COCOAG, SPRING 2025: WEEK 2 QUESTIONS

During "project time", work on problems 2,3 first. We will discuss (some of) the rest in class, 27 Jan 2025

Problem 1: (From last time, in case you didn't do this, do it today!) Consider a random ideal (over, say, the finite field with 101 elements) generated by 4 random quadric polynomials in 6 variables. Using Macaulay2, or another computer algebra system, find the Groebner basis of your ideal, both in lex order, and in the grevlex order.

```
setRandomSeed "42"
R = ZZ/101[a..f]
I = ideal random(R^1, R^{4:-2})
codim I
hilbertSeries I
reduceHilbert oo
tally degrees source groebnerBasis I -- one quintic
Rlex = ZZ/101[gens R, MonomialOrder => Lex]
Ilex = sub(I, Rlex)
groebnerBasis(Ilex, Strategy => "F4");
tally degrees source oo
Ilex = sub(I, Rlex)
groebnerBasis Ilex;
```

Problem 2:

(a) Recall that the (nil)radical of an ideal $I \subseteq R$ is the ideal

$$\sqrt{I} = \langle f \in R \mid f^N \in I, \text{ for some } N \rangle.$$

If $R = k[x_1, \ldots, x_n]$, show how to use Groebner bases and elimination, to decide whether $f \in \sqrt{I}$. (Hint consider adding a new variable t and a polynomial $tf - 1$).

(b) Let $R = k[\{x_{ij} \mid 1 \le i, j \le 3\}]$ be a polynomial ring in the 9 variables x_{ij}. Let M be the 3×3 matrix whose entries are $\{x_{ij}\}$. Let I be the matrix generated by the entries of the matrix M^3. Let $f = \operatorname{tr}(M)$ be the trace of M. Use your methods in Macaulay2 to show that $f \in \sqrt{I}$. What is the minimum power N such that $f^N \in I$?

Problem 3:

(a) Compute (using Groebner bases and elimination orders) the minimal polynomial over $\mathbb{Q}$ of $\sqrt{2}+\sqrt{3}+\sqrt{5}$. Ie. show how to use Groebner bases to do it, and then find the minimal polynomial using your algorithm in Macaulay2 (or another system), using the `groebnerBasis` command (with a ring having a suitable elimination order).

(b) Show that $\mathbb{Q}(\alpha_1, \alpha_2) = \mathbb{Q}(\beta)$, where $\alpha_1^3 - \alpha_1 - 1 = 0$, $\alpha_2^2 = 5$, and $\beta = \frac{\alpha_1\alpha_2+1}{\alpha_1+\alpha_2}$.

Problem 4: For each of the following notions, show how to use Groebner bases and elimination orders to compute each one, and try your method using Macaulay2 on an example. Assume that $R = k[x_1, \ldots, x_n]$, and I, J are ideals, and any polynomials shown are in R (or possibly R/I in (c)).

(a) $I \cap J$.

(b) The kernel of the ring map $\phi : k[y_1, \ldots, y_r] \to R$, given by $y_i \mapsto f_i \in R$.

(c) The kernel of the ring map $\phi : k[y_1, \ldots, y_r] \to R$, given by $y_i \mapsto f_i \in R/\mathrm{I}$.
(d) $I : f$.
(e) $I : f^\infty$.

Problem 5: Let R be a (commutative) ring and let $I \subseteq R$ be an ideal. The **Rees algebra** of I is the graded R-algebra

$$Rees_R(I) := R \oplus I \oplus I^2 \oplus \cdots .$$

(This is also sometimes called the **blowup algebra** Note that $Rees_R(I) = R[It] \subseteq R[t]$, for a new variable t).

(a) For $R = k[x_1, \ldots, x_n]$, and $I = \langle f_1, \ldots, f_r \rangle$, find an algorithm to compute an ideal L such that the induced map $R[y_1, \ldots, y_r]/L \to Rees_I(R)$ is an isomorphism (of graded R-algebras).

(b) For $R = k[x_1, \ldots, x_5]$, let $I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1 \rangle$ be the edge ideal corresponding to a 5-cycle. Compute $Rees_R(I)$.

We will come back to Rees algebras quite a bit in this course!

Problem 6: (Open-ended) Consider $R = k[x_1, \ldots, x_n]$. Investigate, using Macaulay2, ideals of the form

$$\langle x_1^d, x_2^d, \ldots, x_n^d \rangle : f.$$

First try $n = 1$ and $n = 2$, and then experiment with $n = 3$, for various d and homogeneous f. What happens for $n = 3$ turned into one of David Eisenbud's early cool theorems with David Buchsbaum. What can you discover or conjecture? Consider the generators, and the free resolution, perhaps other commutative algebra concepts too.

Problem 7: Let E be the exterior algebra over the field k generated by $e_1, \ldots, e_n$. This is the (associative) k-algebra satisfying: $e_i^2 = 0$, and $e_je_i = -e_ie_j$, for all $1 \leq i, j \leq n$. If $char(k) \neq 2$, then the first set of equations follows from the second. We consider ideals $I \subset E$. For example $I = \langle e_1e_2, e_2e_3e_4 - e_2e_4 \rangle$.

(a) Generalize the notion of monomial order, lead monomial, Gröbner basis to ideals in E.

(b) State a theorem that generalizes Buchbergers theorem. What is the notion of spolynomial needed?

(c) Show that E is a local ring.

(d) (Open-ended) This implies that every ideal (not just homogeneous ideals) has a minimal set of generators. An algorithm to do this is not available in Macaulay2 (yet!). Find such an algorithm! (One also wants this routine for modules over an exterior algebra).

Problem 8: We define a **weight function** w for $R = k[x_1, \ldots, x_n]$ to be a (row) vector $w \in \mathbb{R}^n$. If the entries are all integers, we say that w is an integral weight vector. The weight vector w defines a partial ordering on the monomials of R: $x^\alpha >_w x^\beta$ if and only if $w \cdot \alpha > w \cdot \beta$. In this case the initial "term" $LT_w(f)$ is the sum of all terms with maximal weight.

(a) Fix another "tie breaker" global monomial order, $>_0$. Show that the monomial order which is the lexicographic product of $>_w$ and $>_0$ is a global monomial order if and only if every entry of w is non-negative. (this lexicographic product: $x^\alpha > x^\beta$ if and only if $w \cdot \alpha > w \cdot \beta$, or $w \cdot \alpha = w \cdot \beta$, and $x^\alpha >_0 x^\beta$.

(b) For an ideal I, define $LT_w(I) :=< LT_w(f) \mid f \in I >$. Using Groebner bases, find an algorithm to compute a generating set for $LT_w(I)$.

(c) Do this on a specific example, for example: $w = (1, 0, 0, 0)$, and I is a random ideal in 4 variables defined by 3 quadrics.