C_p(X)$, since X is weakly Grothendieck. Let $\{f_n\}_{n<\omega}$ be a subset of A with closure disjoint from $\overline{A} \cap C_p(X)$. Since $\overline{A}$ is a compact subset of $[0,1]^X$, each ultralimit of the sequence $\{f_n\}_{n<\omega}$ exists, and is discontinuous. Take a non-principal ultrafilter $\mathcal{U}$ over ω and let $\lim_{n\to\mathcal{U}} f_n = g$, where g is discontinuous by assumption. Then there are $\varepsilon > 0$ and $y \in X$ such that $y \in \overline{Y}$, where

$$Y = X \setminus g^{-1}\left(g(y) - \varepsilon, g(y) + \varepsilon\right).$$

Proof continued.

Since $t(X) = \aleph_0$, there is some $Z \subseteq Y$ with $|Z| = \aleph_0$ and $y \in \overline{Z}$. Suppose $Z = \{x_m\}_{m<\omega}$ and for each open neighbourhood U of y, let $M_U = \{m < \omega : x_m \in U\}$. Clearly, the family of all M_U is centred (i.e. all finite subfamilies have non-empty intersections) and so it can be extended to an ultrafilter $\mathcal{V}$ on ω so that

$$\lim_{n \to \mathcal{U}} \lim_{m \to \mathcal{V}} f_n(x_m) = g(y)$$

since each f_n is continuous. On the other hand,

$$\lim_{m \to \mathcal{V}} \lim_{n \to \mathcal{U}} f_n(x_m) = \lim_{m \to \mathcal{V}} g(x_m)$$

exists by compactness of $[0, 1]$. However, by the choice of each x_m, we have $|g(y) - g(x_m)| > \varepsilon$ and so the ultralimits exist but are different, a contradiction. □

Proof of forward direction.

For the forward direction, we will apply the lemma above after verifying its hypotheses. Since X is countably tight, we know X is weakly Grothendieck. Suppose $\{x_m\}_{m<\omega}$ is a sequence in X, $\overline{A} \cap C_p(X)$ is compact, and $\lim_{m\to\mathcal{V}} x_m = y$. Then for any sequence $\{f_n\}_{n<\omega} \subseteq A$ and ultrafilter $\mathcal{U}$ on ω, there is a continuous function $g = \lim_{n\to\mathcal{U}} f_n$. Thus,

$$\lim_{n\to\mathcal{U}} \lim_{m\to\mathcal{V}} f_n(x_m) = \lim_{n\to\mathcal{U}} f_n(y) = g(y),$$

and

$$\lim_{m\to\mathcal{V}} \lim_{n\to\mathcal{U}} f_n(x_m) = \lim_{m\to\mathcal{V}} g(x_m) = g(y). \qquad \square$$

Note we did not need countable tightness for the forward direction.