COCOAG, SPRING 2025: WEEK 3 QUESTIONS

During "project time", work on problems 1,2 first. We will discuss (some of) the rest in class next week

For the problems here, try the 3 ideals:

(a) **The twisted cubic curve**

```
R = ZZ/101[a..d];
I = monomialCurveIdeal(R, {1,2,3})
I = ideal(c^2-b*d,b*c-a*d,b^2-a*c) -- same thing
```

(b) **The rational quartic curve**

```
R = ZZ/101[a..d];
I = monomialCurveIdeal(R, {1,3,4})
I = ideal(b*c-a*d,c^3-b*d^2,a*c^2-b^2*d,b^3-a^2*c)
```

(c) **A monomial ideal**

```
R = ZZ/101[a..d];
I = ideal(b^2, a*b, a*c*d, a^2*d)
```

Problem 1: (This one should be done by hand): Consider the ideal $I = \langle xy, xz \rangle \subset R = k[x, y, z]$.

(a) Find a (graded) free R-resolution of R/I.
(b) Find the modules $Ext^i(R/I, R)$.
(c) Use the free resolution to compute the Hilbert series, and Hilbert polynomial of R/I.
(d) Using this, what is the dimension of $V(I) \subset \mathbb{P}^3$, and what is its degree?

Problem 2: Answer these same questions for (some of) the 3 ideals mentioned above. You may use Macaulay2 if you wish. Also, compute the annihilators of the Ext modules.

Problem 3: Let $R = \mathbb{Z}/101[x, y, z]/(zy^2 - x(x - z)(x - 2z))$. Let $I = ideal(x, y)$. Find a presentation for $I^* := Hom_R(I, R)$. Also, find an ideal of R isomorphic to (a degree shift of) I^*. You may use Macaulay2, and the building blocks we did in class.

Problem 4: Let $M = \text{coker}(m)$ be a module, with presentation matrix m. There is a natural R-map $M \to M^{**}$. If R is a domain, an R-module M is called **reflexive** if this natural map $M \to M^{**}$ is an isomorphism. The **torsion submodule** of M is the kernel of this map.

(a) Theoretically, provide a definition for this natural map.
(b) Find an algorithm (using the building blocks we defined in class today) to find a presentation of M^{**}, as well as a matrix representing this natural map.
(c) Consider the Ext modules M for the three examples. For each, consider them as $R/ann(M)$-modules. Are these reflexive modules? What is their torsion submodule? (All as $R/ann(M)$ modules)

Problem 5: (Open-ended) Consider polynomial rings $R = k[x_1, \ldots, x_n]$ (for various n). Consider homogeneous ideals minimally generated by 4 quadratic polynomials. How many different Betti tables for such ideals can you find? (Even more open-ended) Do the same for ideals generated minimally by 5 quadratics.