COCOAG, SPRING 2025: WEEK 4 QUESTIONS

During "project time", work on problems 1,2 first. We will discuss (some of) the rest in class next week

Problem 1: Let $R = k[x, y, z, w]/(xy - zw)$ be the homogeneous coordinate ring of the quadric surface $Q \subset \mathbb{P}^3$. Let $J = \langle x, z \rangle$. Note that $\mathbb{V}(J) = L \subset Q$ is a line on the quadric Q.

(a) Find a presentation matrix of J.
(b) Compute a free resolution of J over R.
(c) Compute a presentation matrix for $J^* := Hom_R(J, R)$.
(d) Find an ideal $I \subset R$ such that I is isomorphic to J^* (up to a degree shift, if you are paying attention to gradings).
(e) Are I and J R-isomorphic?
(f) What is the geometry of $\mathbb{V}(I) = M \subset Q$? (i.e. what is this, how does it relate to L, Q?)

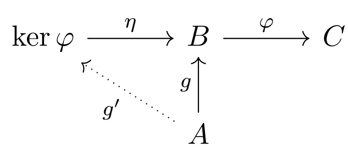
Problem 2: Given presentations for two (finitely generated) R-modules, M, N, consider $M \otimes_R N$.

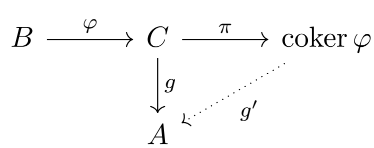
(a) If M and N are free modules, what is a basis of $M \otimes_R N$?
(b) If the presentation matrix of M is a 2×3 matrix, and the presentation matrix of N is a 3×2 matrix, write down the presentation matrix f $M \otimes_R N$ (this requires no computation of syzygies!)
(c) What is the presentation matrix for $M \otimes_R N$ in the general case?

Problem 3: Find algorithms using our building blocks to compute:

(a) The annihilator of the module M, $ann(M) := \{f \in R \mid fM = 0\}$¿
(b) Use this to help find the annihilators for the Ext modules for the 3 ideals from last time: $\langle c^2 - bd, bc - ad, b^2 - ac \rangle$, $\langle bc - ad, c^3 - bd^2, ac^2 - b^2d, b^3 - a^2c \rangle$, and/or $\langle b^2, ab, acd, a^2d \rangle$. (Check your work with the *ann* command in Macaulay2.

For the following problem:





Problem 4: Some universal properties/maps associated to a map of modules Let $\varphi\colon B \to C$ be an R-linear map of R-modules. In this problem, we write down some of the key functions (axioms) which show that the category of R-modules is an Abelian category. In this problem, you are asked to use our building block functions (syz, modulo, lift) to determine how to compute these. (These functions can be useful in practice as well).

(a) (**Kernel of a map**): The kernel of φ is a pair $(\ker\varphi, \eta)$, where $\ker\varphi$ is an R-module and $\eta\colon \ker\varphi \to B$ is an R-linear map, which satisfies the following universal property: Given $g\colon A \to B$ where $\varphi g = 0$, then there exists a (unique) map $g' : A \to \ker\varphi$ saisfying $g = \eta g'$ (see above). **Problem**: Write two functions. The first, called `kernelMap`, takes as input an R-linear map φ, and it returns the map η. The second, called `kernelLift` takes two maps g, and φ as above, and returns the R-linear map g'.

(b) (**Cokernel of a map**): The cokernel of φ is a pair $(\operatorname{coker}\varphi, \pi)$, where $\operatorname{coker}\varphi$ is an R-module and $\pi\colon C \to \operatorname{coker}\varphi$ is an R-linear map, which satisfies the following universal property: Given $g\colon C \to A$ where $g\varphi = 0$, then there exists a (unique) map $g' : \operatorname{coker}\varphi \to A$ saisfying $g = g'\pi$ (see above). **Problem**: Write two functions. The first, called `cokernelMap`, takes as input an R-linear map φ, and it returns the map π. The second, called `cokernelLift` takes two maps g, and φ as above, and returns the R-linear map g'. (**Image and coimage**) The **image** of φ is $\ker\operatorname{coker}\varphi$. The **coimage** of

φ is $\operatorname{coker}\ker\varphi$. Write a function `coimageToImage` which takes as input φ, and returns the natural map $\mu\colon \operatorname{coimage}(\varphi) \to \operatorname{image}(\varphi)$. Then show that this is an isomorphism of R-modules, define the function `imageToCoimage` which takes φ and returns the inverse of the isomorphism μ.

Problem 5: Let $M = \operatorname{coker}(m)$ be a module, with presentation matrix m. There is a natural R-map $M \to M^{**}$. If R is a domain, an R-module M is called **reflexive** if this natural map $M \to M^{**}$ is an isomorphism. The **torsion submodule** of M is the kernel of this map.

(a) Theoretically, provide a definition for this natural map.

(b) Find an algorithm (using the building blocks we defined in class, as well as the new functions created in the last problem) to find a presentation of M^{**}, as well as a matrix representing this natural map.

(c) Consider the Ext modules M for the three examples. For each, consider them as $R/ann(M)$-modules. Are these reflexive modules? What is their torsion submodule? (All as $R/ann(M)$ modules)

More on next page!

Problem 6: Depth and regular sequences Suppose that R is a Noetherian ring, $I \subseteq R$ is an ideal, and M is a finitely generated R-module such that $IM \neq M$. Let $f_1, \ldots, f_r \in I$. Recall that the ordered sequence $(f_1, f_2, \ldots, f_r)$ is called a *regular sequence* on M if $\langle f_1, \ldots, f_r \rangle \neq \langle 1 \rangle$, and for each $1 \leq i \leq r$, f_i is a non-zero divisor of $M/\langle f_1, \ldots, f_{i-1} \rangle M$. Recall also that **the depth of** M **with respect to an ideal** I denoted $depth_I(M)$, is the maximum length of a regular sequence on M contained in I.

We now restrict to the following case: let R be a local ring, or a positively graded ring $k[x_1, \ldots, x_n]$. (positively graded: everything has degree at least zero, and only the elements of k have degree 0). We let m denote the maximal ideal, or the ideal generated by all of the variables, in the graded case.

If R is positively graded or local, then permutations of a regular sequence are also a regular sequence, but that is not always true in more general situations. Some special cases:

(a) $depth_I(R)$ is called the grade of I. If R is a polynomial ring, a theorem in commutative algebra shows that this value is the codimension of I (the height of I).

(b) $depth M := depth_m(M)$ denotes the length of a maximal regular sequence inside the maximal ideal m.

For this problem:

(a) Verify the formula: $depth_I(M) = \min_i \{\mathrm{Ext}^i_R(R/I, M) \neq 0\}$ (This is true in the general setting above).

(b) (More open-ended) How do you find a maximal regular sequence? How can you compute $depth_I(M)$? Can you do better than computing these Ext's?