MATH 6670, SPRING 2025: WEEK 1 QUESTIONS

Problem 1: Suppose that I and J are monomial ideals in $R = \mathbb{K}[x_1, \ldots, x_n]$, each given as a finite number of monomial generators.

(a) Show that $x^\alpha \in I$ if and only if x^α is divisible by some monomial generator of I.
(b) Show how to compute $I \cap J$.
(c) Show how to compute $I : x^\alpha$.
(d) Compute the radical of I (the set of all $f \in R$ such that $f^N \in I$, for some N).
(e) When is I a prime ideal?
(f) When is I a primary ideal?
(g) Compute by hand the Hilbert series of the (quotient by the) monomial ideal

$$I = \langle a^2, ab, b^3, b^2c \rangle.$$

Problem 2: Let G be a finite simple graph (with vertices $0, \ldots, n-1$), with no loops, and at most one edge between vertices). A k-coloring of G consists of k colors, and for each vertex a choice of color, such that for each edge of G, the two vertices have different colors. For this problem, let's restrict to $k = 3$, although the same idea works more generally. The goal is to find an ideal in a ring $k[x_0, \ldots, x_{n-1}]$ whose zeros describe the possible 3-colorings of G, and then to try this on some examples.

One way is to, over the (algebraic closure of the) rationals, let the three "colors" be the 3 roots of unity. Then if x_i is the color at vertex i, so is one of these three values, we know that $x_i^3 - 1 = 0$. Then x_i and x_j are different if $x_i^2 + x_i x_j + x_j^2 = 0$, and the same if $x_i - x_j = 0$. Choose a graph with, say, 7 or 8 vertices, and use this, with Gröbner bases, to find if there is a 3-coloring, and if so, can you tell how many there are?

Another way to approach the problem might be to choose the field to be $\mathbb{Z}/3\mathbb{Z}$, and the three colors to be the three values of the field.

Another interesting problem might be to consider the ideal generated by the $x_i^2 + x_i x_j + x_j^2$, for all the edges of the graph. What can we say about this ideal? We might need more technique to be able to say much, but it might be fun to look at too.

To get you started, here is some Macaulay2 code to perhaps try.

```
needsPackage "NautyGraphs"
Gs = generateGraphs(7, 10)
G = stringToGraph Gs_20
edgesG = (edges G)/toList//sort
R = QQ[x_0..x_6]
I1 = ideal for i from 0 to 6 list x_i^3-1
I2 = ideal for e in edgesG list (a := R_(e#0); b := R_(e#1); a^2 + a*b + b^2)
I = I1 + I2
decompose I
```

Problem 3: Consider a random ideal (over, say, the finite field with 101 elements) generated by 4 random quadric polynomials in 6 variables. Using Macaulay2, or another computer algebra system, find the Groebner basis of your ideal, both in lex order, and in the grevlex order.

And some code for the last problem:

```
setRandomSeed "42"
R = ZZ/101[a..f]
I = ideal random(R^1, R^{4:-2})
codim I
hilbertSeries I
reduceHilbert oo
tally degrees source groebnerBasis I -- one quintic
Rlex = ZZ/101[gens R, MonomialOrder => Lex]
Ilex = sub(I, Rlex)
groebnerBasis(Ilex, Strategy => "F4");
tally degrees source oo
Ilex = sub(I, Rlex)
groebnerBasis Ilex;
```