COCOAG, SPRING 2025: WEEK 5 QUESTIONS

During "project time", work on problems 1,2,3.4 first (or some subset of those!).

Problem 1: Work on finding different betti tables for 5 quadrics, in class, using either methods you thought of earlier, or that we did together for 4 quadrics.
A possibly useful function in Macaulay2 (here B should be a list of monomials):

```
-- B is a list of monomials in a polynomial ring
-- ngens: number of generators you want
-- maxnumterms: number of terms each polynomial should have (could have fewer).
randomSparse = (B, ngens, maxnumterms) -> (
  R := ring B#0;
  kk := coefficientRing R;
  L := for i from 1 to ngens list (
      sum for j from 1 to maxnumterms list (random kk) * B#(random (#B))
      );
  I := ideal L;
  if numcols mingens I != ngens then
      randomSparse(B, ngens, maxnumterms) -- just recursively look for one.
  else
      I
  )
```

Problem 2: Consider one of the following 3 varieties: (a) the cubic surface X in $\mathbb{P}^3$: $x^3 + y^3 + z^3 + w^3 = 0$, (b) the Fermat quartic surface $x^4 + y^4 + z^4 + w^4 = 0$, this is what is called a K3 surface, or (c) the variety $X \subset \mathbb{P}^4$ whose ideal is given by the 3 by 3 minors of a (fairly random) 4 *times*3 matrix of linear forms in 5 variables.

(a) Find the Hilbert polynomial, and the dimension and degree of X.
(b) Find the dimensions of the cohomology vector spaces of $\mathcal{O}_X$ using exact sequences and Serre's result on cohomology of $\mathcal{O}_{\mathbb{P}^n}(d)$.
(c) Use Macaulay2 to compute these cohomologies.
(d) Compute the Hilbert polynomial $p(z)$ of S/I, and the Euler characteristic $\chi(\mathcal{O}_X)$ of $\mathcal{O}_X$. Verify that the Euler characteristic is $p(0)$.

Problem 3: Consider the special case $\mathbb{P}^2$. Compute from the definition in class the cohomologies of $\mathcal{O}_{\mathbb{P}^2}(d)$, for all d, verifying Serre's theorem in this case. Hint: this complex with infinitely generated modules is (over the base field) the direct sum of complexes corresponding to each monomial $x_0^a x_1^b x_2^c$, for $(a, b, c) \in \mathbb{Z}^3$. Once one sees the proof in this special case, it is pretty easy to generalize to prove the entire theorem (and, in fact, even more, that we haven't stated yet!).

Problem 4: In this exercise we prove the important theorems of Serre ((b), (c)). After proving (a), use Serre's "local duality" theorem to prove (b), (c). Here, suppose that $\widetilde{M}$ is a coherent sheaf on $\mathbb{P}^n$ where M is a finitely generated graded S-module.

(a) Show that the k-dual of M is zero in all degrees $d \gg 0$.
(b) Show that $H^i(\mathcal{O}_X(d)) = 0$, for $i > 0$ and $d \gg 0$.
(c) Show that $\mathcal{O}_X(d)$ is generated by global sections for $d \gg 0$. This means that the natural map $M_d \to H^0(\mathcal{O}_X(d))$ is an isomorphism.

Problem 5: Consider the projective variety X which is given by the zeros of $x^3 + y^3 + z^3$, in $\mathbb{P}^2$ (this is an elliptic curve).

(a) Find the dimensions of the cohomology vector spaces of $\mathcal{O}_X$.
(b) Compute "by hand" a module which sheafifies to Ω^1_X.
(c) Find the dimensions of the cohomology vector spaces of 1-forms Ω^1_X.

Problem 6: Compute the sheaf cohomology of the sheaf associated to $\mathrm{Hom}(I/I^2, S/I)$, for some choices of monomial ideals I in a polynomial ring S. (start with e.g. $I = (x^2, xy, y^2) \subset k[x, y, z]$). The H^0 of this turns out to be the tangent space to the point $[I]$ on its Hilbert scheme.

Problem 7: Consider the projective variety X which is given by the 3 by 3 minors of a (random) 3x4 matrix of linear forms in $\mathbb{P}^4$ mentioned earlier. Find the dimensions of the cohomology vector spaces of 1-forms Ω^1_X. Compute the Hodge diamond of X.

```
S = ZZ/32003[a,b,c,d,e]
M = random(S^3, S^{4:-1})
I = minors(3, M)
dim I -- one more than the dimension as a projective variety
codim I
degree I
```